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SPACE-TIME HOLOMORPHIC TIME-PERIODIC SOLUTIONS OF NAVIER-STOKES EQUATIONS

EUGENE TSYGANOV

Abstract. We introduce a concept of space-time holomorphic solutions of partial differential equations and construct a meromorphic solution of Navier-Stokes equations, which can be either space or time periodic.

1. Introduction

We study Navier-Stokes equations in Lagrangean coordinates

$$v_t - u_x = 0, \tag{1.1}$$

$$u_t + p_x = \big(\frac{u_x}{v}\big)_x, \tag{1.2}$$

with Cauchy data

$$v(0,x) = v_0(x), \quad u(0,x) = u_0(x). \tag{1.3}$$

Here $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$ are time and space respectively, dependent variable $v = v(x,t)$ denotes the specific volume, $u = u(t,x)$ - velocity, $p = p(v)$ - pressure. We assume that p satisfies the following conditions:

$$p' < 0, \quad \lim_{v \to 0+} p = +\infty, \quad \lim_{v \to +\infty} p = 0.$$

In addition we assume that p is holomorphic in a neighborhood of $\mathbb{R}^+$.

In this article, we continue our study of solutions of the Navier-Stokes equations having analyticity properties. The issue of analyticity was first addressed in Masuda [5] for Navier-Stokes equations for incompressible fluids and was further investigated in a number of papers (see Foias and Temam [2], Constantin, Foias, Kukavica, Majda [1] etc.). The results show that analyticity arises naturally when solving the equations in their classical form, and can be used to study their properties.

Of special interest is the study of complex solutions of the Navier-Stokes equations. Despite the fact that these models do not have a direct physical significance, they provide new information about the equations themselves (see Li and Sinai [4]).

The study of analytic properties of weak solutions of the Navier-Stokes equations for a compressible gas dynamic was initiated in Tsyganov [6] and was further developed for the multi-dimensional case in Hoff and Tsyganov [3]. We can point out that analyticity plays a critical role in the proof of backward uniqueness and

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in the derivation of exact rates of regularization and asymptotic behavior of weak solutions (see Tsyganov [7]).

In this article, we study a special class of analytic solutions, which we call space-time holomorphic. The basic idea is to merge t and x in one complex variable by the equality $z = it + x$, where i is the imaginary unit. After that, in order to find solutions of the original partial differential equation, we need to solve an ordinary differential equation in the complex plane. We give an example of a family of solutions, which are elementary meromorphic functions on the whole complex plane. These functions may blow-up (have a pole) in finite time, however, they become smooth again once we go through these positive values of time. We also prove that these solutions are space and time meromorphic.

It is important to point out that our results have no immediate applications to the pure real case of Navier-Stokes equations because of their strong non-linearity, but, nonetheless, they are significant for several reasons. Firstly, we offer a robust method of constructing explicit complex solutions of partial differential equations with two independent variables. This may be particularly interesting in studying systems where solutions are implied to be complex-valued from the beginning. Secondly, our approach gives new insight into Navier-Stokes equations, as the study of weak solutions far away from the real axis t is a difficult task. Our findings give an idea of what one can expect from the solutions of the Navier-Stokes equations with real initial data on the entire complex plane. Another important result is the construction of a solution where singularities can be described explicitly.

This article is structured as follows: in Section 2 we introduce the notion of space-time holomorphic solutions of partial differential equations. Then, in Section 3, we give an example of such a solution of the Navier-Stokes equations and study its properties.

2. Space-time holomorphic solutions

In this section we introduce a concept of space-time holomorphic solutions.

Definition 2.1. Consider a partial differential equation

$$F\Big(u, \frac{\partial u}{\partial t}, \dots, \frac{\partial^n u}{\partial t^n}, \frac{\partial u}{\partial x}, \dots, \frac{\partial^m u}{\partial x^m}\Big) = 0, \tag{2.1}$$

where $t \in \mathbb{R}, x \in \mathbb{R}, u \in \mathbb{C}^l$, and $F \in \mathbb{R}^k$ is a holomorphic function of its arguments.

We say that solution u of equation (2.1) is space-time holomorphic, if, in addition, it satisfies the equation

$$\frac{\partial u}{\partial t} = i \frac{\partial u}{\partial x}, \tag{2.2}$$

where i is the imaginary unit.

Then it follows from (2.2) that function u satisfies the Cauchy-Riemann condition. So, if we set $z = x + it$, then u becomes a holomorphic function of z and the following equalities hold:

$$\frac{\partial u}{\partial t} = i \frac{du}{dz}, \quad \frac{\partial u}{\partial x} = \frac{du}{dz}. \tag{2.3}$$

Now we can give another definition of space-time holomorphy.

Definition 2.2. We say that a function u is a space-time holomorphic solution of equation (2.1), if it satisfies an ordinary differential equation

$$F\Big(u, i\frac{du}{dz}, \dots, i^n\frac{d^nu}{dz^n}, \frac{du}{dz}, \dots, \frac{d^mu}{dz^m}\Big) = 0 \tag{2.4}$$

in some domain of the complex plane.

Remark 2.3. Instead of condition (2.2) we can set

$$i\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}. \tag{2.5}$$

If we now let $z = t + ix$, then we will come to a different equation for the space-time holomorphy:

$$F\Big(u, \frac{du}{dz}, \dots, \frac{d^nu}{dz^n}, i\frac{du}{dz}, \dots, i^m\frac{d^mu}{dz^m}\Big) = 0. \tag{2.6}$$

In the rest of the article, we will consider only equation (2.4).

Remark 2.4. Let $u = u(z)$ satisfy (2.4) in some domain of the complex plane. If we set $x = \mathrm{Re}(z)$, $t = \mathrm{Im}(z)$, then $u = u(x, t)$ satisfies partial differential equation (2.1) in some domain in $\mathbb{R}^2$.

3. Meromorphic solutions of Navier-Stokes equations

The equations of the space-time holomorphy (2.4) for Navier-Stokes equations (1.1), (1.2) are the following:

$$iv_z - u_z = 0; \tag{3.1}$$

$$iu_z + p_z = \big(\frac{u_z}{v}\big)_z. \tag{3.2}$$

We express v_z in terms of u_z and then substitute it into the second equation:

$$-v_z + p_z = \big(\frac{iv_z}{v}\big)_z. \tag{3.3}$$

Then we integrate the above equality to obtain

$$-v + p = i\frac{v_z}{v} + C, \tag{3.4}$$

where C is an arbitrary constant of integration. We rewrite the equation and integrate it once again:

$$z + C_1 = i\int \frac{dv}{-v^2 + vp + C_2v} \tag{3.5}$$

We will primarily be interested in those functions p, for which the equation is integrable by quadratures. Setting $p = \frac{1}{v}$, $C_2 = 0$ and $C_1 = C$, we can obtained the answer in closed form:

$$z + C = i\frac{1}{2}(\ln(v+1) - \ln(1-v)),$$

from which we obtain v:

$$v = \frac{e^{-2i(z+C)} - 1}{e^{-2i(z+C)} + 1}. \tag{3.6}$$

Now we can find u:

$$u = i\frac{e^{-2i(z+C)} - 1}{e^{-2i(z+C)} + 1} + C_2 \tag{3.7}$$

for a new constant C_2. We point out that the functions v and u are meromorphic on the whole complex plane and have the following properties:

$$\lim_{z\to\infty,\,\mathrm{Im}(z)>0} v = 1, \qquad \lim_{z\to\infty,\,\mathrm{Im}(z)>0} u = i + C_2. \tag{3.8}$$

Remark 3.1. We can see that the corresponding solution $v = v(x,t)$, $u = u(x,t)$ of (2.1) is periodic with respect to the space variable x.

Remark 3.2. If we use condition (2.5) instead of (2.2), then space-time holomorphic solutions for Navier-Stokes equations with $p = 1/v$ will be

$$v = \frac{e^{2(z+C)} - 1}{e^{2(z+C)} + 1}, \quad u = i\frac{e^{2(z+C)} - 1}{e^{2(z+C)} + 1} + C_2. \tag{3.9}$$

The corresponding functions $v = v(x,t)$, $u = u(x,t)$ are periodic with respect to the time variable t.

We are interested primarily in the zeros and poles of v:

$$\text{zeroes: } z = \pi k - C, \quad \text{poles: } z = \frac{\pi}{2} + \pi k - C, \quad k \in \mathbb{Z}.$$

Depending on the value of C we can have the following situation in the half-plane $\mathrm{Im}(z) \geq 0$:

(1) $\mathrm{Im}(C) > 0$. Then the functions v and u are holomorphic in $\mathrm{Im}(z) > 0$ and continuous on $\mathrm{Im}(z) \geq 0$. In this case, the pair of functions $v = v(x,t)$, $u = u(x,t)$ is a smooth solution of the Navier-Stokes equations (1.1), (1.2) in the half-plane $\mathbb{R} \times \mathbb{R}^+$ with smooth initial data;

(2) $\mathrm{Im}(C) = 0$. Then (v,u) is a smooth solution of (1.1),(1.2) in the half-plane $t > 0$ with meromorphic initial data;

(3) $\mathrm{Im}(C) < 0$. In this case, (v,u) is a meromorphic solution for $t > 0$ with smooth initial data at $t = 0$.

Remark 3.3. Equation (1.2) does not hold at the points where $v = 0$. These singularities are, however, removable, since (v,u) exist and is smooth at such points.

Remark 3.4. Functions $v = v(t,x)$, $u = u(t,x)$ are meromorphic in t for any fixed $x \in \mathbb{R}$. To prove this we use the following identities:

$$v(t,x) := v(t+x), \quad u(t,x) := u(t+x), \quad t \in \mathbb{C},\ x \in \mathbb{R}. \tag{3.10}$$

The solution is also meromorphic in x for any fixed $t \in \mathbb{R}$. This follows from the obvious definitions

$$v(t,x) := v(it+x), \quad u(t,x) := u(it+x), \quad x \in \mathbb{C},\ t \in \mathbb{R}. \tag{3.11}$$

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Eugene Tsyganov
Department of Mathematics and Informatics Technology, Bashkir State University, Russia
E-mail address: entsyganov@yahoo.com