Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 222, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ENERGY DECAY FOR DEGENERATE KIRCHHOFF EQUATIONS WITH WEAKLY NONLINEAR DISSIPATION

MAMA ABDELLI, SALIM A. MESSAOUDI

Abstract. In this article we consider a degenerate Kirchhoff equation wave equation with a weak frictional damping,

$$(|u_t|^{l-2}u_t)_t - \Big(\int_\Omega |\nabla_x u|^2\,dx\Big)^\gamma \Delta_x u + \alpha(t)g(u_t) = 0.$$

We prove general stability estimates using some properties of convex functions, without imposing any growth condition at the frictional damping term.

1. Introduction

In this article, we consider the initial-boundary value problem for the nonlinear Kirchhoff equation

$$(|u_t|^{l-2}u_t)_t - \Big(\int_\Omega |\nabla u|^2\,dx\Big)^\gamma \Delta u + \alpha(t)g(u_t) = 0, \quad \text{in } \Omega\times(0,\infty) \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega\times(0,\infty) \tag{1.2}$$

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x\in\Omega, \tag{1.3}$$

where $l \ge 2$, $\gamma \ge 0$ are given constants, Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and g and α are one-variable functions satisfying some conditions to be specified later. This problem has been studied by many authors and several existence, nonexistence, and decay results have appeared. For instance, when $l = 2$ and $\gamma = 0$, the problem was treated by Mustafa and Massaoudi [11]. By using some properties of convex functions, they established a general decay result without imposing any growth condition on g at the origin. Abdelli and Benaissa [1] treated system (1.1)-(1.3) for g having a polynomial growth near the origin and established energy decay results depending on α and g under appropriate relations between l and γ. In a realted work, Amroun and Benaissa [3] constructed an exact solution of (1.1)-(1.3) in the presence of a nonlinear source term and for $\alpha \equiv 0$. They also proved a finite-time blow-up result for some specific initial data. Benaissa and Guesmia [5] proved the existence of global solution, as well as, a general stability result for the following equation

$$(|u'|^{l-2}u')' - \Delta_\phi u + \alpha(t)g(u') = 0, \quad \text{in } \Omega\times\mathbb{R}_+,$$

2000 *Mathematics Subject Classification.* 35B37, 35L55, 74D05, 93D15, 93D20.
Key words and phrases. Decay of solutions; nonlinear; degenerate; Kirchhoff equation.

Submitted September 18, 2013. Published October 11, 2013.

where $\Delta_\phi = \sum_{i=1}^n \partial_{x_i}(\phi(|\partial_{x_i}|^2)\partial_{x_i})$.

In this article, we use some technique from [11] to establish an explicit and general decay result, depending on g and α. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasiecka and co-workers ([7, 8, 9]) and used, with appropriate modifications, by Liu and Zuazua [10], Alabau-Boussouira [2] and others.

The paper is organized as follows: in section 2, we give our hypotheses and establish a useful lemma. In section 3, we state and prove our main result.

2. Preliminaries

To state and prove our result, we need the following hypotheses:

(H1) $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function.

(H2) $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exist $\varepsilon, c_1, c_2 > 0$, and a convex and increasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying $G(0) = G'(0) = 0$ or G is linear on $[0, \varepsilon]$ such that

$$\begin{gathered} c_1|s|^{l-1} \le |g(s)| \le c_2|s|^p, \quad \text{if } |s| \ge \varepsilon; \\ |s|^l + |g(s)|^{\frac{l}{l-1}} \le G^{-1}(sg(s)), \quad \text{if } |s| \le \varepsilon; \end{gathered}$$

with p satisfying

$$\begin{gathered} l-1 \le p \le \frac{n+2}{n-2}, \quad \text{if } n > 2; \\ l-1 \le p < \infty, \quad \text{if } n \le 2\,. \end{gathered}$$

Now we define the energy associated to the solution of the system (1.1) -(1.3) by

$$E(t) = \frac{l-1}{l}\|u_t\|_l^l + \frac{1}{1+\gamma}\|\nabla_x u\|_2^{2(\gamma+1)} \tag{2.1}$$

Lemma 2.1. *Let u be the solution of* (1.1)-(1.3)*. Then*

$$E'(t) = -\alpha(t)\int_\Omega u_t g(u_t)\,dx \le 0. \tag{2.2}$$

Proof. Multiplying (1.1) by u_t and integrating over Ω, using the boundary conditions, the assertion of the lemma follows. □

3. Main Result

To prove our main result, first prove the following lemma.

Lemma 3.1. *Assume that* (H1), (H2) *hold and that $l \ge 2(\gamma+1)$. Then the functional*

$$F(t) = ME(t) + \int_\Omega u|u_t|^{l-2}u_t\,dx,$$

defined along the solution of (1.1)-(1.3)*, satisfies the following estimate, for some positive constants M, c, m:*

$$F'(t) \le -mE(t) + c\int_\Omega (|u_t|^l + |ug(u_t)|^{\frac{l}{l-1}})\,dx$$

and $F(t) \sim E(t)$.

Proof. Using system (1.1)-(1.3), (2.1) and (2.2), we obtain

$$
\begin{aligned}
F'(t) &= ME'(t) + \int_\Omega |u_t|^l\,dx + \int_\Omega u(|u_t|^l u_t)_t\,dx \\
&\le \int_\Omega |u_t|^l\,dx + \Big(\int_\Omega |\nabla u|^2\,dx\Big)^\gamma \int_\Omega u\Delta u\,dx - \alpha(t)\int_\Omega ug(u_t)\,dx \\
&\le \int_\Omega |u_t|^l\,dx - \int_\Omega |\nabla u|^{2(\gamma+1)}\,dx - \alpha(t)\int_\Omega ug(u_t)\,dx \\
&\le -mE(t) + c\int_\Omega [|u_t|^l + |ug(u_t)|]\,dx
\end{aligned}
$$

To prove that $F(t) \sim E(t)$, we show that for some positive constants λ_1 and λ_2,

$$
\lambda_1 E(t) \le E(t) \le \lambda_2 E(t)\,. \tag{3.1}
$$

We use (2.1), Poincaré's and Young's inequalities with exponents $\frac{l}{l-1}$ and $\frac{1}{l}$ and recall that $2 \le l \le p+1 \le \frac{2n}{n+2}$, to obtain

$$
\begin{aligned}
\int_\Omega u|u_t|^{l-2}u_t\,dx &\le C_\varepsilon \int_\Omega |u|^l\,dx + \varepsilon\int_\Omega |u_t|^l\,dx \\
&\le C_\varepsilon \|\nabla u\|_2^l + \varepsilon\|u_t\|_l^l \\
&\le C_\varepsilon E^{\frac{l}{2(\gamma+1)}}(t) + c\varepsilon E(t) \\
&\le C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(t)E(t) + c\varepsilon E(t).
\end{aligned}
$$

By noting that $l \ge 2(\gamma+1)$ and using (2.2), we have

$$
\int_\Omega u|u_t|^{l-2}u_t\,dx \le C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0)E(t) + c\varepsilon E(t),
$$

and

$$
\begin{aligned}
\int_\Omega u|u_t|^{l-2}u_t\,dx &\ge -C_\varepsilon \int_\Omega |u|^l\,dx - \varepsilon\int_\Omega |u_t|^l\,dx \\
&\ge -C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(t)E(t) - c\varepsilon E(t) \\
&\ge -C_\varepsilon E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0)E(t) - c\varepsilon E(t)
\end{aligned}
$$

Then, for M large enough, we obtain (3.1). This completes the proof. □

Taking $0 < \varepsilon_1 < \varepsilon$ such that

$$
sg(s) \le \min\{\varepsilon, G(\varepsilon)\}, \quad \text{if } |s| \le \varepsilon_1, \tag{3.2}
$$

and

$$
\begin{cases}
c_1'|s|^{l-1} \le |g(s)| \le c_2'|s|^p, & \text{if } |s| \ge \varepsilon_1 \\
|s|^l + |g(s)|^{\frac{l}{l-1}} \le G^{-1}(sg(s)), & \text{if } |s| \le \varepsilon_1.
\end{cases} \tag{3.3}
$$

Considering the following partition of Ω,

$$
\Omega_1 = \{x \in \Omega : |u_t| \le \varepsilon_1\}, \quad \Omega_2 = \{x \in \Omega : |u_t| > \varepsilon_1\}
$$

and using the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and Hölder's inequality, we obtain

$$
\int_{\Omega_2} |ug(u_t)|\,dx \le \Big(\int_{\Omega_2} |u|^{p+1}\,dx\Big)^{\frac{1}{p+1}} \Big(\int_{\Omega_2} |g(u_t)|^{1+\frac{1}{p}}\,dx\Big)^{p/(p+1)}
$$

$$\le c\|u\|_{H_0^1(\Omega)}\Big(\int_{\Omega_2}|g(u_t)|^{1+\frac{1}{p}}\,dx\Big)^{p/(p+1)}$$

Using Poincaré's inequality and (3.3) yields

$$\begin{aligned}
&\int_{\Omega_2}[|u_t|^l+|ug(u_t)|]\,dx\\
&\le c\int_{\Omega_2}|u_t|^{l-1}|u_t|\,dx+c\Big(\int_\Omega|\nabla u|^2\,dx\Big)^{1/2}\Big(\int_{\Omega_2}|g(u_t)|^{1+\frac{1}{p}}\,dx\Big)^{p/(p+1)}\\
&\le c\int_{\Omega_2}u_tg(u_t)\,dx+c\Big(\int_\Omega|\nabla u|^2\,dx\Big)^{1/2}\Big(\int_{\Omega_2}u_tg(u_t)\,dx\Big)^{p/(p+1)}\\
&\le -cE'(t)+cE^{\frac{1}{2(\gamma+1)}}(-E'(t))^{p/(p+1)}.
\end{aligned}$$

Then, we use Young's inequality and the fact that $p\ge l-1\ge 2\gamma+1$, for any $\delta>0$, we have

$$\begin{aligned}
\int_{\Omega_2}[|u_t|^l+|ug(u_t)|]\,dx&\le -cE'(t)+c\delta E^{\frac{p+1}{2(\gamma+1)}}(t)+C_\delta(-E'(t))\\
&\le c\delta E^{\frac{p+1}{2(\gamma+1)}}(t)-C_\delta E'(t)\\
&\le c\delta E^{\frac{p-(2\gamma+1)}{2(\gamma+1)}}(0)E(t)-C_\delta E'(t)
\end{aligned}\tag{3.4}$$

Similarly, using (2.1) and Young's inequality, we have, for any $\delta>0$,

$$\begin{aligned}
\int_{\Omega_1}[|u_t|^l+|ug(u_t)|]\,dx&\le\int_{\Omega_1}|u_t|^l\,dx+\delta\int_{\Omega_1}|u|^l\,dx+C_\delta\int_{\Omega_1}|g(u_t)|^{\frac{l}{l-1}}\,dx\\
&\le\int_{\Omega_1}|u_t|^l\,dx+c\delta E^{\frac{l}{2(\gamma+1)}}(t)+C_\delta\int_{\Omega_1}|g(u_t)|^{\frac{l}{l-1}}\,dx
\end{aligned}\tag{3.5}$$

By Lemma 3.1, (3.4) and (3.5), for δ small enough, the function $L=F+C_\delta E$ satisfies

$$\begin{aligned}
L'(t)&\le\Big(-m+c\delta E^{\frac{p-(2\gamma+1)}{2}}(0)+c\delta E^{\frac{l-2(\gamma+1)}{2(\gamma+1)}}(0)\Big)E(t)\\
&\quad+\int_{\Omega_1}|u_t|^l\,dx+C_\delta\int_{\Omega_1}|g(u_t)|^{\frac{l}{l-1}}\,dx\\
&\le -dE(t)+c\int_{\Omega_1}\Big(|u_t|^l+|g(u_t)|^{\frac{l}{l-1}}\Big)\,dx
\end{aligned}\tag{3.6}$$

and

$$L(t)\sim E(t).\tag{3.7}$$

Theorem 3.2. *Assume that* (H1), (H2) *hold and* $l\ge 2(\gamma+1)$. *Then there exist positive constants* k_1,k_2,k_3 *and* ε_0 *such that the solution of* (1.1)-*1.3 satisfies*

$$E(t)\le k_3G_1^{-1}\Big(k_1\int_0^t\alpha(s)\,ds+k_2\Big)\quad\forall t\ge 0,\tag{3.8}$$

where

$$G_1(t)=\int_t^1\frac{1}{G_2(s)}\,ds,\quad G_2(t)=tG'(\varepsilon_0t).\tag{3.9}$$

Here, G_1 *is strictly decreasing and convex on* $(0,1]$ *with* $\lim_{t\to0}G_1(t)=+\infty$.

Proof. Multiplying (3.6) by $\alpha(t)$, we have

$$\alpha(t)L'(t) \le -d\alpha(t)E(t) + c\alpha(t)\int_{\Omega_1}\Big(|u_t|^l + |g(u_t)|^{\frac{l}{l-1}}\Big)\,dx \tag{3.10}$$

Case 1. G is linear on $[0,\varepsilon]$, then we deduce that

$$\alpha(t)L'(t) \le -d\alpha(t)E(t) + c\alpha(t)\int_{\Omega_1} u_t g(u_t)\,dx = -d\alpha(t)E(t) - cE'(t)$$

Consequently, we arrive at

$$(\alpha L + cE)'(t) \le -d\alpha(t)E(t).$$

Recalling that

$$\alpha L + cE \sim E, \tag{3.11}$$

we obtain

$$E(t) \le c' e^{-c''\int_0^t \alpha(s)\,ds}$$

Thus, we have

$$E(t) \le c' e^{-c''\int_0^t \alpha(s)\,ds} = c' G_1^{-1}(c''\int_0^t \alpha(s)\,ds)$$

by a simple computation.

Case 2. G is nonlinear on $[0,\varepsilon]$. In this case, we define

$$I(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u_t g(u_t)\,dx.$$

and exploit Jensen's inequality and the concavity of G^{-1} to obtain

$$G^{-1}(I(t)) \ge c\int_{\Omega_1} G^{-1}(u_t g(u_t))\,dx.$$

By using this inequality and (3.3), we obtain

$$\alpha(t)\int_{\Omega_1}[|u_t|^l + |g(u_t)|^{\frac{l}{l-1}}]\,dx \le \alpha(t)\int_{\Omega_1} G^{-1}(u_t g(u_t))\,dx \le c\alpha(t)G^{-1}(I(t)). \tag{3.12}$$

Let us set $H_0 = \alpha L + E$ and exploit (2.2), (3.10), (3.12), and α being nonincreasing, to obtain

$$\begin{aligned} H_0'(t) &\le -d\alpha(t)E(t) + c\alpha(t)G^{-1}(I(t)) + E'(t) \\ &\le -d\alpha(t)E(t) + c\alpha(t)G^{-1}(I(t)), \end{aligned} \tag{3.13}$$

and recall (3.7), to deduce that $H_0 \sim E$.

For $\varepsilon_0 < \varepsilon$ and $c_0 > 0$, we define H_1 by

$$H_1(t) = G'(\varepsilon_0\frac{E(t)}{E(0)})H_0(t) + c_0E(t).$$

Then, we see easily that, for $a_1, a_2 > 0$,

$$a_1H_1(t) \le E(t) \le a_2H_1(t), \tag{3.14}$$

By recalling that $E' \le 0$, $G' > 0$, $G'' > 0$ on $(0,\varepsilon]$ and making use of (2.1) and (3.13), we obtain

$$\begin{aligned} H_1'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}G''(\varepsilon_0\frac{E(t)}{E(0)})H_0(t) + G'(\varepsilon_0\frac{E(t)}{E(0)})H_0'(t) + c_0E'(t) \\ &\le -d\alpha(t)E(t)G'(\varepsilon_0\frac{E(t)}{E(0)}) + c\alpha(t)G'(\varepsilon_0\frac{E(t)}{E(0)})G^{-1}(I(t)) + c_0E'(t). \end{aligned} \tag{3.15}$$

Let G^* be the convex conjugate of G in the sense of Young (see Arnold [4, p. 61-64]), then

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad \text{if } s \in (0, G'(\varepsilon)], \tag{3.16}$$

and G^* satisfies the generalized Young's inequality

$$AB \le G^*(A) + G(B), \quad \text{if } A \in (0, G'(\varepsilon)],\ B \in (0, \varepsilon]. \tag{3.17}$$

with $A = G'(\varepsilon_0 E(t)/E(0))$ and $B = G^{-1}(I(t))$, using (2.2), (3.2) and (3.15)–(3.16), we obtain

$$\begin{aligned} H_1'(t) &\le -d\alpha(t)E(t)G'(\varepsilon_0\frac{E(t)}{E(0)}) + c\alpha(t)G^*\Big(G'(\varepsilon_0\frac{E(t)}{E(0)})\Big) + c\alpha(t)I(t) + c_0E'(t) \\ &\le -d\alpha(t)E(t)G'(\varepsilon_0\frac{E(t)}{E(0)}) + c\varepsilon_0\alpha(t)\frac{E(t)}{E(0)}G'(\varepsilon_0\frac{E(t)}{E(0)}) - cE'(t) + c_0E'(t). \end{aligned}$$

Choosing $c_0 > c$ and ε_0 small enough, we obtain

$$H_1'(t) \le -k\alpha(t)\frac{E(t)}{E(0)}G'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) = -k\alpha(t)G_2\Big(\frac{E(t)}{E(0)}\Big) \tag{3.18}$$

where $G_2(t) = tG'(\varepsilon_0 t)$. Since

$$G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 t G''(\varepsilon_0 t).$$

and G is convex on $(0, \varepsilon]$, we find that $G_2'(t) > 0$ and $G_2(t) > 0$ on $(0, 1]$. By setting $H(t) = \frac{a_1 H_1(t)}{E(0)}$ (a_1 is given in (3.14)), we easily see that, by (3.14), we have

$$H(t) \sim E(t) \tag{3.19}$$

Using (3.18), we arrive at

$$H'(t) \le -k_1\alpha(t)G_2(H(t))$$

By recalling (3.9), we deduce $G_2(t) = -1/G_1'(t)$, hence

$$H'(t) \le k_1\alpha(t)\frac{1}{G_1'(H(t))},$$

which gives

$$[G_1(H(t))]' = H'(t)G_1'(H(t)) \le k_1\alpha(t)$$

A simple integration leads to

$$G_1(H(s)) \le k_1\int_0^t \alpha(s)\,ds + G_1(H(0))$$

Consequently,

$$H(t) \le G_1^{-1}(k_1\int_0^t \alpha(s)\,ds + k_2) \tag{3.20}$$

Using (3.19) and (3.20) we obtain (3.8). The proof is complete. □

Acknowledgments. The second author wants to thank KFUPM for its financial support.

References

[1] M. Abdelli, A. Benaissa; *Energy decay of solutions of a degenerate Kirchhoff equation with a weak nonlinear dissipation*, Nonlinear Analysis **69** (2008), 1999-2008.

[2] F. Alabau-Boussouira; *Convexity and weighted intgral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems*, Appl. Math. Optim. **51** (2005), 61-105.

[3] N. D. Amroun, A. Benaissa; *On exact solutions of a quasilinear wave equation with source term*, Electron. J. Differential Equations **31** (2006), 231-238.

[4] V. I. Arnold; *Mathematical methods of classical mechanics*, Springer-Verlag, New York 1989.

[5] A. Benaissa, A. Guesmia; *Energy decay for wave equations of ϕ-Laplacian type with weakly nonlinear dissipation*, Electron. J. Differential Equations **2008** (2008), 1-22.

[6] M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka; *Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction*, J. Differential Equations **236** (2007), 407-459.

[7] I. Lasiecka; *Stabilization of wave and plate-like equation with nonlinear dissipation on the boundary*, J. Differential Equations **79** (1989), 340-381.

[8] I. Lasiecka, D. Toundykov; *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms*, Nonlinear Anal. **64** (2006), 1757-1797.

[9] I. Lasiecka, D. Toundykov; *Regularity of higher energies of wave equation with nonlinear localized damping and a nonlinear source*, Nonlinear Anal. **69** (2008), 898-910.

[10] W. J. Liu, E. Zuazua; *Decay rates for dissipative wave equations*, Ricerche Mat **48** (1999), 61-75.

[11] M. I. Mustafa, S. A. Messaoudi; *General energy decay for a weakly damped wave equation*, Communications in Mathematical Analysis **9** (2010), 1938-9787.

Mama Abdelli
Université Djillali Liabés, Laboratoire de Mathématique, B.P. 89. Sidi Bel Abbés 22000, Algeria
E-mail address: abdelli_mama@yahoo.fr

Salim A. Messaoudi
King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran 31261, Saudi Arabia
E-mail address: messaoud@kfupm.edu.sa