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THIRD-ORDER OPERATOR-DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND OPERATORS IN THE BOUNDARY CONDITIONS

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Abstract. We study a third-order operator-differential equation on the semi-axis with a discontinuous coefficient and boundary conditions which include an abstract linear operator. Sufficient conditions for the well-posed and unique solvability are found by means of properties of the operator coefficients in a Sobolev-type space.

1. Introduction

It is known that many problems of partial differential equations can be reduced to problems for differential equations whose coefficients are unbounded operators in a Hilbert space. Many articles are dedicated to the study of problems with operators in the boundary conditions for operator-differential equations of second order (see, for example, [1, 7, 10, 13, 16, 17, 18, 25] and the references therein); however, these studies are far from the full completion. Note that only a few papers are dedicated to the study of such boundary-value problems for operator-differential equations of third order (see, for example, [5]).

This article is dedicated to the study of boundary-value problem for a class of third-order operator-differential equations with a discontinuous coefficient; one of the boundary conditions includes an abstract linear operator. Such equations cover some non-classical problems of mathematical physics (see [8]), investigated in inhomogeneous environments.

Let H be a separable Hilbert space with the scalar product (x, y), $x, y \in H$ and let A be a self-adjoint positive-definite operator in H ($A = A^* \geq cE$, $c > 0$, E is the identity operator). By H_γ ($\gamma \geq 0$) we denote the scale of Hilbert spaces generated by the operator A; i.e., $H_\gamma = D(A^\gamma)$, $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$, for $\gamma = 0$ we consider that $H_0 = H$, $(x, y)_0 = (x, y)$, $x, y \in H$.

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We denote by $L_2([a,b];H)$, $-\infty \le a < b \le +\infty$, the Hilbert space of all vector functions defined on $[a,b]$ with values in H and endowed with the norm

$$\|f\|_{L_2([a,b];H)} = \Big(\int_a^b \|f(t)\|_H^2 dt\Big)^{1/2}.$$

Following the book [14], we introduce the Hilbert space

$$W_2^3([a,b];H) = \{u(t) : u'''(t) \in L_2([a,b];H), A^3u(t) \in L_2([a,b];H)\}$$

endowed with the norm

$$\|u\|_{W_2^3([a,b];H)} = \big(\|u'''\|^2_{L_2([a,b];H)} + \|A^3u\|^2_{L_2([a,b];H)}\big)^{1/2}.$$

Hereafter, derivatives are understood in the sense of distributions in a Hilbert space [14]. The spaces $L_2((-\infty,+\infty);H)$, $W_2^3((-\infty,+\infty);H)$, $L_2([0,+\infty);H)$ and $W_2^3([0,+\infty);H)$ will be denoted by $L_2(R;H)$, $W_2^3(R;H)$, $L_2(R_+;H)$ and $W_2^3(R_+;H)$, respectively.

Further, we denote by $L(X,Y)$ the space of all linear bounded operators acting from a Hilbert space X to another Hilbert space Y, and we denote by $\sigma(\cdot)$ the spectrum of the operator $(\cdot)$.

Consider the boundary value problem in the Hilbert space H

$$-u'''(t) + \rho(t)A^3u(t) + \sum_{j=1}^{3} A_j \frac{d^{3-j}u(t)}{dt^{3-j}} = f(t), t \in R_+, \tag{1.1}$$

$$u'(0) = 0, \quad u''(0) = Ku(0), \tag{1.2}$$

where $A = A^* \ge cE$, $c > 0$, $K \in L(H_{5/2}, H_{1/2})$, $\rho(t) = \alpha$, if $0 \le t \le 1$, $\rho(t) = \beta$, if $1 < t < +\infty$, here α, β are positive numbers, $f(t) \in L_2(R_+;H)$, $u(t) \in W_2^3(R_+;H)$.

Definition 1.1. If a vector function $u(t) \in W_2^3(R_+;H)$ satisfies (1.1) almost everywhere in R_+, then it is called a regular solution of equation (1.1).

Definition 1.2. If for any $f(t) \in L_2(R_+;H)$ there exists a regular solution of (1.1), which satisfies the boundary conditions (1.2) in the sense that

$$\lim_{t\to 0} \|u'(t)\|_{H_{3/2}} = 0, \quad \lim_{t\to 0} \|u''(t) - Ku(t)\|_{H_{1/2}} = 0$$

and the following inequality holds

$$\|u\|_{W_2^3(R_+;H)} \le \text{const } \|f\|_{L_2(R_+;H)},$$

then we say that the problem (1.1), (1.2) is regularly solvable.

Similar kind of problems on a semi-axis for elliptic operator-differential equations of the second order is considered in papers [13, 16, 17]. We should especially note the work [21] which considers the non-local boundary value problems for second order elliptic operator-differential equations on the interval with the coefficients belonging to a broader class of discontinuous functions, while the coefficients in the boundary conditions are complex numbers. In [2, 8, 11, 12, 15, 19, 20, 22, 23] along with other problems investigated the solvability of boundary-value problems for elliptic operator-differential equations of the general form when the coefficients in the boundary conditions are complex numbers and the equations do not contain discontinuous coefficients. Such case also is considered in [3, 6] for the third and fourth orders equations with multiple characteristics. Note that problem (1.1), (1.2) is investigated in the case $A_3 = 0$, $K = 0$ in [4] and when $\rho(t) \equiv 1$, $t \in R_+$, $K = 0$ in [15].

In this article, we obtain the conditions of regular solvability of the boundary-value problem (1.1), (1.2) by means of properties of operator coefficients.

2. Main results

Before proceeding to the consideration of the question posed, let us introduce additional notation. Let

$$W^3_{2,K}(R_+;H) = \{u(t) : u(t) \in W^3_2(R_+;H), u'(0) = 0, u''(0) = Ku(0)\}$$

and denote by P_0, P_1 and P the operators acting from the space $W^3_{2,K}(R_+;H)$ into the space $L_2(R_+;H)$ by the following rules, respectively:

$$\begin{gathered} P_0u(t) = -u'''(t) + \rho(t)A^3u(t), \\ P_1u(t) = A_1u''(t) + A_2u'(t) + A_3u(t), \\ Pu(t) = P_0u(t) + P_1u(t), \quad u(t) \in W^3_{2,K}(R_+;H). \end{gathered}$$

Put $B = A^{1/2}KA^{-5/2}$, $\kappa(c_1, c_2, c_3) = c_1\sqrt[3]{\beta^2} + c_2\sqrt[3]{\alpha\beta} + c_3\sqrt[3]{\alpha^2}$ and

$$\begin{aligned} K_{\alpha,\beta} &= (E + \frac{1}{\sqrt[3]{\alpha^2}}\omega_2A^{-2}K)(\kappa(1,1,1)\omega_2E - \kappa(1,\omega_2,\omega_1)e^{\sqrt[3]{\alpha}(\omega_2-1)A}) \\ &\quad + \Big(E + \frac{1}{\sqrt[3]{\alpha^2}}\omega_1A^{-2}K\Big)\Big(\kappa(1,\omega_1,\omega_2)e^{\sqrt[3]{\alpha}(\omega_1-1)A} - \kappa(1,1,1)\omega_1E\Big), \end{aligned}$$

where $\omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\omega_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Lemma 2.1. *Let $A = A^* \geq cE$, $c > 0$, $K \in L(H_{5/2}, H_{1/2})$, $-\sqrt[3]{\alpha^2}\omega_2 \notin \sigma(B)$ and the operator $K_{\alpha,\beta}$ have a bounded inverse operator in $H_{5/2}$. Then the equation $P_0u = 0$ has only the trivial solution in the space $W^3_{2,K}(R_+;H)$.*

Proof. The general solution of the equation $P_0u(t) = 0$ in the space $W^3_2(R_+;H)$ has the following form [24]:

$$u_0(t) = \begin{cases} u_{0,1}(t) = e^{\sqrt[3]{\alpha}\omega_1tA}\varphi_0 + e^{\sqrt[3]{\alpha}\omega_2tA}\varphi_1 + e^{-\sqrt[3]{\alpha}(1-t)A}\varphi_2, & 0 \leq t < 1, \\ u_{0,2}(t) = e^{\sqrt[3]{\beta}\omega_1(t-1)A}\varphi_3 + e^{\sqrt[3]{\beta}\omega_2(t-1)A}\varphi_4, & 1 < t < +\infty, \end{cases}$$

where the vectors $\varphi_k \in H_{5/2}$, $k = 0,1,2,3,4$, are determined from the boundary conditions (1.2) and the condition $u_0(t) \in W^3_2(R_+;H)$. Therefore, to determine the vectors φ_k, $k = 0,1,2,3,4$, we have the following relations:

$$\begin{gathered} u'_{0,1}(0) = 0, \quad u''_{0,1}(0) = Ku_{0,1}(0), \quad u_{0,1}(1) = u_{0,2}(1), \\ u'_{0,1}(1) = u'_{0,2}(1), \quad u''_{0,1}(1) = u''_{0,2}(1). \end{gathered}$$

From these relations we obtain the following system of equations with respect to φ_k, $k = 0, 1, 2, 3, 4$:

$$
\begin{gathered}
\omega_1\varphi_0 + \omega_2\varphi_1 + e^{-\sqrt[3]{\alpha}A}\varphi_2 = 0\,, \\
\omega_1^2\varphi_0 + \omega_2^2\varphi_1 + e^{-\sqrt[3]{\alpha}A}\varphi_2 = \frac{1}{\sqrt[3]{\alpha^2}}A^{-2}K(\varphi_0 + \varphi_1 + e^{-\sqrt[3]{\alpha}A}\varphi_2), \\
e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0 + e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1 + \varphi_2 = \varphi_3 + \varphi_4\,, \\
\sqrt[3]{\alpha}\omega_1 e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0 + \sqrt[3]{\alpha}\omega_2 e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1 + \sqrt[3]{\alpha}\varphi_2 = \sqrt[3]{\beta}\omega_1\varphi_3 + \sqrt[3]{\beta}\omega_2\varphi_4\,, \\
\sqrt[3]{\alpha^2}\omega_1^2 e^{\sqrt[3]{\alpha}\omega_1 A}\varphi_0 + \sqrt[3]{\alpha^2}\omega_2^2 e^{\sqrt[3]{\alpha}\omega_2 A}\varphi_1 + \sqrt[3]{\alpha^2}\varphi_2 = \sqrt[3]{\beta^2}\omega_1^2\varphi_3 + \sqrt[3]{\beta^2}\omega_2^2\varphi_4\,.
\end{gathered}
\tag{2.1}
$$

Taking into account $\omega_1\omega_2 = 1$, $\omega_1 + \omega_2 = -1$, $\omega_1^2 = \omega_2$, $\omega_2^2 = \omega_1$, from the system (2.1) after simple transformations with respect to φ_0 we have

$$
\begin{aligned}
&(E + \frac{1}{\sqrt[3]{\alpha^2}}\omega_2 A^{-2}K)(\kappa(1,1,1)\omega_2 E - \kappa(1,\omega_2,\omega_1)e^{\sqrt[3]{\alpha}(\omega_2-1)A})\varphi_0 \\
&+ (E + \frac{1}{\sqrt[3]{\alpha^2}}\omega_1 A^{-2}K)(\kappa(1,\omega_1,\omega_2)e^{\sqrt[3]{\alpha}(\omega_1-1)A} - \kappa(1,1,1)\omega_1 E)\varphi_0 = 0.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
K_{\alpha,\beta}\varphi_0 \equiv \Big[&\big(E + \frac{1}{\sqrt[3]{\alpha^2}}\omega_2 A^{-2}K\big)\big(\kappa(1,1,1)\omega_2 E - \kappa(1,\omega_2,\omega_1)e^{\sqrt[3]{\alpha}(\omega_2-1)A}\big) \\
&+ \big(E + \frac{1}{\sqrt[3]{\alpha^2}}\omega_1 A^{-2}K\big)\big(\kappa(1,\omega_1,\omega_2)e^{\sqrt[3]{\alpha}(\omega_1-1)A} - \kappa(1,1,1)\omega_1 E\big)\Big]\varphi_0 = 0.
\end{aligned}
\tag{2.2}
$$

By the assumption of this lemma, $K_{\alpha,\beta}$ has a bounded inverse operator in the space $H_{5/2}$, then from equation (2.2) follows that $\varphi_0 = 0$. Considering $\varphi_0 = 0$ in the first and second equations of (2.1), we obtain

$$
\varphi_1 = -\omega_1 e^{-\sqrt[3]{\alpha}A}\varphi_2, \tag{2.3}
$$

$$
(E + \frac{1}{\sqrt[3]{\alpha^2}}\omega_1 A^{-2}K)e^{-\sqrt[3]{\alpha}A}\varphi_2 = 0. \tag{2.4}
$$

In turn, by assumption $-\sqrt[3]{\alpha^2}\omega_2 \notin \sigma(B)$ from (2.4) it follows that $\varphi_2 = 0$, and therefore, from (2.3) $\varphi_1 = 0$. Now, considering $\varphi_0 = \varphi_1 = \varphi_2 = 0$ in the fourth and fifth equations of (2.1), we obtain:

$$
\begin{gathered}
\omega_1\varphi_3 + \omega_2\varphi_4 = 0, \\
\omega_2\varphi_3 + \omega_1\varphi_4 = 0.
\end{gathered}
$$

And from these equations we have that $\varphi_3 = \varphi_4 = 0$. Thus, $u_0(t) = 0$. The proof is complete. □

Let us consider the question of regular solvability of problem (1.1), (1.2) when $A_1 = A_2 = A_3 = 0$.

Lemma 2.2. *In the conditions of Lemma 2.1, the problem*

$$
-u'''(t) + \rho(t)A^3 u(t) = f(t), \quad t \in R_+, \tag{2.5}
$$

$$
u'(0) = 0, \quad u''(0) = Ku(0) \tag{2.6}
$$

is regularly solvable.

Proof. We show that the equation $P_0u(t) = f(t)$ has a solution $u(t) \in W^3_{2,K}(R_+; H)$ for any $f(t) \in L_2(R_+; H)$. First, we continue the vector-function $f(t)$ in such a way that $f(t) = 0$ for $t < 0$. We denote the new function by $g(t)$. Let $\hat{g}(\xi)$ be the Fourier transform of the vector-function $g(t)$; i.e.,

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-i\xi t} dt,$$

where the integral on the right side is understood in the sense of convergence on the average in H. Then, using the direct and inverse Fourier transforms, it is clear that the vector-functions

$$\upsilon_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\xi^3 E + \alpha A^3)^{-1} \Big(\int_0^{+\infty} f(s) e^{-i\xi s} ds \Big) e^{it\xi} d\xi, \quad t \in R,$$
$$\upsilon_2(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\xi^3 E + \beta A^3)^{-1} \Big(\int_0^{+\infty} f(s) e^{-i\xi s} ds \Big) e^{it\xi} d\xi, \quad t \in R,$$

satisfy the equations

$$-\frac{d^3\upsilon(t)}{dt^3} + \alpha A^3 \upsilon(t) = g(t),$$
$$-\frac{d^3\upsilon(t)}{dt^3} + \beta A^3 \upsilon(t) = g(t),$$

respectively, almost everywhere in R. We prove that $\upsilon_1(t)$ and $\upsilon_2(t)$ belong to $W^3_2(R; H)$. By Plancherel's theorem

$$\begin{aligned} \|\upsilon_1(t)\|^2_{W^3_2(R;H)} &= \|\upsilon_1'''(t)\|^2_{L_2(R;H)} + \|A^3\upsilon_1(t)\|^2_{L_2(R;H)} \\ &= \| -i\xi^3 \hat{\upsilon}_1(\xi)\|^2_{L_2(R;H)} + \|A^3 \hat{\upsilon}_1(\xi)\|^2_{L_2(R;H)}, \end{aligned}$$

where $\hat{\upsilon}_1(\xi)$ is the Fourier transform of the function $\upsilon_1(t)$. Since

$$\hat{\upsilon}_1(\xi) = (i\xi^3 E + \alpha A^3)^{-1} \hat{g}(\xi),$$

we have

$$\begin{aligned} \| -i\xi^3 \hat{\upsilon}_1(\xi)\|_{L_2(R;H)} &= \| -i\xi^3 (i\xi^3 E + \alpha A^3)^{-1} \hat{g}(\xi)\|_{L_2(R;H)} \\ &\le \sup_{\xi\in R} \| -i\xi^3 (i\xi^3 E + \alpha A^3)^{-1}\|_{H\to H} \|\hat{g}(\xi)\|_{L_2(R;H)} \\ &= \sup_{\xi\in R} \| -i\xi^3 (i\xi^3 E + \alpha A^3)^{-1}\|_{H\to H} \|g(t)\|_{L_2(R;H)}. \end{aligned} \tag{2.7}$$

It follows from the spectral theory of self-adjoint operators that

$$\begin{aligned} \| -i\xi^3 (i\xi^3 E + \alpha A^3)^{-1}\| &= \sup_{\sigma\in\sigma(A)} \big| -i\xi^3 (i\xi^3 + \alpha\sigma^3)^{-1}\big| \\ &= \sup_{\sigma\in\sigma(A)} \frac{|\xi|^3}{(\xi^6 + \alpha^2\sigma^6)^{1/2}} \le 1. \end{aligned} \tag{2.8}$$

Therefore, from (2.7) it follows that $-i\xi^3\hat{\upsilon}_1(\xi) \in L_2(R;\, H)$. Since

$$\begin{aligned} \|A^3 \hat{\upsilon}_1(\xi)\|_{L_2(R;H)} &= \|A^3 (i\xi^3 E + \alpha A^3)^{-1} \hat{g}(\xi)\|_{L_2(R;H)} \\ &\le \sup_{\xi\in R} \|A^3 (i\xi^3 E + \alpha A^3)^{-1}\|_{H\to H} \|\hat{g}(\xi)\|_{L_2(R;H)} \\ &= \sup_{\xi\in R} \|A^3 (i\xi^3 E + \alpha A^3)^{-1}\|_{H\to H} \|g(t)\|_{L_2(R;H)}, \end{aligned} \tag{2.9}$$

again from the spectral theory of self-adjoint operators, we have

$$\|A^3(i\xi^3E+\alpha A^3)^{-1}\| = \sup_{\sigma\in\sigma(A)} |\sigma^3(i\xi^3+\alpha\sigma^3)^{-1}| = \sup_{\sigma\in\sigma(A)} \frac{\sigma^3}{(\xi^6+\alpha^2\sigma^6)^{1/2}} \le \frac{1}{\alpha}.$$

Thus, from (2.9) it follows that $A^3\hat{\upsilon}_1(\xi)\in L_2(R;H)$. Hence, $\upsilon_1(t)\in W_2^3(R;H)$. Thus $\upsilon_2(t)\in W_2^3(R;H)$.

Let us denote the restriction of the vector-function $\upsilon_1(t)$ on $[0,1)$ by $u_\alpha(t)$ and the restriction of the vector-function $\upsilon_2(t)$ on $(1,+\infty)$ by $u_\beta(t)$. It is obvious, that $u_\alpha(t)\in W_2^3([0,1);H)$, $u_\beta(t)\in W_2^3((1,+\infty);H)$. Then, from the theorem on traces [14, ch. 1] it follows that $\frac{d^su_\alpha(0)}{dt^s}$, $\frac{d^su_\alpha(1)}{dt^s}$, $\frac{d^su_\beta(0)}{dt^s}$, $\frac{d^su_\beta(1)}{dt^s}\in H_{5/2-s}$, $s=0,1,2$.

Now, we denote

$$u(t)=\begin{cases} u_1(t)=u_\alpha(t)+e^{\sqrt[3]{\alpha}\omega_1tA}\psi_0+e^{\sqrt[3]{\alpha}\omega_2tA}\psi_1+e^{-\sqrt[3]{\alpha}(1-t)A}\psi_2, & 0\le t<1,\\ u_2(t)=u_\beta(t)+e^{\sqrt[3]{\beta}\omega_1(t-1)A}\psi_3+e^{\sqrt[3]{\beta}\omega_2(t-1)A}\psi_4, & 1<t<+\infty,\end{cases}$$

where $\psi_k\in H_{5/2}$, $k=0,1,2,3,4$. The function $u(t)$ belongs to $W_{2,K}^3(R_+;H)$, so the vectors ψ_k, $k=0,1,2,3,4$, can be determined from the following relations:

$$u_1'(0)=0,\quad u_1''(0)=Ku_1(0),\quad u_1(1)=u_2(1),\quad u_1'(1)=u_2'(1),\quad u_1''(1)=u_2''(1).$$

From here with respect to ψ_k, $k=0,1,2,3,4$, we have the system of equations

$$\begin{gathered} u_\alpha'(0)+\sqrt[3]{\alpha}\omega_1A\psi_0+\sqrt[3]{\alpha}\omega_2A\psi_1+\sqrt[3]{\alpha}Ae^{-\sqrt[3]{\alpha}A}\psi_2=0\,,\\ u_\alpha''(0)+\sqrt[3]{\alpha^2}A^2\big(\omega_1^2\psi_0+\omega_2^2\psi_1+e^{-\sqrt[3]{\alpha}A}\psi_2\big)=Ku_\alpha(0)+K\big(\psi_0+\psi_1+e^{-\sqrt[3]{\alpha}A}\psi_2\big),\\ u_\alpha(1)+e^{\sqrt[3]{\alpha}\omega_1A}\psi_0+e^{\sqrt[3]{\alpha}\omega_2A}\psi_1+\psi_2=u_\beta(1)+\psi_3+\psi_4\,,\\ u_\alpha'(1)+\sqrt[3]{\alpha}\omega_1Ae^{\sqrt[3]{\alpha}\omega_1A}\psi_0+\sqrt[3]{\alpha}\omega_2Ae^{\sqrt[3]{\alpha}\omega_2A}\psi_1+\sqrt[3]{\alpha}A\psi_2\\ =u_\beta'(1)+\sqrt[3]{\beta}\omega_1A\psi_3+\sqrt[3]{\beta}\omega_2A\psi_4\,,\\ u_\alpha''(1)+\sqrt[3]{\alpha^2}\omega_1^2A^2e^{\sqrt[3]{\alpha}\omega_1A}\psi_0+\sqrt[3]{\alpha^2}\omega_2^2A^2e^{\sqrt[3]{\alpha}\omega_2A}\psi_1+\sqrt[3]{\alpha^2}A^2\psi_2\\ =u_\beta''(1)+\sqrt[3]{\beta^2}\omega_1^2A^2\psi_3+\sqrt[3]{\beta^2}\omega_2^2A^2\psi_4\,. \end{gathered}$$

From this system we obtain:

$$\begin{gathered} \sqrt[3]{\alpha}\omega_1\psi_0+\sqrt[3]{\alpha}\omega_2\psi_1+\sqrt[3]{\alpha}e^{-\sqrt[3]{\alpha}A}\psi_2=-A^{-1}u_\alpha'(0),\\ \sqrt[3]{\alpha^2}(\omega_1^2\psi_0+\omega_2^2\psi_1+e^{-\sqrt[3]{\alpha}A}\psi_2)-A^{-2}K(\psi_0+\psi_1+e^{-\sqrt[3]{\alpha}A}\psi_2)\\ =A^{-2}(Ku_\alpha(0)-u_\alpha''(0))\,,\\ e^{\sqrt[3]{\alpha}\omega_1A}\psi_0+e^{\sqrt[3]{\alpha}\omega_2A}\psi_1+\psi_2-\psi_3-\psi_4=u_\beta(1)-u_\alpha(1),\\ \sqrt[3]{\alpha}\omega_1e^{\sqrt[3]{\alpha}\omega_1A}\psi_0+\sqrt[3]{\alpha}\omega_2e^{\sqrt[3]{\alpha}\omega_2A}\psi_1+\sqrt[3]{\alpha}\psi_2-\sqrt[3]{\beta}\omega_1\psi_3-\sqrt[3]{\beta}\omega_2\psi_4\\ =A^{-1}(u_\beta'(1)-u_\alpha'(1))\,,\\ \sqrt[3]{\alpha^2}\omega_1^2e^{\sqrt[3]{\alpha}\omega_1A}\psi_0+\sqrt[3]{\alpha^2}\omega_2^2e^{\sqrt[3]{\alpha}\omega_2A}\psi_1+\sqrt[3]{\alpha^2}\psi_2-\sqrt[3]{\beta^2}\omega_1^2\psi_3-\sqrt[3]{\beta^2}\omega_2^2\psi_4\\ =A^{-2}(u_\beta''(1)-u_\alpha''(1))\,. \end{gathered} \tag{2.10}$$

Since $u_\alpha(t)\in W_2^3([0,1);H)$ and $u_\beta(t)\in W_2^3((1,+\infty);H)$, by the theorem on traces [14, ch. 1] $A^{-1}u_\alpha'(0)$, $A^{-2}(Ku_\alpha(0)-u_\alpha''(0))$, $u_\beta(1)-u_\alpha(1)$, $A^{-1}(u_\beta'(1)-u_\alpha'(1))$ and $A^{-2}(u_\beta''(1)-u_\alpha''(1))$ belong to $H_{5/2}$. Then by these values acting also as in the system (2.1), in this case, taking into account that the operator $K_{\alpha,\beta}$ has a bounded inverse operator in the space $H_{5/2}$ and $-\sqrt[3]{\alpha^2}\omega_2\notin\sigma(B)$, obviously, from (2.10) it is possible to find the vectors ψ_k, $k=0,1,2,3,4$, where all $\psi_k \in H_{5/2}$, $k=0,1,2,3,4$. Therefore, $u(t)\in W_2^3(R_+;H)$ satisfies equation (2.5) almost everywhere in R_+ and conditions (2.6).

By lemma 2.1, the problem

$$-u'''(t)+\rho(t)A^3u(t)=0,$$
$$u'(0)=0,\quad u''(0)=Ku(0)$$

has only the trivial solution in the space $W^3_{2,K}(R_+;H)$.

Now we show that the operator $P_0 : W^3_{2,K}(R_+;H)\to L_2(R_+;H)$ is bounded. Indeed, for $u(t)\in W^3_{2,K}(R_+;H)$ we have

$$\begin{aligned}
&\|P_0u\|^2_{L_2(R_+;H)}\\
&=\|u'''\|^2_{L_2(R_+;H)}+\|\rho(t)A^3u\|^2_{L_2(R_+;H)}-2\operatorname{Re}(u''',\rho(t)A^3u)_{L_2(R_+;H)}\\
&\le \|u'''\|^2_{L_2(R_+;H)}+\|\rho(t)A^3u\|^2_{L_2(R_+;H)}+2\|u'''\|_{L_2(R_+;H)}\|\rho(t)A^3u\|_{L_2(R_+;H)}\\
&\le 2\big(\|u'''\|^2_{L_2(R_+;H)}+\|\rho(t)A^3u\|^2_{L_2(R_+;H)}\big)\\
&\le 2\max(1;\alpha^2;\beta^2)\|u\|^2_{W_2^3(R_+;H)}.
\end{aligned}$$

Thus, according to the Banach theorem on the inverse operator, there exists P_0^{-1} : $L_2(R_+;H)\to W^3_{2,K}(R_+;H)$ and it is bounded. Hence, it follows that

$$\|u\|_{W_2^3(R_+;H)}\le \text{const}\|f\|_{L_2(R_+\,H)}.$$

The proof is complete. □

On the basis of Lemmas 2.1 and 2.2 we obtain the following conclusion.

Theorem 2.3. *Let the conditions of Lemma 2.1 be satisfied. Then the operator P_0 is an isomorphism between the spaces $W^3_{2,K}(R_+;H)$ and $L_2(R_+;H)$.*

Let us prove the following coercive inequality which will be used further.

Lemma 2.4. *Let $\operatorname{Re}(B)\ge 0$. Then for any $u(t)\in W^3_{2,K}(R_+;H)$, the following inequality holds*

$$\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)}+\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)}\le \frac{1}{\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}. \tag{2.11}$$

Proof. Consider the following equalities:

$$\begin{aligned}
(P_0u,A^3u)_{L_2(R_+;H)}&=(-u'''+\rho(t)A^3u,A^3u)_{L_2(R_+;H)}\\
&=(-u''',A^3u)_{L_2(R_+;H)}+(\rho(t)A^3u,A^3u)_{L_2(R_+;H)}\\
&=(-u''',A^3u)_{L_2(R_+;H)}+\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)},
\end{aligned}\tag{2.12}$$

$$\begin{aligned}
(P_0u,-\rho^{-1}(t)u''')_{L_2(R_+;H)}&=(-u'''+\rho(t)A^3u,-\rho^{-1}(t)u''')_{L_2(R_+;H)}\\
&=(-u''',-\rho^{-1}(t)u''')_{L_2(R_+;H)}-(A^3u,u''')_{L_2(R_+;H)}\\
&=\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)}-(A^3u,u''')_{L_2(R_+;H)}.
\end{aligned}\tag{2.13}$$

Note that by integrating by parts for $u(t)\in W^3_{2,K}(R_+;H)$, we have

$$-\operatorname{Re}(u''',A^3u)_{L_2(R_+;H)}=\operatorname{Re}(BA^{5/2}u(0),A^{5/2}u(0)). \tag{2.14}$$

By (2.12) and (2.13) and taking into account (2.14), we obtain

$$
\begin{aligned}
&(P_0u, A^3u - \rho^{-1}(t)u''')_{L_2(R_+;H)} \\
&= \|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)} + \|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)} + 2\operatorname{Re}(BA^{5/2}u(0), A^{5/2}u(0)).
\end{aligned} \tag{2.15}
$$

Applying the Cauchy-Schwarz inequality to the left side and then the Young's inequality and taking into account (2.14), we obtain

$$
\begin{aligned}
&(P_0u, A^3u - \rho^{-1}(t)u''')_{L_2(R_+;H)} \\
&\le \|\rho^{-1/2}(t)P_0u\|_{L_2(R_+;H)}\|\rho^{1/2}(t)A^3u - \rho^{-1/2}(t)u'''\|_{L_2(R_+;H)} \\
&\le \frac{1}{2\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)} + \frac{1}{2}\|\rho^{1/2}(t)A^3u - \rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)} \\
&= \frac{1}{2\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)} + \frac{1}{2}\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)} \\
&\quad + \frac{1}{2}\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)} + \operatorname{Re}\big(BA^{5/2}u(0), A^{5/2}u(0)\big).
\end{aligned} \tag{2.16}
$$

Taking into account (2.16) into (2.15), we have

$$
\begin{aligned}
&\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)} + \|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)} + 2\operatorname{Re}(BA^{5/2}u(0), A^{5/2}u(0)) \\
&\le \frac{1}{\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}.
\end{aligned} \tag{2.17}
$$

Since $\operatorname{Re} B \ge 0$, then from inequality (2.17), we obtain the validity of inequality (2.11). The proof is complete. □

Theorem 2.3 implies that the norm $\|P_0u\|_{L_2(R_+;H)}$ is equivalent to the norm $\|u\|_{W_2^3(R_+;H)}$ in the space $W^3_{2,K}(R_+;H)$. Therefore, the norms of the intermediate derivative operators $A^j\frac{d^{3-j}}{dt^{3-j}} : W^3_{2,K}(R_+;H) \to L_2(R_+;H)$, $j = 1,2,3$, can be estimated with respect to $\|P_0u\|_{L_2(R_+;H)}$ (by the continuity of these operators [14]). Methods for solution of equations with scalar boundary conditions are often inapplicable to the problems with boundary conditions which include abstract operators. For example, when $K = 0$, operator pencil factorization method for the estimation of the norms of intermediate derivative operators has been developed in [4] (this method was first mentioned in [15] when considering operator-differential equations with constant coefficients). The estimates for the norms of intermediate derivative operators are playing an important role in obtaining solvability conditions. But, the method of [4] is not applicable to the boundary value problems for odd order operator-differential equations with the boundary conditions which include abstract operators. In this work, to estimate the norms of intermediate derivative operators we use the classical inequalities of mathematical analysis and the coercive inequality (2.11).

Theorem 2.5. *Let* $\operatorname{Re} B \ge 0$. *Then for any* $u(t) \in W^3_{2,K}(R_+;H)$ *the following inequalities hold:*

$$
\|A^j\frac{d^{3-j}u}{dt^{3-j}}\|_{L_2(R_+;H)} \le a_j\|P_0u\|_{L_2(R_+;H)}, \quad j = 1,2,3, \tag{2.18}
$$

where

$$a_1 = \frac{2^{1/3}\max^{1/3}(\alpha;\beta)}{3^{1/2}\min^{2/3}(\alpha;\beta)}, \quad a_2 = \frac{2^{1/3}\max^{1/6}(\alpha;\beta)}{3^{1/2}\min^{5/6}(\alpha;\beta)}, \quad a_3 = \frac{1}{\min(\alpha;\beta)}.$$

Proof. Let $u(t) \in W^3_{2,K}(R_+;H)$. Integrating by parts and applying the Cauchy-Schwarz inequality, and then the Young's inequality, we obtain

$$\begin{aligned}
\|Au''\|^2_{L_2(R_+;H)} &= \int_0^{+\infty} (Au'', Au'')_H dt \\
&= (Au', Au'')_H|_0^{+\infty} - \int_0^{+\infty} (Au', Au''')_H dt \\
&= -\int_0^{+\infty} (A^2u', u''')_H dt \le \|A^2u'\|_{L_2(R_+;H)}\|u'''\|_{L_2(R_+;H)} \\
&\le \max_t \rho^{1/2}(t)\|A^2u'\|_{L_2(R_+;H)}\|\rho^{-1/2}(t)u'''\|_{L_2(R_+;H)} \\
&\le \frac{\varepsilon}{2}\max(\alpha;\beta)\|A^2u'\|^2_{L_2(R_+;H)} + \frac{1}{2\varepsilon}\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)},
\end{aligned} \tag{2.19}$$

with $\varepsilon > 0$. Proceeding in a similar manner, we have

$$\begin{aligned}
\|A^2u'\|^2_{L_2(R_+;H)} &= \int_0^{+\infty} (A^2u', A^2u')_H dt \\
&= (A^2u, A^2u')_H|_0^{+\infty} - \int_0^{+\infty} (A^2u, A^2u'')_H dt \\
&= -\int_0^{+\infty} (A^3u, Au'')_H dt \\
&\le \|A^3u\|_{L_2(R_+;H)}\|Au''\|_{L_2(R_+;H)} \\
&\le \max_t \rho^{-1/2}(t)\|Au''\|_{L_2(R_+;H)}\|\rho^{1/2}(t)A^3u\|_{L_2(R_+;H)} \\
&\le \frac{\eta}{2}\frac{1}{\min(\alpha;\beta)}\|Au''\|^2_{L_2(R_+;H)} + \frac{1}{2\eta}\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)},
\end{aligned} \tag{2.20}$$

with $\eta > 0$. Taking into account inequality (2.20) in (2.19):

$$\begin{aligned}
\|Au''\|^2_{L_2(R_+;H)} &\le \frac{\varepsilon}{2}\max(\alpha;\beta)\Big(\frac{\eta}{2}\frac{1}{\min(\alpha;\beta)}\|Au''\|^2_{L_2(R_+;H)} \\
&\quad + \frac{1}{2\eta}\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)}\Big) + \frac{1}{2\varepsilon}\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)}.
\end{aligned} \tag{2.21}$$

From this inequality we obtain

$$\begin{aligned}
\Big(1 - \frac{\varepsilon\eta\max(\alpha;\beta)}{4\min(\alpha;\beta)}\Big)\|Au''\|^2_{L_2(R_+;H)} &\le \frac{\varepsilon\max(\alpha;\beta)}{4\eta}\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)} \\
&\quad + \frac{1}{2\varepsilon}\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)}.
\end{aligned} \tag{2.22}$$

Choosing $\eta = \frac{\varepsilon^2\max(\alpha;\beta)}{2}$, from inequality (2.22) we have

$$\begin{aligned}
&\|Au''\|^2_{L_2(R_+;H)} \\
&\le \frac{4\min(\alpha;\beta)}{8\varepsilon\min(\alpha;\beta) - \varepsilon^4\max^2(\alpha;\beta)}\big[\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)} + \|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)}\big].
\end{aligned}$$

Then, by minimizing ε, we find $\varepsilon = \sqrt[3]{2\min(\alpha;\beta)/\max^2(\alpha;\beta)}$. Therefore,

$$\begin{aligned} &\|Au''\|^2_{L_2(R_+;H)} \\ &\le \frac{2^{2/3}\max^{2/3}(\alpha;\beta)}{3\min^{1/3}(\alpha;\beta)}\big[\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)} + \|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)}\big]. \end{aligned} \tag{2.23}$$

Now, taking into account inequality (2.11), from inequality (2.23) we obtain

$$\|Au''\|^2_{L_2(R_+;H)} \le \frac{2^{2/3}\max^{2/3}(\alpha;\beta)}{3\min^{4/3}(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}.$$

As a result,

$$\|Au''\|_{L_2(R_+;H)} \le \frac{2^{1/3}\max^{1/3}(\alpha;\beta)}{3^{1/2}\min^{2/3}(\alpha;\beta)}\|P_0u\|_{L_2(R_+;H)}.$$

To estimate the norm $\|A^2u'\|_{L_2(R_+;H)}$, we take into account (2.19) in (2.20):

$$\begin{aligned} &\big(1 - \frac{\varepsilon\eta\max(\alpha;\beta)}{4\min(\alpha;\beta)}\big)\|A^2u'\|^2_{L_2(R_+;H)} \\ &\le \frac{\eta}{4\varepsilon\min(\alpha;\beta)}\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)} + \frac{1}{2\eta}\|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)}. \end{aligned} \tag{2.24}$$

Choosing $\varepsilon = \eta^2/(2\min(\alpha;\beta))$, from inequality (2.24) we have

$$\begin{aligned} &\|A^2u'\|^2_{L_2(R_+;H)} \\ &\le \frac{4\min^2(\alpha;\beta)}{8\eta\min^2(\alpha;\beta) - \eta^4\max(\alpha;\beta)}\big[\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)} + \|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)}\big]. \end{aligned}$$

In this case, minimizing η, we find $\eta = \sqrt[3]{2\min^2(\alpha;\beta)/\max(\alpha;\beta)}$. Therefore,

$$\begin{aligned} &\|A^2u'\|^2_{L_2(R_+;H)} \\ &\le \frac{2^{2/3}\max^{1/3}(\alpha;\beta)}{3\min^{2/3}(\alpha;\beta)}[\|\rho^{-1/2}(t)u'''\|^2_{L_2(R_+;H)} + \|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)}]. \end{aligned} \tag{2.25}$$

From this inequality, taking into account inequality (2.11), we obtain

$$\|A^2u'\|^2_{L_2(R_+;H)} \le \frac{2^{2/3}\max^{1/3}(\alpha;\beta)}{3\min^{5/3}(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}.$$

Thus,

$$\|A^2u'\|_{L_2(R_+;H)} \le \frac{2^{1/3}\max^{1/6}(\alpha;\beta)}{3^{1/2}\min^{5/6}(\alpha;\beta)}\|P_0u\|_{L_2(R_+;H)}.$$

Now we estimate the norm $\|A^3u\|_{L_2(R_+;H)}$. From inequality (2.11) we have

$$\frac{1}{\min(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)} \ge \|\rho^{1/2}(t)A^3u\|^2_{L_2(R_+;H)} \ge \min(\alpha;\beta)\|A^3u\|^2_{L_2(R_+;H)}.$$

Hence, we obtain

$$\|A^3u\|^2_{L_2(R_+;H)} \le \frac{1}{\min^2(\alpha;\beta)}\|P_0u\|^2_{L_2(R_+;H)}$$

or

$$\|A^3u\|_{L_2(R_+;H)} \le \frac{1}{\min(\alpha;\beta)}\|P_0u\|_{L_2(R_+;H)}.$$

The proof is complete. □

Now, we prove the boundedness of the operator $P_1 : W^3_{2,K}(R_+; H) \to L_2(R_+; H)$.

Lemma 2.6. *Let $A_j A^{-j} \in L(H, H)$, $j = 1, 2, 3$. Then P_1 is a bounded operator from the space $W^3_{2,K}(R_+; H)$ into the space $L_2(R_+; H)$.*

Proof. For any $u(t) \in W^3_{2,K}(R_+; H)$ we have

$$\begin{aligned}
\|P_1 u\|_{L_2(R_+;H)} &= \|A_1 u'' + A_2 u' + A_3 u\|_{L_2(R_+;H)} \\
&\le \|A_1 A^{-1}\|_{H\to H} \|A u''\|_{L_2(R_+;H)} + \|A_2 A^{-2}\|_{H\to H} \|A^2 u'\|_{L_2(R_+;H)} \\
&\quad + \|A_3 A^{-3}\|_{H\to H} \|A^3 u\|_{L_2(R_+;H)}.
\end{aligned}$$

Applying the theorem on intermediate derivatives [14, ch. 1], we obtain from the last inequality that

$$\|P_1 u\|_{L_2(R_+;H)} \le const \|u\|_{W^3_2(R_+;H)}.$$

The proof is complete. □

Let us consider the question of regular solvability of problem (1.1), (1.2).

Theorem 2.7. *Let $A = A^* \ge cE$, $c > 0$, $K \in L(H_{5/2}, H_{1/2})$, $-\sqrt[3]{\alpha^2}\omega_2 \notin \sigma(B)$, the operator $K_{\alpha,\beta}$ has a bounded inverse in the space $H_{5/2}$, $\operatorname{Re} B \ge 0$ and $A_j A^{-j} \in L(H, H)$, $j = 1, 2, 3$, moreover, the following inequality holds*

$$a_1 \|A_1 A^{-1}\|_{H\to H} + a_2 \|A_2 A^{-2}\|_{H\to H} + a_3 \|A_3 A^{-3}\|_{H\to H} < 1,$$

where the numbers a_j, $j = 1, 2, 3$, are defined in Theorem 2.5. Then the boundary value problem (1.1), (1.2) *is regularly solvable.*

Proof. Boundary value problem (1.1), (1.2) can be represented in the operator form

$$P_0 u(t) + P_1 u(t) = f(t),$$

where $f(t) \in L_2(R_+; H)$, $u(t) \in W^3_{2,K}(R_+; H)$.

Under conditions $A = A^* \ge cE$, $c > 0$, $K \in L(H_{5/2}, H_{1/2})$, $-\sqrt[3]{\alpha^2}\omega_2 \notin \sigma(B)$, the operator $K_{\alpha,\beta}$ has a bounded inverse in the space $H_{5/2}$, by Theorem 2.3 the operator P_0 has a bounded inverse P_0^{-1} acting from the space $L_2(R_+; H)$ into the space $W^3_{2,K}(R_+; H)$. If we put $v(t) = P_0 u(t)$ we obtain the following equation in $L_2(R_+; H)$:

$$(E + P_1 P_0^{-1}) v(t) = f(t).$$

We show that under the conditions of the theorem, the norm of the operator $P_1 P_0^{-1}$ is less than unity. Taking into account inequalities (2.18), we have

$$\begin{aligned}
&\|P_1 P_0^{-1} v\|_{L_2(R_+;H)} \\
&= \|P_1 u\|_{L_2(R_+;H)} \\
&\le \|A_1 u''\|_{L_2(R_+;H)} + \|A_2 u'\|_{L_2(R_+;H)} + \|A_3 u\|_{L_2(R_+;H)} \\
&\le \|A_1 A^{-1}\|_{H\to H} \|A u''\|_{L_2(R_+;H)} + \|A_2 A^{-2}\|_{H\to H} \|A^2 u'\|_{L_2(R_+;H)} \\
&\quad + \|A_3 A^{-3}\|_{H\to H} \|A^3 u\|_{L_2(R_+;H)} \\
&\le a_1 \|A_1 A^{-1}\|_{H\to H} \|P_0 u\|_{L_2(R_+;H)} + a_2 \|A_2 A^{-2}\|_{H\to H} \|P_0 u\|_{L_2(R_+;H)} \\
&\quad + a_3 \|A_3 A^{-3}\|_{H\to H} \|P_0 u\|_{L_2(R_+;H)} \\
&= \sum_{j=1}^{3} a_j \|A_j A^{-j}\|_{H\to H} \|v\|_{L_2(R_+;H)}.
\end{aligned}$$

Thus,

$$\|P_1P_0^{-1}\|_{L_2(R_+;H)\to L_2(R_+;H)} \le \sum_{j=1}^{3} a_j\|A_jA^{-j}\|_{H\to H} < 1.$$

Therefore, the operator $E + P_1P_0^{-1}$ is invertible in the space $L_2(R_+;H)$ and $u(t)$ is defined by the formula

$$u(t) = P_0^{-1}(E + P_1P_0^{-1})^{-1}f(t),$$

moreover

$$\begin{aligned} &\|u\|_{W_2^3(R_+;H)} \\ &\le \|P_0^{-1}\|_{L_2(R_+;H)\to W_2^3(R_+;H)}\|(E + P_1P_0^{-1})^{-1}\|_{L_2(R_+;H)\to L_2(R_+;H)}\|f\|_{L_2(R_+;H)} \\ &\le \text{const}\|f\|_{L_2(R_+;H)}. \end{aligned}$$

The proof is complete. □

Corollary 2.8. *In the conditions of Theorem 2.7, the operator P is an isomorphism between the spaces $W_{2,K}^3(R_+;H)$ and $L_2(R_+;H)$.*

In conclusion, we remark that our solvability results imply the results of [4] when $K = 0$ and $A_3 = 0$, and the results of [15] when $K = 0$ and $\alpha = \beta = 1$.

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