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PERSISTENCE AND EXTINCTION OF A STOCHASTIC SINGLE-SPECIES POPULATION MODEL IN A POLLUTED ENVIRONMENT WITH IMPULSIVE TOXICANT INPUT

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Abstract. A stochastic single-species population system in a polluted environment with impulsive toxicant input is proposed and studied. Sufficient conditions for extinction, non-persistence in the mean, strong persistence in the mean and stochastic permanence of the population are established. The threshold between strong persistence in the mean and extinction is obtained. Some simulation figures are introduced to illustrate the main results.

1. Introduction

In the world today, a large quantity of toxicants and contaminants are emitted into ecosystems by various industries and other activities of human. Uncontrolled contribution of toxicants to the environment has led many populations to extinction and several others to be on the verge of extinction. This motivates scholars to investigate the effect of toxins on the populations and to find a theoretical threshold value which determines extinction or persistence of a species or community. These investigations are becoming more and more important.

Recently, Hallam et al [6, 7, 8] proposed some deterministic models to study the effect of toxicants on a single species initially. From then on, many important deterministic models were proposed and studied, see e.g. [2, 3, 4, 9, 10, 11],[13]-[18],[28, 29, 31],[33]-[39] and the references cited therein. Particularly, Liu, Chen and Zhang [16] proposed a single-species population model in a polluted environment with impulsive toxicant input. The authors obtained the survival threshold and investigated the globally asymptotical stability of the positive periodic solution of the model.

In the real world, population systems are inevitably affected by the environmental noises. Taking the random perturbations into account, Gard [5] proposed and studied the following stochastic model

$$dx = \frac{x}{g(x)}\Big[(r_0 - r_1 C_0(t) - f(x)x)dt + \alpha_1 dB_1(t)\Big], \tag{1.1}$$

where $x = x(t)$ is the population size at time t; r_0 stands for the intrinsic growth rate of the population without toxicant; r_1 denotes the population response to

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the pollutant present in the organism; $C_0(t)$ is the concentration of toxicant in the organism at time t; The functions f and g in equation (1.1) represent intra specific density dependent interactions and are assumed to be in $C[\mathbb{R}_+, \mathbb{R}_+]$ and $C[\mathbb{R}_+, \mathbb{R}_+ - \{0\}]$ respectively. $B_1(t)$ is a standard Brownian motion and α_1 is a constant representing the intensity of the white noise. Under the assumption that the toxicant stress $C_0(t)$ in equation (1.1) was a constant, the author obtained the conditions for the existence of an invariant distribution on $(0, +\infty)$. Then Liu and Wang [19]-[27] studied the persistence and extinction of system (1.1) and its generalized forms without the constant assumption. Recently, Liu and Wang [21, 24, 20] investigated the stochastic Lotka-Volterra competitive model and cooperation model in a polluted environment respectively. For each species, the authors established the survival threshold. However, in all of these stochastic models, it is supposed that the exogenous input of toxicant is continuous. In fact, in practical situation, many toxicants are emitted in regular pulses. For example, pesticides can be sprayed instantaneously and regularly. Chemical plant and artificial industry often termly let sewage or other pollutant into rivers, soil and air. So far as our knowledge is concerned, there are no studies of the stochastic system in a polluted environment with impulsive toxicant input. So in this paper, we try to study this problem. The contributions of this paper are therefore clear.

To obtain our results, we need the following widely used concepts (see e.g. [9]-[11, 18]-[29], [31, 34, 38]).

Definition 1.1.

(I) The population, $x(t)$, is said to go to extinction if $\lim_{t\to+\infty} x(t) = 0$.
(II) $x(t)$ is said to be non-persistent in the mean if $\lim_{t\to+\infty}\langle x(t)\rangle = 0$, where $\langle f(t)\rangle = t^{-1}\int_0^t f(s)ds$.
(III) $x(t)$ is said to be strongly persistent in the mean if $\langle x(t)\rangle_* > 0$, where $f_* = \liminf_{t\to+\infty} f(t)$, $f^* = \limsup_{t\to+\infty} f(t)$.
(IV) $x(t)$ is said to be stochastically permanent if for arbitrary $\varepsilon > 0$, there are two positive constants σ and δ such that

$$\mathcal{P}_*\{x(t) \geq \sigma\} \geq 1 - \varepsilon, \quad \mathcal{P}_*\{x(t) \leq \delta\} \geq 1 - \varepsilon.$$

The rest of the paper is arranged as follows. In Section 2, we propose our stochastic model. In Section 3, we carry out the survival analysis for our system. Sufficient conditions for extinction, non-persistence in the mean, strong persistence in the mean and stochastic permanence are established. The threshold between strong persistence in the mean and extinction is obtained. In Section 4, we work out some figures to illustrate our main results. We close the paper with conclusions in Section 5.

2. Model Formulation

The model we consider is based on the following single species model with pulse toxicant input at fixed moment (Liu, Chen and Zhang [16])

$$
\left.\begin{aligned}
\frac{dx(t)}{dt} &= x(t)[r_0 - r_1C_0(t) - fx] \\
\frac{dC_0(t)}{dt} &= kC_e(t) - (g+m)C_0(t) \\
\frac{dC_e(t)}{dt} &= -hC_e(t)
\end{aligned}\right\}\quad t \neq n\tau,\ n \in Z^+.
\tag{2.1}
$$
$$
\Delta x(t) = 0, \quad \Delta C_0(t) = 0, \quad \Delta C_e(t) = b, \quad t = n\tau, \quad n \in Z^+.
$$

Here, the state variable $x(t)$ is the population size, $C_0(t)$ is the concentration of toxicant in the organism and $C_e(t)$ is the concentration of toxicant in the environment, $x(0) > 0$, $0 \le C_0(0) \le 1$, $0 \le C_e(0) \le 1$, $\Delta x(t) = x(t^+) - x(t)$, $\Delta C_0(t) = C_0(t^+) - C_0(t)$, $\Delta C_e(t) = C_e(t^+) - C_e(t)$, $r_0, r_1, f, k, g, m, h, b$ are positive constants and $Z^+ = \{1, 2, \dots\}$; τ is the period of the impulsive effect about the exogenous input of toxicant and b is the toxicant input amount at every time; $kC_e(t)$ stands for the organism's net uptake of toxicant from the environment, and $gC_0(t)$ and $mC_0(t)$ represent the egestion and depuration rates of the toxicant in the organism, respectively; $hC_e(t)$ represents the toxicant loss from the environment itself by volatilization and so on.

As said above, population system is inevitably affected by environmental noises. In fact, it has been noted that (see e.g. [1]) if systems assume that parameters are deterministic, they would have some limitations in mathematical modeling of ecological systems, besides they would be difficult to fit data perfectly and to predict the future dynamics of the system accurately. May [30] has also pointed out that due to environmental noise, the birth rate should exhibit random fluctuation.

Recall that the parameter r_0 denotes the intrinsic growth rate of the population. In practice we usually estimate it by an average growth rate plus an error term. If we still use r_0 to denote the average growth rate, then the growth rate becomes

$$
r_0 + \text{error}.
$$

Let us consider a small subsequent time interval dt, during which $x(t)$ becomes $x(t) + dx(t)$. Therefore the first equation in (2.1) becomes

$$
\frac{dx(t)}{dt} = x(t)[r_0 + \text{error} - r_1C_0(t) - fx].
$$

By the well-known central limit theorem, the error term follows a normal distribution. Thus we can approximate it by a normal distribution with mean zero and variance γ^2; that is,

$$
\text{error}dt \backsim N(0, \gamma^2 dt).
$$

Taking into account that the random noise in the environment may not be single, thus we may describe them by an n-dimensional Brownian motion $B(t) = (B_1(t), \dots, B_n(t))^T$ as follows

$$
\text{error}\, dt = \sum_{i=1}^{n} \alpha_i dB_i(t),
$$

where $\sum_{i=1}^n \alpha_i^2 = \gamma$. Hence we obtain the following stochastic system

$$\left.\begin{aligned} dx(t) &= x(t)[r_0 - r_1C_0(t) - fx(t)]dt + \sum_{i=1}^n \alpha_i x(t)dB_i(t) \\ \frac{dC_0(t)}{dt} &= kC_e(t) - (g+m)C_0(t) \\ \frac{dC_e(t)}{dt} &= -hC_e(t) \end{aligned}\right\} \quad t \neq n\tau,\ n \in Z^+. \\ \Delta x(t) = 0, \quad \Delta C_0(t) = 0, \quad \Delta C_e(t) = b, \quad t = n\tau, \quad n \in Z^+. \\ x(0) > 0, \quad 0 \le C_0(0) \le 1, \quad 0 \le C_e(0) \le 1. \tag{2.2}$$

There are many methods to analysis deterministic system, such as Lyapunov functions, coincidence degree theory, Jacobian matrix and so on. But there is lack of mathematical machinery available to analyze stochastic system. One of current approaches for studying stochastic system is to make use of Fokker-Planck equation (see e.g. [32]). However, (2.2) is a non-autonomous stochastic system, whose corresponding Fokker-Planck equation is not an ordinary differential equations but a partial differential equation. Moreover, the ultimate boundedness of $x(t)$ in deterministic model (2.1) is destroyed in model (2.2) by stochastic disturbance. It is well-known that boundedness is a very important property in the proof. In this work, we mainly use Itô's formula, theory of stochastic differential equation and Lyapunov function to analyze the properties of system (2.2).

Since Each of $C_0(t)$ and $C_e(t)$ is a concentration, thus the inequalities $0 \le C_0(t) \le 1$ and $0 \le C_e(t) \le 1$ must be satisfied. Now let us prepare an important lemma to close this section.

Lemma 2.1 ([16]). *For system* (2.2), *if* $k \le g+m$, $b \le 1-e^{-h\tau}$, *then* $0 \le C_0(t) \le 1$ *and* $0 \le C_e(t) \le 1$ *for all* $t \ge 0$.

For the rest of this article, we impose $k \le g+m$ and $b \le 1-e^{-h\tau}$ in this paper.

3. Persistence and extinction

To begin with, let us give some basic properties of the following subsystem of (2.2):

$$\left.\begin{aligned} \frac{dC_0(t)}{dt} &= kC_e(t) - (g+m)C_0(t) \\ \frac{dC_e(t)}{dt} &= -hC_e(t) \end{aligned}\right\} \quad t \neq n\tau,\ n \in Z^+. \\ \Delta C_0(t) = 0, \quad \Delta C_e(t) = b, \quad t = n\tau, \quad n \in Z^+. \\ 0 \le C_0(0) \le 1, \quad 0 \le C_e(0) \le 1. \tag{3.1}$$

Lemma 3.1 ([16]). *System* (3.1) *has a unique positive* τ*-periodic solution* $(\tilde{C}_0(t), \tilde{C}_e(t))^T$ *and for every solution* $(C_0(t), C_e(t))^T$ *of* (3.1), $C_0(t) \to \tilde{C}_0(t)$ *and* $C_e(t) \to \tilde{C}_e(t)$ *as* $t \to \infty$. *Moreover,* $C_0(t) > \tilde{C}_0(t)$ *and* $C_e(t) > \tilde{C}_e(t)$ *for all* $t \ge 0$ *if* $C_0(0) > \tilde{C}_0(0)$ *and* $C_e(0) > \tilde{C}_e(0)$, *where*

$$\tilde{C}_0(t) = \tilde{C}_0(0)e^{-(g+m)(t-n\tau)} + \frac{kb(e^{-(g+m)(t-n\tau)} - e^{-h(t-n\tau)})}{(h-g-m)(1-e^{-h\tau})},$$

$$\tilde{C}_e(t) = \frac{be^{-h(t-n\tau)}}{1-e^{-h\tau}},$$
$$\tilde{C}_0(0) = \frac{kb(e^{-(g+m)\tau} - e^{-h\tau})}{(h-g-m)(1-e^{-(g+m)\tau})(1-e^{-h\tau})},$$
$$\tilde{C}_e(0) = \frac{b}{1-e^{-h\tau}}$$

for $t \in (n\tau, (n+1)\tau]$ and $n \in Z^+$. In addition,

$$\lim_{t\to+\infty} t^{-1}\int_0^t \tilde{C}_0(s)ds = \frac{kb}{h(g+m)\tau}. \tag{3.2}$$

Now we can give our main results.

Theorem 3.2. *For model (2.2), if*

$$r_0 < \frac{r_1 kb}{h(g+m)\tau} + 0.5\sum_{i=1}^n \alpha_i^2,$$

then the population goes to extinction almost surely (a.s.).

Proof. From Lemma 3.1, for for all $\varepsilon > 0$, there exists a constant $T > 0$ such that

$$\tilde{C}_0(t) - \varepsilon \le C_0(t) \le \tilde{C}_0(t) + \varepsilon, \quad t > T.$$

Then for $t > T$,

$$\begin{aligned} t^{-1}\int_T^t (\tilde{C}_0(s)-\varepsilon)ds &\le t^{-1}\int_0^t C_0(s)ds = t^{-1}\Big[\int_0^T C_0(s)ds + \int_T^t C_0(s)ds\Big] \\ &\le t^{-1}\Big[T + \int_T^t (\tilde{C}_0(s)+\varepsilon)ds\Big]. \end{aligned}$$

By (3.2),

$$\lim_{t\to+\infty} t^{-1}\int_0^t C_0(s)ds = \frac{kb}{h(g+m)\tau}. \tag{3.3}$$

Applying Itô's formula to the first equation of (2.2) gives

$$d\ln x = \frac{dx}{x} - \frac{(dx)^2}{2x^2} = \Big[r_0 - r_1C_0(t) - fx - 0.5\sum_{i=1}^n \alpha_i^2\Big]dt + \sum_{i=1}^n \alpha_i dB_i(t).$$

In other words, we have shown that

$$\ln(x(t)/x_0) = \int_0^t \Big[r_0 - r_1C_0(s) - fx(s) - 0.5\sum_{i=1}^n \alpha_i^2\Big]ds + \sum_{i=1}^n \alpha_i B_i(t)$$

Then for sufficiently large t, we have

$$t^{-1}\ln(x(t)/x_0) = r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - r_1\langle C_0(t)\rangle - f\langle x(t)\rangle + \sum_{i=1}^n \alpha_i B_i(t)/t. \tag{3.4}$$

Using the strong law of large numbers for martingales (see e.g. Mao [32] on page 16) leads to

$$\lim_{t\to+\infty} B_i(t)/t = 0, \quad i = 1, \dots, n. \tag{3.5}$$

Substituting (3.3) and (3.5) into (3.4) and then using the arbitrariness of ε, one can obtain

$$\big[t^{-1}\ln\frac{x(t)}{x_0}\big]^* \le r_0 - \frac{r_1 kb}{h(g+m)\tau} - 0.5\sum_{i=1}^n \alpha_i^2 < 0.$$

Thus $\lim_{t\to+\infty} x(t) = 0$, a.s. □

Theorem 3.3. *For model (2.2), if* $r_0 = \frac{r_1 kb}{h(g+m)\tau} + 0.5\sum_{i=1}^n \alpha_i^2$, *then the population is nonpersistent in the mean a.s..*

Proof. For arbitrarily fixed $\varepsilon > 0$, there is a constant T such that

$$r_1 t^{-1}\int_0^t C_0(s)ds > \frac{r_1 kb}{h(g+m)\tau} - \varepsilon/2, \quad \sum_{i=1}^n \alpha_i B_i(t)/t < \varepsilon/2$$

for $t > T$. Substituting these inequalities into (3.4) results in

$$\begin{aligned}\ln(x(t)/x_0) &= [r_0 - 0.5\sum_{i=1}^n \alpha_i^2]t - r_1\int_0^t C_0(s)ds - f\int_0^t x(s)ds + \sum_{i=1}^n \alpha_i B_i(t)\\ &< \Big[r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau} + \varepsilon\Big]t - f\int_0^t x(s)ds\end{aligned}$$

for all $t \ge T$ almost surely. Set $h(t) = \int_0^t x(s)ds$, $\psi = r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau} + \varepsilon$, then we obtain

$$\ln(dh/dt) < \psi t - fh(t) + \ln\ x_0.$$

In other words, for $t \ge T$, we have

$$e^{fh(t)}(dh/dt) < x_0 e^{\psi t}.$$

Integrating this inequality from T to t leads to

$$f^{-1}[e^{fh(t)} - e^{fh(T)}] < x_0\psi^{-1}[e^{\psi t} - e^{\psi T}].$$

Rewriting this inequality, one can obtain that

$$e^{fh(t)} < e^{fh(T)} + x_0 f\psi^{-1}e^{\psi t} - x_0 f\psi^{-1}e^{\psi T}.$$

Taking the logarithm of both sides yields

$$h(t) < f^{-1}\ln\{x_0 f\psi^{-1}e^{\psi t} + e^{fh(T)} - x_0 f\psi^{-1}e^{\psi T}\}.$$

In other words, we have shown that

$$\Big\{t^{-1}\int_0^t x(s)ds\Big\}^* \le f^{-1}\Big\{t^{-1}\ln\Big\{x_0 f\psi^{-1}e^{\psi t} + e^{fh(T)} - x_0 f\psi^{-1}e^{\psi T}\Big\}\Big\}^*.$$

An application of the L'Hospital's rule results in

$$\langle x(t)\rangle^* \le f^{-1}\Big\{t^{-1}\ln\big[x_0 f\psi^{-1}e^{\psi t}\big]\Big\}^* = \psi/f.$$

Thus it follows from the arbitrariness of ε that

$$\langle x(t)\rangle^* \le \Big[r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau}\Big]/f. \tag{3.6}$$

Then the required assertion follows from $r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau} = 0$. □

Theorem 3.4. *If*

$$r_0 > \frac{r_1 kb}{h(g+m)\tau} + 0.5\sum_{i=1}^{n} \alpha_i^2,$$

then the population $x(t)$ *represented by system* (2.2) *is strongly persistent in the mean. Moreover,*

$$\lim_{t\to+\infty} \langle x(t)\rangle = \Big[r_0 - \frac{r_1 kb}{h(g+m)\tau} - 0.5\sum_{i=1}^{n} \alpha_i^2\Big]/f.$$

Proof. For all $\varepsilon > 0$, there exists a constant T such that

$$\sum_{i=1}^{n} \alpha_i B_i(t)/t > -\varepsilon/2, \quad t^{-1}\int_0^t r_1 C_0(s)ds < \frac{r_1 kb}{h(g+m)\tau} + \varepsilon/2$$

for all $t > T$. Substituting these inequalities into (3.4) results in

$$\ln(x(t)) - \ln x_0 > \rho t - f\int_0^t x(s)ds; \quad t > T,$$

where

$$\rho = r_0 - \frac{r_1 kb}{h(g+m)\tau} - 0.5\sum_{i=1}^{n} \alpha_i^2 - \varepsilon.$$

Let $g(t) = \int_0^t x(s)ds$, then we have

$$\ln\frac{dg}{dt} > \rho t - fg(t) + \ln x_0; \quad t > T.$$

In other words, for $t \geq T$, one can see that

$$e^{fg(t)}(dg/dt) > x_0 e^{\rho t}.$$

Integrating this inequality from T to t yields

$$f^{-1}[e^{fg(t)} - e^{fg(T)}] > x_0\rho^{-1}[e^{\rho t} - e^{\rho T}].$$

Rewriting this inequality one can observe that

$$e^{fg(t)} > e^{fg(T)} + x_0 f\rho^{-1}e^{\rho t} - x_0 f\rho^{-1}e^{\rho T}.$$

Taking the logarithm of both sides leads to

$$g(t) > f^{-1}\ln\{x_0 f\rho^{-1}e^{\rho t} + e^{fg(T)} - x_0 f\rho^{-1}e^{\rho T}\}.$$

In other words, we have already shown that

$$\Big\{t^{-1}\int_0^t x(s)ds\Big\}_* \geq f^{-1}\Big\{t^{-1}\ln\Big\{x_0 f\rho^{-1}e^{\rho t} + e^{fg(T)} - x_0 f\rho^{-1}e^{\rho T}\Big\}\Big\}_*.$$

An application of the L'Hospital's rule gives

$$\langle x(t)\rangle_* \geq f^{-1}\Big\{t^{-1}\ln\Big[x_0 f\rho^{-1}e^{\rho t}\Big]\Big\}_* = \rho/f.$$

Thus it follows from the arbitrariness of ε that

$$\langle x\rangle_* \geq \Big[r_0 - \frac{r_1 kb}{h(g+m)\tau} - 0.5\sum_{i=1}^{n} \alpha_i^2\Big]/f.$$

This, togethers with (3.6), complete the proof. □

It follows from Theorems 3.2-3.4 that

$$r_0 - \frac{r_1 kb}{h(g+m)\tau} - 0.5\sum_{i=1}^{n}\alpha_i^2$$

is the threshold between strong persistence in the mean and extinction for $x(t)$. Now let us turn to establishing the stochastic permanence of model (2.2). To this end, we need the following useful lemma.

Lemma 3.5. *For all $p > 0$, there exists a positive constant $K = K(p)$ such that*

$$\limsup_{t\to+\infty} E[x^p(t)] \le K(p).$$

The proof of the above lemma is a modification of Liu and Wang [19] (the second part of Theorem 4.5) and hence is omitted.

Theorem 3.6. *If*

$$r_0 > (r_1\tilde{C}_0(t))^* + 0.5\sum_{i=1}^{n}\alpha_i^2,$$

then the population $x(t)$ is stochastically permanent.

Proof. Firstly, let us demonstrate that for given $\varepsilon > 0$, there is a positive constant σ such that $\mathcal{P}_*\{x(t) \ge \sigma\} \ge 1-\varepsilon$. Define $V_1(x) = 1/x^2$ for $x \in R_+$. Applying Itô's formula to the first equation of (2.2) we have

$$\begin{aligned}
dV_1(x) &= -2x^{-3}dx + 3x^{-4}(dx)^2\\
&= 2V_1(x)[fx - r_0 + r_1C_0(t)]dt + 3V_1(x)\sum_{i=1}^{n}\alpha_i^2 dt - 2V_1(x)\sum_{i=1}^{n}\alpha_i dB_i(t)\\
&= 2V_1(x)\Big[fx - r_0 + r_1C_0(t) + 1.5\sum_{i=1}^{n}\alpha_i^2\Big]dt - 2V_1(x)\sum_{i=1}^{n}\alpha_i dB_i(t),
\end{aligned}$$

For arbitrarily small ε satisfying $r_0 - 0.5\sum_{i=1}^{n}\alpha_i^2 - (r_1C_0(t))^* > \varepsilon > 0$, we can choose a positive constant θ such that

$$r_0 - 0.5\sum_{i=1}^{n}\alpha_i^2 - (r_1C_0(t))^* - \varepsilon - \theta\sum_{i=1}^{n}\alpha_i^2 > 0.$$

Define

$$V_2(x) = (1+V_1(x))^\theta.$$

An application of Itô's formula gives

$$\begin{aligned}
dV_2(x) &= \theta(1+V_1(x))^{\theta-1}dV_1 + 0.5\theta(\theta-1)(1+V_1(x))^{\theta-2}(dV_1)^2\\
&= \theta(1+V_1(x))^{\theta-2}\Big\{(1+V_1(x))2V_1(x)\\
&\quad\times\Big[fx - r_0 + r_1C_0(t) + 1.5\sum_{i=1}^{n}\alpha_i^2\Big] + 2(\theta-1)V_1^2(x)\sum_{i=1}^{n}\alpha_i^2\Big\}dt\\
&\quad - 2\theta(1+V_1(x))^{\theta-1}V_1(x)\sum_{i=1}^{n}\alpha_i dB_i(t)\\
&= \theta(1+V_1(x))^{\theta-2}\Big\{-2\Big[r_0 - r_1C_0(t) - 0.5\sum_{i=1}^{n}\alpha_i^2 - \theta\sum_{i=1}^{n}\alpha_i^2\Big]V_1^2(x)
\end{aligned}$$

$$
\begin{aligned}
&+2fV_1^{1.5}(x)+\Big[-2r_0+2r_1C_0(t)+3\sum_{i=1}^n\alpha_i^2\Big]V_1(x)+2fV_1^{0.5}(x)\Big\}dt\\
&-2\theta(1+V_1(x))^{\theta-1}V_1(x)\sum_{i=1}^n\alpha_i dB_i(t)\\
\le{}& \theta(1+V_1(x))^{\theta-2}\Big\{-2\Big[r_0-0.5\sum_{i=1}^n\alpha_i^2-(r_1\tilde{C}_0(t))^*-\theta\sum_{i=1}^n\alpha_i^2-\varepsilon\Big]V_1^2(x)\\
&+2fV_1^{1.5}(x)+\Big[2r_1+3\sum_{i=1}^n\alpha_i^2\Big]V_1(x)+2fV_1^{0.5}(x)\Big\}dt\\
&-2\theta(1+V_1(x))^{\theta-1}V_1(x)\sum_{i=1}^n\alpha_i dB_i(t)
\end{aligned}
$$

for sufficiently large t. Now, choose $\eta>0$ sufficiently small to satisfy

$$
0<\eta/\theta<2\Big[r_0-0.5\sum_{i=1}^n\alpha_i^2-(r_1\tilde{C}_0(t))^*-\theta\sum_{i=1}^n\alpha_i^2-\varepsilon\Big]. \tag{3.7}
$$

Define

$$
V_3(x)=e^{\eta t}V_2(x)=e^{\eta t}(1+V_1(x))^\theta.
$$

An application of Itô's formula yields

$$
\begin{aligned}
dV_3(x(t))&=\eta e^{\eta t}V_2(x)dt+e^{\eta t}dV_2(x)\\
&\le\theta e^{\eta t}(1+V_1(x))^{\theta-2}\Big\{\eta(1+V_1(x))^2/\theta\\
&\quad-2\Big[r_0-0.5\sum_{i=1}^n\alpha_i^2-(r_1\tilde{C}_0(t))^*-\theta\sum_{i=1}^n\alpha_i^2-\varepsilon\Big]V_1^2(x)\\
&\quad+2fV_1^{1.5}(x)+\Big[2r_1+3\sum_{i=1}^n\alpha_i^2\Big]V_1(x)+2fV_1^{0.5}(x)\Big\}dt\\
&\quad-2e^{\eta t}\theta(1+V_1(x))^{\theta-1}V_1(x)\sum_{i=1}^n\alpha_i dB_i(t)\\
&=\theta e^{\eta t}(1+V_1(x))^{\theta-2}\Big\{-2\Big[r_0-0.5\sum_{i=1}^n\alpha_i^2-(r_1\tilde{C}_0(t))^*\\
&\quad-\theta\sum_{i=1}^n\alpha_i^2-\varepsilon-0.5\eta/\theta\Big]V_1^2(x)\\
&\quad+2fV_1^{1.5}(x)+\Big[2r_1+3\sum_{i=1}^n\alpha_i^2+2\eta/\theta\Big]V_1(x)+2fV_1^{0.5}(x)+\eta/\theta\Big\}dt\\
&\quad-2e^{\eta t}\theta(1+V_1(x))^{\theta-1}V_1(x)\sum_{i=1}^n\alpha_i dB_i(t)\\
&=:e^{\eta t}F(x)dt-2e^{\eta t}\theta(1+V_1(x))^{\theta-1}V_1(x)\sum_{i=1}^n\alpha_i dB_i(t)
\end{aligned}
$$

for sufficiently large t, where

$$\begin{aligned} F(x) = \theta(1+V_1(x))^{\theta-2}\Big\{ &-2\Big[r_0 - 0.5\sum_{i=1}^{n}\alpha_i^2 - (r_1\tilde{C}_0(t))^* - \theta\sum_{i=1}^{n}\alpha_i^2 - \varepsilon \\ &- 0.5\eta/\theta\Big]V_1^2(x) + 2fV_1^{1.5}(x) + \Big[2r_1 + 3\sum_{i=1}^{n}\alpha_i^2 + 2\eta/\theta\Big]V_1(x) \\ &t \neq n\tau,\ n \in Z^+. + 2fV_1^{0.5}(x) + \eta/\theta\Big\}. \end{aligned}$$

It follows from (3.7) that $F(x)$ is bounded form above in R_+, namely $F_1 := \sup_{x\in R_+} F(x) < +\infty$. Thus,

$$dV_3(x(t)) \le F_1 e^{\eta t}dt - 2e^{\eta t}\theta(1+V_1(x))^{\theta-1}V_1(x)\sum_{i=1}^{n}\alpha_i dB_i(t).$$

Integrating both sides and then taking expectations, one can see that

$$E\Big[e^{\eta t}\Big(1+V_1(x(t))\Big)^{\theta}\Big] \le \Big(1+V_1(x(0))\Big)^{\theta} + F_1(e^{\eta t}-1)/\eta.$$

In other words, we have already shown that

$$\limsup_{t\to+\infty} E[V_1^{\theta}(x(t))] \le \limsup_{t\to+\infty} E\Big[\Big(1+V_1(x(t))\Big)^{\theta}\Big] \le F_1/\eta.$$

That is to say

$$\limsup_{t\to+\infty} E[x^{-2\theta}(t)] \le F_1/\eta =: F_2.$$

Thus for any given $\varepsilon > 0$, denote $\sigma = \varepsilon^{0.5/\theta}/F_2^{0.5/\theta}$. By Chebyshev's inequality (see e.g. Mao [32, page 7]), one can derive that

$$\mathcal{P}\{x(t) < \sigma\} = \mathcal{P}\{x^{-2\theta}(t) > \sigma^{-2\theta}\} \le E[x^{-2\theta}(t)]/\sigma^{-2\theta} = \sigma^{2\theta}E[x^{-2\theta}(t)],$$

which is to say $\mathcal{P}^*\{x(t) < \sigma\} \le \sigma^{2\theta}F_2 = \varepsilon$. Thus $\mathcal{P}_*\{x(t) \ge \sigma\} \ge 1-\varepsilon$.

Next we prove that for arbitrary fixed $\varepsilon > 0$, there exists a positive constant δ such that $\mathcal{P}_*(x(t) \le \delta) \ge 1-\varepsilon$. The proof follows from Lemma 3.1 and Chebyshev's inequality immediately. □

4. NUMERICAL SIMULATIONS

In this section we will introduce some figures to illustrate our main results.

In Figure 1, we choose $r_0 = 0.75$, $r_1 = k = h = g = m = \tau = 1$, $b = 0.1$, $f = 0.18$, $n = 2$, $\alpha_2^2/2 = 0.1$. The only difference between conditions of Figure 1 are the values of $\alpha_1^2/2$. In Figure 1(a), we choose $\alpha_1^2/2 = 0.605$. It then follows from Theorem 3.2 that population goes to extinction. Figure 1(a) confirms this. In Figure 1(b), we choose $\alpha_1^2/2 = 0.6$. Making use of Theorem 3.3 yields that population x is nonpersistent in the mean. See Figure 1(b). In Figure 1(c), we choose $\alpha_1^2/2 = 0.55$. By Theorem 3.4, population x is strongly persistent in the mean and

$$\limsup_{t\to+\infty}\langle x(t)\rangle = \Big[r_0 - \alpha_1^2/2 - \alpha_2^2/2 - \frac{r_1kb}{h(g+m)\tau}\Big]/f = 0.2778.$$

See Figure 1(c). In Figure 1(d), we choose $\alpha_1^2/2 = 0.41$. By virtue of Theorem 3.6, we can observe that population x is stochastically permanent. See Figure 1(d).

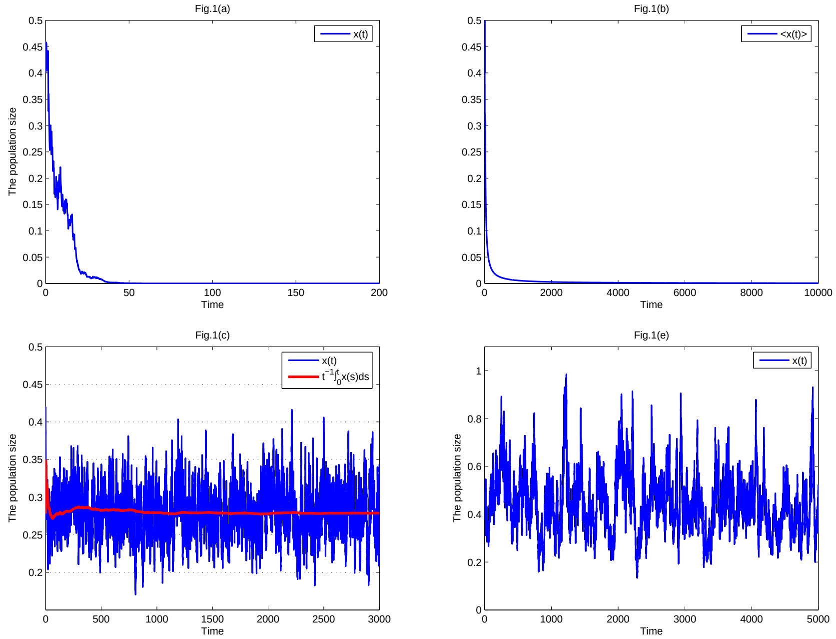

FIGURE 1. Solutions of system (2.2) for $r_0 = 0.75$, $r_1 = k = h = g = m = \tau = 1$, $b = 0.1$, $f = 0.18$, $n = 2$, $\alpha_2^2/2 = 0.1$, $x(0) = 0.4$, $C_0(0) = 0.5$ and $C_e(0) = 0.5$. The horizontal axis represents the time t. (a) is with $\alpha_1^2/2 = 0.605$; (b) is with $\alpha_1^2/2 = 0.6$; (c) is with $\alpha_1^2/2 = 0.55$; (d) is with $\alpha_1^2/2 = 0.51$

5. CONCLUSIONS AND FURTHER RESEARCH

In this paper, a stochastic single-species population system in a polluted environment with impulsive toxicant input is proposed and studied. Owing to its theoretical and practical significance, population model in a polluted environment with pulse toxicant input has deserved a lot of attention (see e.g. [3, 13, 14, 15, 16, 17, 33, 35, 38, 39]), but mainly in deterministic case. The present paper is the first attempt, up to our knowledge, of such a study in a stochastic setting.

It is shown that

(A) If $r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau} < 0$, then the population is extinctive with probability one;

(B) If $r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau} = 0$, then the population is non-persistent in the mean with probability one;

(C) If $r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau} > 0$, then the population is strongly persistent in the mean with probability one and
$$\lim_{t\to+\infty} \langle x(t)\rangle = \Big[r_0 - 0.5\sum_{i=1}^n \alpha_i^2 - \frac{r_1 kb}{h(g+m)\tau}\Big]/f, \text{ a.s.};$$

(D) If $r_0 > (r_1\tilde{C}_0(t))^* + 0.5\sum_{i=1}^n \alpha_i^2$, then the population is stochastically permanent.

From our results, we can see that both the white noises (i.e., α_i^2) and the impulsive effect period (i.e., τ) play very important roles in determining the extinction and persistence of the population.

Since many population models are inevitably affected by some stochastic noises, so the studies of these stochastic models are important and useful for better understanding of the real world. This paper devotes to studying a stochastic single-species model in a polluted environment with impulsive toxicant input which is basic and important, and the methods developed in this paper can be referred when one further studies other stochastic models, for example, Lotka-Volterra system with stochastic disturbance.

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