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OSGOOD TYPE REGULARITY CRITERION FOR THE 3D NEWTON-BOUSSINESQ EQUATION

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Abstract. In this article, we show an Osgood type regularity criterion for the three-dimensional Newton-Boussinesq equations, which improves the recent results in [4].

1. Introduction

In this article, we consider the three-dimensional Newton-Boussinesq equation

$$\begin{gathered} \boldsymbol{\omega}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - \Delta\boldsymbol{\omega} = \nabla\times(\theta\mathbf{e}_3), \\ \theta_t + (\mathbf{u}\cdot\nabla)\theta - \Delta\theta = 0, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \end{gathered} \tag{1.1}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$, and $\mathbf{u}$ is the velocity field, θ is the scalar temperature, while $\mathbf{u}_0$, θ_0 are the prescribed initial data with $\nabla\cdot\mathbf{u}_0 = 0$ in distributional sense.

System (1.1) arises from the study of Bénard flow [1]. Guo [2, 3] investigated the two-dimensional (2D) periodic case by using spectral methods and nonlinear Galerkin methods. Meanwhile, the existence and regularity of a global attractor for the 2D Newton-Boussinesq equations were obtained in [5]. Consequently, it is desirable to consider the regularity criteria for (1.1). Noticing that the convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$, $(\mathbf{u}\cdot\nabla)\theta$ are the same as that in the 3D Boussinesq equations

$$\begin{gathered} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} + \nabla\pi = \theta\mathbf{e}_3, \\ \theta_t + (\mathbf{u}\cdot\nabla)\theta - \Delta\theta = 0, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0, \end{gathered}$$

we could prove many regularity conditions as that for the Boussinesq equations.

For the 3D Boussinesq equations, Ishimura nad Morimoto [6] showed that if

$$\nabla\mathbf{u} \in L^1(0,T;L^\infty(\mathbb{R}^3)), \tag{1.2}$$

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then the solution is smooth on $(0,T)$. Fan and Zhou [7] established the regularity of the solution provided that

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \in L^1\big(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\big), \tag{1.3}$$

where $B^0_{\infty,\infty}(\mathbb{R}^3)$ is the homogeneous Besov spaces which will be introduced in Section 2. The interested readers can find more result in [8, 9] and references cited therein.

For the 3D Newton-Boussinesq equations (1.1), Guo and Gala [4] obtained some regularity criteria in terms of Morrey spaces and Besov spaces. One of them reads

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)). \tag{1.4}$$

A blow-up criterion for the 2D Newton-Boussinesq equations was established in [10].

As we know, Osgood type conditions play an important role in solving uniqueness of solutions to the incompressible fluid equations. Motivated by the recent result [12] for the 3D MHD equations

$$\begin{gathered}
\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\ \cdot\nabla) + \nabla\pi = \nvdash,\\
{}_t - \Delta + (\mathbf{u}\cdot\nabla) - (\ \cdot\nabla)\mathbf{u} = \nvdash,\\
\nabla\cdot\mathbf{u} = \nabla\cdot\ = 0,\\
\mathbf{u}(0) = \mathbf{u}_0, \quad (0) = {}_0,
\end{gathered}$$

we would like to improve (1.4). Precisely, we will prove the following theorem.

Theorem 1.1. *Let $(\mathbf{u}_0,\theta_0) \in H^1(\mathbb{R}^3)$ with $\nabla\cdot\mathbf{u}_0 = 0$ in distributional sense. Assume that*

$$\sup_{q\ge 2}\int_0^T \frac{\|\bar{S}_q\nabla\mathbf{u}\|_{L^\infty}}{q\ln q}\,\mathrm{d}\tau < \infty, \tag{1.5}$$

with $\bar{S}_q = \sum_{l=-q}^{q}\dot{\Delta}_l$, $\dot{\Delta}_l$ being the Fourier localization operator. Then the solution pair $(\mathbf{u},\theta)$ to (1.1) with initial data $(\mathbf{u}_0,\theta_0)$ is smooth on $[0,T]$.

Remark 1.2. Since

$$\frac{\|\bar{S}_q\nabla\mathbf{u}\|_{L^\infty}}{q\ln q} \le \frac{1}{q\ln q}\sum_{l=-q}^{q}\|\dot{\Delta}_l\nabla\mathbf{u}\|_{L^\infty} \le C\|\nabla\times\mathbf{u}\|_{\dot{B}^0_{\infty,\infty}},$$

we indeed improve the regularity condition (1.4) established in [4].

Remark 1.3. When $\theta = 0$, (1.1) reduces to the Navier-Stokes equations, thus our result covers the case for the Navier-Stokes equations.

The rest of this article is organized as follows. In Section 2, we recall the definition of Besov spaces, and some interpolation inequalities. Section 3 is devoted to proving Theorem 1.1.

2. Preliminaries

Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathscr{S}(\mathbb{R}^3)$, its Fourier transform $\mathscr{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x.$$

Let us choose a nonnegative radial function $\varphi \in \mathscr{S}(\mathbb{R}^3)$ such that

$$0 \le \hat{\varphi}(\xi) \le 1, \quad \hat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 1, \\ 0, & \text{if } |\xi| \ge 2, \end{cases}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi(x/2), \quad \varphi_j(x) = 2^{3j}\varphi(2^j x), \quad \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and $\dot{\Delta}_j$ are, respectively, defined by

$$S_j f = \varphi_j * f, \quad \dot{\Delta}_j f = \psi_j * f.$$

Observe that $\dot{\Delta}_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \to 0, \text{ as } j \to -\infty; \quad S_j f \to f, \text{ as } j \to +\infty,$$

in the L^2 sense. By telescoping the series, we thus have the following Littlewood-Paley decomposition

$$f = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j f, \tag{2.1}$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is the L^2 sense. Note that

$$\dot{\Delta}_j f = \sum_{l=j-2}^{j+2} \dot{\Delta}_l \dot{\Delta}_j f = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f,$$

then from Young's inequality, it readily follows that

$$\|\dot{\Delta}_j f\|_{L^q} \le C 2^{3j(1/p-1/q)} \|\dot{\Delta}_j f\|_{L^p}, \tag{2.2}$$

where $1 \le p \le q \le \infty$, and C is an absolute constant independent of f and j.

Let $-\infty < s < \infty,\ 1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by the full-dyadic decomposition such as

$$\dot{B}^s_{p,q} = \{f \in \mathscr{Z}'(\mathbb{R}^3);\ \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \|\{2^{js}\|\dot{\Delta}_j f\|_{L^p}\}_{j=-\infty}^{+\infty}\|_{\ell^q},$$

and $\mathscr{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathscr{Z}(\mathbb{R}^3) = \{f \in \mathscr{S}(\mathbb{R}^3); D^\alpha \hat{f}(0) = 0,\ \forall\ \alpha \in \mathbb{N}^3\}.$$

Also, it is well-known that

$$\dot{H}^s(\mathbb{R}^3) = \dot{B}^s_{2,2}(\mathbb{R}^3), \quad \forall s \in \mathbb{R}. \tag{2.3}$$

We refer the reader to [11] for more detailed properties.

3. Proof of Theorem 1.1

This section is devoted to proving Theorem (1.1). Taking the inner products of $(1.1)_1$, $(1.1)_2$ with $\boldsymbol{\omega}$, $-\Delta\theta$ in $L^2(\mathbb{R}^3)$ respectively, we have

$$\begin{gathered}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\omega}\|_{L^2}^2+\|\nabla\boldsymbol{\omega}\|_{L^2}^2=\int_{\mathbb{R}^3}\nabla\times(\theta\mathbf{e}_3)\cdot\boldsymbol{\omega}\,\mathrm{d}x,\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2=\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\theta]\cdot\Delta\theta\,\mathrm{d}x.\end{gathered}$$

Adding together yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{\omega}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2]+[\|\nabla\boldsymbol{\omega}\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2]\\ &=\int_{\mathbb{R}^3}\nabla\times(\theta\mathbf{e}_3)\cdot\boldsymbol{\omega}\,\mathrm{d}x+\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)\theta]\cdot\Delta\theta\,\mathrm{d}x\\ &\le\|\nabla\theta\|_{L^2}\|\nabla\boldsymbol{\omega}\|_{L^2}-\int_{\mathbb{R}^3}[(\nabla\mathbf{u}\cdot\nabla)\theta]\cdot\nabla\theta\,\mathrm{d}x\\ &\le\frac{1}{2}\|\nabla\theta\|_{L^2}^2+\frac{1}{2}\|\nabla\boldsymbol{\omega}\|_{L^2}^2-\int_{\mathbb{R}^3}[(\nabla\mathbf{u}\cdot\nabla)\theta]\cdot\nabla\theta\,\mathrm{d}x.\end{aligned}\tag{3.1}$$

We are now in a position to estimate

$$I=-\int_{\mathbb{R}^3}[(\nabla\mathbf{u}\cdot\nabla)\theta]\cdot\nabla\theta\,\mathrm{d}x.\tag{3.2}$$

Applying the Littlewood-Paley decomposition as in (2.1),

$$\nabla\mathbf{u}=\sum_{l<-q}\dot{\Delta}\nabla\mathbf{u}+\sum_{l=-q}^{q}\dot{\Delta}\nabla\mathbf{u}+\sum_{l>q}\dot{\Delta}\nabla\mathbf{u},\tag{3.3}$$

where q is a positive integer to be determined later on. Substituting (3.3) in I, we see that

$$\begin{aligned}I\le&\sum_{l<-q}\int_{\mathbb{R}^3}\|\dot{\Delta}_l\nabla\mathbf{u}\|\cdot\|\nabla\theta\|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\Big|\sum_{l=-q}^{q}\dot{\Delta}_l\nabla\mathbf{u}\Big|\cdot\|\nabla\theta\|^2\,\mathrm{d}x\\ &+\sum_{l>q}\int_{\mathbb{R}^3}\|\dot{\Delta}_l\nabla\mathbf{u}\|\cdot\|\nabla\theta\|^2\,\mathrm{d}x\\ \equiv&\, I_1+I_2+I_3.\end{aligned}\tag{3.4}$$

For I_1, we have

$$\begin{aligned}I_1&\le\sum_{l<-q}\|\dot{\Delta}_l\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^2}^2\\ &\le C\sum_{l<-q}2^{3l/2}\|\dot{\Delta}_l\nabla\mathbf{u}\|_{L^2}\|\nabla\theta\|_{L^2}^2\quad\text{(by (2.2))}\\ &\le C\Big(\sum_{l<-q}2^{\frac{3l}{2}\cdot2}\Big)^{1/2}\cdot\Big(\sum_{l<-q}\|\dot{\Delta}_l\nabla\mathbf{u}\|_{L^2}^2\Big)^{1/2}\|\nabla\theta\|_{L^2}^2\\ &\le C2^{-3q/2}|\nabla\mathbf{u}|_{L^2}\|\nabla\theta\|_{L^2}^2\quad\text{(by (2.3))}\\ &=[C2^{-q/2}\|\nabla\theta\|_{L^2}]^3.\end{aligned}\tag{3.5}$$

For I_2, we have

$$I_2 = \int_{\mathbb{R}^3} |\bar{S}_q \nabla \mathbf{u}| \cdot |\nabla\theta|^2 \, \mathrm{d}x \le \|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty} \|\nabla\theta\|_{L^2}^2. \tag{3.6}$$

Finally, for I_3, we have

$$\begin{aligned} I_3 &\le \sum_{l>q} \|\Delta_l \nabla \mathbf{u}\|_{L^3} \|\nabla\theta\|_{L^3}^2 \\ &\le C \sum_{l>q} 2^{1/2} \|\Delta_l \nabla \mathbf{u}\|_{L^2} \|\nabla\theta\|_{L^2} \|\Delta\theta\|_{L^2} \\ &\quad \text{by (2.2) and Gagliardo-Nireberg inequality} \\ &\le C\Big(\sum_{l>q} 2^{-\frac{l}{2}\cdot 2}\Big)^{1/2} \cdot \Big(\sum_{l>q} 2^{l\cdot 2} \|\dot{\Delta}_l \nabla \mathbf{u}\|_{L^2}^2\Big)^{1/2} \|\nabla\theta\|_{L^2} \|\Delta\theta_{L^2\|} \\ &\le [C2^{-q/2} \|\nabla\theta\|_{L^2}][\|\nabla\boldsymbol{\omega}\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2] \quad (\text{by (2.3)}). \end{aligned} \tag{3.7}$$

Gathering (3.5), (3.6) and (3.7) together, and plugging them into (3.8), we deduce

$$I \le [C2^{-q/2}\|\nabla\theta\|_{L^2}]^3 + \|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty} \|\nabla\theta\|_{L^2}^2 + [C2^{-q/2}\|\nabla\theta\|_{L^2}] \cdot [\|\nabla\boldsymbol{\omega}\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2]. \tag{3.8}$$

Substituting (3.8) into (3.1), we find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{\omega}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2] + [\|\nabla\boldsymbol{\omega}\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2] \\ &\le \|\nabla\theta\|_{L^2}^2 + [C2^{-q/2}\|\nabla\theta\|_{L^2}]^3 \\ &\quad + \frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} \cdot q \ln q \|\nabla\theta\|_{L^2}^2 + [C2^{-q/2}\|\nabla\theta\|_{L^2}][\|\nabla\boldsymbol{\omega}\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2]. \end{aligned} \tag{3.9}$$

Taking

$$q = [\frac{2}{\ln 2} \ln^+(C\|\nabla\theta\|_{L^2})] + 3,$$

where $[t]$ is the largest integer smaller that $t \in \mathbb{R}$, and $\ln^+ t = \ln(e+t)$, then (3.9) implies that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\|\boldsymbol{\omega}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2] + \frac{1}{2}[\|\nabla\boldsymbol{\omega}\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2] \\ &\le \|\nabla\theta\|_{L^2}^2 + C + \frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} \ln^+(\|\nabla\theta\|_{L^2}) \ln^+ \ln^+(\|\nabla\theta\|_{L^2}) \|\nabla\theta\|_{L^2}^2. \end{aligned}$$

Applying Gronwall inequality three times, we deduce

$$\begin{aligned} &[\|\boldsymbol{\omega}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2] + \int_0^t [\|\nabla\boldsymbol{\omega}\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2] \, \mathrm{d}\tau \\ &\le C \exp\exp\exp\Big(\int_0^t \frac{\|\bar{S}_q \nabla \mathbf{u}\|_{L^\infty}}{q \ln q} \, \mathrm{d}\tau\Big). \end{aligned}$$

By (1.5), the solutions $(\mathbf{u}, \theta)$ are uniformly bounded in $L^\infty(0, T; H^1(\mathbb{R}^3))$, and thus smooth. This completes the proof of Theorem 1.1.

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