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SELFADJOINT EXTENSIONS OF MULTIPOINT SINGULAR DIFFERENTIAL OPERATORS

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Abstract. This article describes all selfadjoint extensions of the minimal operator generated by a linear singular multipoint symmetric differential-operator expression for first order in the direct sum of Hilbert spaces of vector-functions. This description is done in terms of the boundary values, and it uses the Everitt-Zettl and the Calkin-Gorbachuk methods. Also the structure of the spectrum of these extensions is studied.

1. Introduction

The general theory of selfadjoint extensions of symmetric operators in Hilbert spaces and their spectral theory have deeply been investigated by many mathematicians; see for example [2, 4, 5, 7, 8, 9]. Applications of this theory to two point differential operators in Hilbert spaces of functions have been even continued up to date. It is well-known that for the existence of selfadjoint extension of any linear closed densely defined symmetric operator in a Hilbert space, a necessary and sufficient condition is an equality of deficiency indices [9]. However multipoint situations may occur in different tables in the following sense: Let L_1 and L_2 be minimal operators generated by the linear differential expression $l(u) = i\frac{d}{dt}$ in the Hilbert space of functions $L^2(-\infty, a_1)$ and $L^2(a_2, +\infty)$, $a_1, a_2 \in \mathbb{R}$, respectively. In this case, it is known that deficiency indices of these minimal operators are in form $(m(L_1), n(L_1)) = (0, 1)$, $(m(L_2), n(L_2)) = (1, 0)$. Consequently, L_1 and L_2 are maximal symmetric operators, but they have no selfadjoint extensions. However, direct sum $L = L_1 \oplus L_2$ of operators in the direct sum $L^2(-\infty, a_1) \oplus L^2(a_2, +\infty)$ of Hilbert spaces have equal defect numbers (1,1). Then by the general theory [9] it has a selfadjoint extension. On the other hand, it can be easily shown [1] that all selfadjoint extensions of L are in the form

$$u_2(a_2) = e^{i\varphi} u_1(a_1), \quad \varphi \in [0, 2\pi), \quad u = (u_1, u_2), \quad u_1 \in D(L_1^*), \quad u_2 \in D(L_2^*).$$

Note that in the multiinterval linear ordinary differential expression case the deficiency indices may be different for each interval, but equal in the direct sum Hilbert spaces from the different intervals. The selfadjoint extension theory of linear ordinary differential expression of any order is known from famous work of

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Everitt and Zettl [3] for any number of intervals, finite or infinite, of real-axis. This theory is based on the Glassman-Krein-Naimark Theorem. Information on the selfadjoint extensions, the direct and complete characterizations for the Sturm-Liouville differential expression in finite or infinite interval with interior points or endpoints singularities can be found in the significant monograph of Zettl [10]. Special cases of problems considered in this paper has been investigated in [1, 6].

Lastly, note that many problems arising in the modeling processes, multi-particle quantum mechanics, quantum field theory, the physics of rigid bodies and etc support to study selfadjoint extension of symmetric differential operators in direct sum of Hilbert spaces (see [10] and references in it).

In section 2 in this work, by the method of Calkin-Gorbachuk Theory, all selfadjoint extensions of the minimal operator generated by linear multipoint singular symmetric differential-operator expression of first order in the direct sum of Hilbert spaces $L^2(H_1,(-\infty,a_1))\oplus L^2(H_2,(a_2,+\infty))\oplus L^2(H_3,(a_3,+\infty))$, where H_1, H_2, H_3 are a separable Hilbert spaces with condition $0<\dim H_1=\dim H_2+\dim H_3\le\infty$ and $a_1,a_2,a_3\in\mathbb{R}$, in terms of boundary values are described. In section 3, the spectrum of such extensions is investigated.

In this article, let

$$\Delta_1=(-\infty,a_1),\quad \Delta_2=(a_2,+\infty),\quad \Delta_3=(a_3,+\infty)\quad \text{for } a_k\in\mathbb{R},\quad k=1,2,3,$$
$$\mathfrak{L}(k)=L^2(H_k,\Delta_k),\quad k=1,2,3;\quad \mathfrak{L}=\mathfrak{L}(1)\oplus\mathfrak{L}(2)\oplus\mathfrak{L}(3).$$

2. Description of selfadjoint extensions

In the Hilbert space $\mathfrak{L}$ of vector-functions let us consider the linear multipoint singular symmetric differential-operator expression

$$l(u)=(l_1(u_1),l_2(u_2),l_3)(u_3)),$$

where $u=(u_1,u_2,u_3)$, $l_k(u_k)=iu_k'(t)+A_ku_k(t)$, $t\in\Delta_k$, $k=1,2,3$.

$A_k:D(A_k)\subset H_k\to H_k$, $k=1,2,3$ are linear selfadjoint operators in H_k. In the linear manifold $D(A_k)\subset H_k$ introduce the inner product

$$(f,g)_{k,+}=(A_kf,A_kg)_{H_k}+(f,g)_{H_k},\quad f,g\in D(A_k),\quad k=1,2,3.$$

For $k=1,2,3$, $D(A_k)$ is a Hilbert space under the positive norm $\|\cdot\|_{k,+}$ with respect to the Hilbert space H_k. It is denoted by $H_{k,+}$, $k=1,2,3$. Denote $H_{k,-}$, $k=1,2,3$ a Hilbert space with the negative norm (for information on Hilbert spaces with positive and negative norms, see for example [5]). It is clear that an operator A_k is continuous from $H_{k,+}$ to H_k and that its adjoint operator $\tilde{A}_k:H_k\to H_{k,-}$ is a extension of the operator A_k, $k=1,2,3$. On the other hand, $\tilde{A}_k:H_k\subset H_{k,-}\to H_{k,-}$, $k=1,2,3$ is a linear selfadjoint operator.

In the direct sum $\mathfrak{L}$ let us define

$$\tilde{l}(u)=(\tilde{l}_1(u_1),\tilde{l}_2(u_2),\tilde{l}_3(u_3)),\tag{2.1}$$

where $u=(u_1,u_2,u_3)$ and $\tilde{l}_k(u_k)=iu_k'(t)+\tilde{A}_ku_k(t)$, $t\in\Delta_k$, $k=1,2,3$.

The minimal operator L_{10} (L_{20}, L_{30}) and maximal operator L_1 (L_2, L_3) generated by differential-operator expression $\tilde{l}_1(\cdot)$ ($\tilde{l}_2(\cdot)$, $\tilde{l}_3(\cdot)$) in $\mathfrak{L}$ have been investigated in [4] and here established that the minimal operator L_{10} (L_{20}, L_{30}) is not selfadjoint in $\mathfrak{L}$. The operators $L_0=L_{10}\oplus L_{20}\oplus L_{30}$ and $L=L_1\oplus L_2\oplus L_3$ in the space $\mathfrak{L}$ are called minimal and maximal (multipoint) operators generated by the differential expression (2.1), respectively. Note that the operator L_0 is symmetric in $\mathfrak{L}$. On the other hand, it is clear that, deficiency indices $m(L_{10}) = \dim H_1$, $n(L_{10}) = 0$, $m(L_{20}) = 0$, $n(L_{20}) = \dim H_2$. Consequently, $m(L_0) = \dim H_1$, $n(L_0) = \dim H_2 + \dim H_3$. Hence, the minimal operator L_0 in the direct sum $\mathfrak{L}$ has a selfadjoint extension [9].

In this section all selfadjoint extensions of the minimal operator L_0 in $\mathfrak{L}$ in terms of the boundary values are described, using Calkin-Gorbachuk method. Note that in this theory, the space of boundary values is important for the description of selfadjoint extensions of linear symmetric differential operators [4, 5, 8]. Now give their definition.

Definition 2.1. Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$, having equal finite or infinite deficiency indices. A triplet $(\mathfrak{H}, \gamma_1, \gamma_2)$, where $\mathfrak{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings of $D(T^*)$ into $\mathfrak{H}$, is called a space of boundary values for the operator T if for any $f, g \in D(T^*)$

$$(T^*f, g)_{\mathcal{H}} - (f, T^*g)_{\mathcal{H}} = (\gamma_1(f), \gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f), \gamma_1(g))_{\mathfrak{H}},$$

while for any $F, G \in \mathfrak{H}$, there exists an element $f \in D(T^*)$, such that $\gamma_1(f) = F$ and $\gamma_2(f) = G$.

Note that any symmetric operator with equal deficiency indices has at least one space of boundary values [5].

Since H_1, H_2, H_3 are separable Hilbert spaces and $\dim H_1 = \dim H_2 + \dim H_3$, then it is known that there exist an isometric isomorphism $V : H_2 \oplus H_3 \to H_1$ such that $V(H_2 \oplus H_3) = H_1$. In this case the following statement is true.

Lemma 2.2. *The triplet $(H_1, \gamma_1, \gamma_2)$, where*

$$\gamma_1 : D(L_0^*) \to H_1, \quad \gamma_1(u) = \frac{1}{i\sqrt{2}}(u_1(a_1) + V(u_2(a_2), u_3(a_3))), \quad u \in D(L_0^*),$$
$$\gamma_2 : D(L_0^*) \to H_1, \quad \gamma_2(u) = \frac{1}{\sqrt{2}}(u_1(a_1) - V(u_2(a_2), u_3(a_3))), \quad u \in D(L_0^*)$$

is a space of boundary values of the minimal operator L_0 in direct sum $\mathfrak{L}$.

Proof. For arbitrary $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in D(L_0)$ the validity of the equality

$$(Lu, v)_{\mathfrak{L}} - (u, Lv)_{\mathfrak{L}} = (\gamma_1(u), \gamma_2(v))_{H_1} - (\gamma_2(u), \gamma_1(v))_{H_1}$$

can be easily verified. Now for any given elements $F, G \in H_1$, we will find the function $u = (u_1, u_2, u_3) \in D(L_0)$ such that

$$\gamma_1(u) = \frac{1}{i\sqrt{2}}(u_1(a_1) + V(u_2(a_2), u_3(a_3))) = F,$$
$$\gamma_2(u) = \frac{1}{\sqrt{2}}(u_1(a_1) - V(u_2(a_2), u_3(a_3))) = G.$$

Indeed, in this case

$$V(u_2(a_2), u_3(a_3)) = (iF - G)/\sqrt{2}, \quad u_1(a_1) = (iF + G)/\sqrt{2}$$

and from this, since V is the isometric mapping from $H_2 \oplus H_3$ onto H_1, then it implies that there exists unique elements $v_1(F, G) \in H_2$ and $v_2(F, G) \in H_3$ such that

$$(u_2(a_2), u_3(a_3)) = \frac{1}{\sqrt{2}} V^{-1}(iF - G) = (v_1(F, G), v_2(F, G)) \in H_2 \oplus H_3,$$

$$u_1(a_1) = \frac{1}{\sqrt{2}}(iF + G) \in H_1.$$

If we choose the functions $u(t) = (u_1(t), u_2(t), u_3(t))$ in the form

$$\begin{aligned} u_1(t) &= e^{\frac{t-a_1}{2}} \frac{1}{\sqrt{2}}(iF + G), \quad t \in \Delta_1, \\ u_2(t) &= e^{\frac{a_2-t}{2}} v_1(F, G), \quad t \in \Delta_2, \\ u_3(t) &= e^{\frac{a_3-t}{2}} v_2(F, G), \quad t \in \Delta_3, \end{aligned}$$

then it is clear that $u(t) = (u_1(t), u_2(t), u_3(t)) \in D(L_0)$ and $\gamma_1(u) = F$, $\gamma_2(u) = G$. $\square$

Using the method in [5] the following result can be deduced.

Theorem 2.3. *If $\tilde{L}$ is a selfadjoint extension of the minimal operator L_0 in direct sum $\mathfrak{L}$, then it is generated by differential-operator expression* (2.1) *and boundary condition*

$$u_1(a_1) = WV(u_2(a_2), u_3(a_3)),$$

where $W : H_1 \to H_1$ is a unitary operator. Moreover, the unitary operator W is determined uniquely by extension $\tilde{L}$, i.e. $\tilde{L} = L_W$ and vice versa.

Remark 2.4. In a similar way, the selfadjoint extensions can be described of minimal operator generated by multipoint differential-operator expression

$$l(u) = (l_1(u_1), l_2(u_2), \dots, l_n(u_n); m_1(v_1), m_2(v_2), \dots, m_k(v_k)),$$

where $u = (u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_k)$,

$$\begin{aligned} l_p(u_p) &= iu_p'(t) + A_p u_p(t), \quad t \in (-\infty, a_p),\ p = 1, 2, \dots, n; \\ m_j(v_j) &= iu_j'(t) + B_j u_j(t), \quad t \in (b_j, +\infty),\ j = 1, 2, \dots, k, \end{aligned}$$

$A_p : D(A_p) \subset H_p \to H_p$ and $B_j : D(B_j) \subset G_j \to G_j$ are linear selfadjoint operators in Hilbert spaces H_p, $p = 1, 2, \dots, n$ and G_j, $j = 1, 2, \dots, k$ respectively, in direct sum spaces $\bigoplus_{p=1}^n L^2(H_p, (-\infty, a_p)) \oplus \bigoplus_{j=1}^k L^2(G_j, (b_j, +\infty))$ with condition $0 < \sum_{p=1}^n \dim H_p = \sum_{j=1}^k \dim G_j \leq \infty$.

3. Spectrum of the selfadjoint extensions

In this section, we study the structure of the spectrum of the selfadjoint extension L_W in a direct sum $\mathfrak{L}$. First, we have to prove the following result.

Theorem 3.1. *The point spectrum of any selfadjoint extension L_W is empty; i.e.,*

$$\sigma_p(L_W) = \emptyset.$$

Proof. Let us consider the problem for the spectrum of the selfadjoint extension L_W,

$$\begin{aligned} \tilde{l}(u) &= \lambda u(t), \quad \lambda \in \mathbb{R}, \\ u_1(a_1) &= WV(u_2(a_2), u_3(a_3)), \end{aligned}$$

where $W : H_1 \to H_1$ is a unitary operator. Then

$$(\tilde{l}_1(u_1), \tilde{l}_2(u_2), \tilde{l}_3(u_3)) = \lambda(u_1, u_2, u_3), \quad u_1(a_1) = WV(u_2(a_2), u_3(a_3))$$

and from this we have

$$\begin{gathered}\tilde{l}_k(u_k) = iu'_k(t) + \tilde{A}_k u_k(t) = \lambda u_k(t), \quad t \in \Delta_1,\ k = 1,2,3; \\ u_1(a_1) = WV(u_2(a_2), u_3(a_3)), \quad \lambda \in \mathbb{R}.\end{gathered}$$

The general solution of the this problem is

$$u_k(\lambda;t) = e^{i(\tilde{A}_k - \lambda)(t-a_k)} f_{k,\lambda}, \quad f_{k,\lambda} \in H_k, t \in \Delta_k, \quad k = 1,2,3.$$

The boundary condition is in form $f_{1,\lambda} = WV(f_{2,\lambda}, f_{3,\lambda})$. In order for $u_1(\lambda;t) \in \mathfrak{L}(1)$, $u_2(\lambda;t) \in \mathfrak{L}(2)$, $u_3(\lambda;t) \in \mathfrak{L}(3)$, necessary and sufficient conditions are $f_{k,\lambda} = 0$, $k = 1,2,3$. So for every operator W we have $\sigma_p(L_W) = \emptyset$. $\square$

Since the residual spectrum of any selfadjoint operator in any Hilbert space is empty, it is sufficient to investigate the continuous spectrum of the selfadjoint extensions L_W of the minimal operator L_0 in the Hilbert space $\mathfrak{L}$. First of all, we prove the following result.

Theorem 3.2. *For the resolvent set $\rho(L_W)$ it holds*

$$\rho(L_W) \supset \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \neq 0\}.$$

Proof. For this, we research the existence of the resolvent operator of L_W generated by the differential-operator expression $\tilde{l}(\cdot)$ and boundary condition $u_1(a_1) = WV(u_2(a_2), u_3(a_3))$ in $\mathfrak{L}$, in case when $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda \neq 0$. Firstly, consider the spectral problem in form

$$\begin{gathered}\tilde{l}(u) = \lambda u(t) + f(t), \quad f = (f_1, f_2, f_3) \quad \lambda \in \mathbb{C}, \quad \lambda_i = \operatorname{Im}\lambda > 0 \\ u_1(a_1) = WV(u_2(a_2), u_3(a_3))\end{gathered} \tag{3.1}$$

Now, we will show that the function

$$u(\lambda;t) = (u_1(\lambda;t), u_2(\lambda;t), u_3(\lambda;t)),$$

where

$$\begin{gathered}u_1(\lambda;t) = e^{i(\tilde{A}_1-\lambda)(t-a_1)} f_\lambda + i\int_t^{a_1} e^{i(\tilde{A}_1-\lambda)(t-s)} f_1(s)ds, \quad t \in \Delta_1, \\ u_2(\lambda;t) = i\int_t^\infty \mathrm{e}^{i(\tilde{A}_2-\lambda)(t-s)} f_2(s)ds, \quad t \in \Delta_2, \\ u_3(\lambda;t) = i\int_t^\infty \mathrm{e}^{i(\tilde{A}_3-\lambda)(t-s)} f_3(s)ds, \quad t \in \Delta_3, \\ f_\lambda = WV\Big(i\int_{a_2}^\infty \mathrm{e}^{i(\tilde{A}_2-\lambda)(a_2-s)} f_2(s)ds, i\int_{a_3}^\infty \mathrm{e}^{i(\tilde{A}_3-\lambda)(a_3-s)} f_3(s)ds\Big),\end{gathered}$$

is a solution of boundary value problem (3.1) in $\mathfrak{L}$. It is sufficient to show that $u_1(\lambda;t) \in \mathfrak{L}(1)$, $u_2(\lambda;t) \in \mathfrak{L}(2)$, $u_3(\lambda;t) \in \mathfrak{L}(3)$, for $\lambda_i > 0$. Indeed, in this case

$$\begin{aligned}\|f_\lambda\|_{H_1}^2 &= \Big\| \int_{a_2}^\infty \mathrm{e}^{i(\tilde{A}_2-\lambda)(a_2-s)} f_2(s)ds \Big\|_{H_2}^2 + \Big\| \int_{a_3}^\infty \mathrm{e}^{i(\tilde{A}_3-\lambda)(a_3-s)} f_3(s)ds \Big\|_{H_3}^2 \\ &\le \Big(\int_{a_2}^\infty e^{\lambda_i(a_2-s)} \|f_2(s)\|_{H_2} ds \Big)^2 + \Big(\int_{a_3}^\infty e^{\lambda_i(a_3-s)} \|f_3(s)\|_{H_3} ds \Big)^2 \\ &\le \Big(\int_{a_2}^\infty e^{2\lambda_i(a_2-s)} ds \Big) \Big(\int_{a_2}^\infty \|f_2(s)\|_{H_2}^2 ds \Big)\end{aligned}$$

$$
\begin{aligned}
&+\Big(\int_{a_3}^{\infty} e^{2\lambda_i(a_3-s)}ds\Big)\Big(\int_{a_3}^{\infty}\|f_3(s)\|_{H_3}^2 ds\Big)\\
&=\frac{1}{2\lambda_i}(\|f_2\|_{\mathfrak{L}(2)}^2+\|f_3\|_{\mathfrak{L}(3)}^2)<\infty,
\end{aligned}
$$

$$
\begin{aligned}
\|e^{i(\tilde{A}_1-\lambda)(t-a_1)}f_\lambda\|_{\mathfrak{L}(1)}^2 = \|e^{-i\lambda(t-a_1)}f_\lambda\|_{\mathfrak{L}(1)}^2 &= \int_{-\infty}^{a_1}\|e^{-i\lambda(t-a_1)}f_\lambda\|_{H_1}^2 dt\\
&= \int_{-\infty}^{a_1} e^{2\lambda_i(t-a_1)}dt\|f_\lambda\|_{H_1}^2 = \frac{1}{2\lambda_i}\|f_\lambda\|_{H_1}^2<\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|i\int_t^{a_1} e^{i(\tilde{A}_1-\lambda)(t-s)}f_1(s)ds\Big\|_{\mathfrak{L}(1)}^2\\
&\le \int_{-\infty}^{a_1}\Big(\int_t^{a_1} e^{\lambda_i(t-s)}\|f_1(s)\|_{H_1}ds\Big)^2dt\\
&\le \int_{-\infty}^{a_1}\Big(\int_t^{a_1} e^{\lambda_i(t-s)}ds\Big)\Big(\int_t^{a_1} e^{\lambda_i(t-s)}\|f_1(s)\|_{H_1}^2ds\Big)dt\\
&= \frac{1}{\lambda_i}\int_{-\infty}^{a_1}\int_t^{a_1} e^{\lambda_i(t-s)}\|f_1(s)\|_{H_1}^2dsdt = \frac{1}{\lambda_i}\int_{-\infty}^{a_1}\Big(\int_{-\infty}^{s} e^{\lambda_i(t-s)}\|f_1(s)\|_{H_1}^2dt\Big)ds\\
&= \frac{1}{\lambda_i}\int_{-\infty}^{a_1}\Big(\int_{-\infty}^{s} e^{\lambda_i(t-s)}dt\Big)\|f_1(s)\|_{H_1}^2ds\\
&= \frac{1}{\lambda_i^2}\int_{-\infty}^{a_1}\|f_1(s)\|_{H_1}^2ds = \frac{1}{\lambda_i^2}\|f_1\|_{\mathfrak{L}(1)}^2<\infty.
\end{aligned}
$$

Hence, $\|u_1(\lambda;t)\|_{\mathfrak{L}(1)}<\infty$. Furthermore,

$$
\begin{aligned}
\|u_2(\lambda;t)\|_{\mathfrak{L}(2)}^2 &= \Big\|i\int_t^{\infty} e^{i(\tilde{A}_2-\lambda)(t-s)}f_2(s)ds\Big\|_{\mathfrak{L}(2)}^2\\
&\le \int_{a_2}^{\infty}\Big(\int_t^{\infty} e^{\lambda_i(t-s)}\|f_2(s)\|_{H_2}ds\Big)^2dt\\
&\le \int_{a_2}^{\infty}\Big(\int_t^{\infty} e^{\lambda_i(t-s)}ds\Big)\Big(\int_t^{\infty} e^{\lambda_i(t-s)}\|f_2(s)\|_{H_2}^2ds\Big)dt\\
&= \frac{1}{\lambda_i}\int_{a_2}^{\infty}\Big(\int_t^{\infty} e^{\lambda_i(t-s)}\|f_2(s)\|_{H_2}^2ds\Big)dt\\
&= \frac{1}{\lambda_i}\int_{a_2}^{\infty}\Big(\int_{a_2}^{s} e^{\lambda_i(t-s)}\|f_2(s)\|_{H_2}^2dt\Big)ds\\
&= \frac{1}{\lambda_i}\int_{a_2}^{\infty}\Big(\int_{a_2}^{s} e^{\lambda_i(t-s)}dt\Big)\|f_2(s)\|_{H_2}^2ds\\
&= \frac{1}{\lambda_i^2}\int_{a_2}^{\infty}(1-e^{\lambda_i(a_2-s)})\|f_2(s)\|_{H_2}^2ds\\
&\le \frac{1}{\lambda_i^2}\|f_2\|_{\mathfrak{L}(2)}^2<\infty
\end{aligned}
$$

and

$$
\|u_3(\lambda;t)\|_{\mathfrak{L}(3)}^2 = \Big\|i\int_t^{\infty} e^{i(\tilde{A}_3-\lambda)(t-s)}f_3(s)ds\Big\|_{\mathfrak{L}(3)}^2
$$

$$
\begin{aligned}
&\le \int_{a_3}^{\infty}\Big(\int_t^{\infty} e^{\lambda_i(t-s)}\|f_3(s)\|_{H_3}ds\Big)^2dt\\
&\le \int_{a_3}^{\infty}\Big(\int_t^{\infty} e^{\lambda_i(t-s)}ds\Big)\Big(\int_t^{\infty} e^{\lambda_i(t-s)}\|f_3(s)\|_{H_3}^2ds\Big)dt\\
&= \frac{1}{\lambda_i}\int_{a_3}^{\infty}\Big(\int_t^{\infty} e^{\lambda_i(t-s)}\|f_3(s)\|_{H_3}^2ds\Big)dt\\
&= \frac{1}{\lambda_i}\int_{a_3}^{\infty}\Big(\int_{a_3}^{s} e^{\lambda_i(t-s)}\|f_3(s)\|_{H_3}^2dt\Big)ds\\
&= \frac{1}{\lambda_i}\int_{a_3}^{\infty}\Big(\int_{a_3}^{s} e^{\lambda_i(t-s)}dt\Big)\|f_3(s)\|_{H_3}^2ds\\
&= \frac{1}{\lambda_i^2}\int_{a_3}^{\infty}(1-e^{\lambda_i(a_3-s)})\|f_3(s)\|_{H_3}^2ds\\
&\le \frac{1}{\lambda_i^2}\|f_3\|_{\mathfrak{L}(3)}^2<\infty.
\end{aligned}
$$

The above calculations imply that $u_1(\lambda;t)\in\mathfrak{L}(1)$, $u_2(\lambda;t)\in\mathfrak{L}(2)$ and $u_3(\lambda;t)\in\mathfrak{L}(1)$ for $\lambda\in\mathbb{C}$, $\lambda_i=\operatorname{Im}\lambda>0$. On the other hand, one can easily verify that $u(\lambda;t)=(u_1(\lambda;t),u_2(\lambda;t),u_3(\lambda;t))$ is a solution of boundary value problem (3.1).

When $\lambda\in\mathbb{C}$, $\lambda_i=\operatorname{Im}\lambda<0$ is true solution of the boundary value problem (3.1) is in the form $u(\lambda;t)=(u_1(\lambda;t),u_2(\lambda;t),u_3(\lambda;t))$,

$$
\begin{gathered}
u_1(\lambda;t)=-i\int_{-\infty}^{t} e^{i(\tilde A_1-\lambda)(t-s)}f_1(s)ds,\quad t\in\Delta_1\\
u_2(\lambda;t)=e^{i(\tilde A_2-\lambda)(t-a_2)}g_\lambda-i\int_{a_2}^{t} e^{i(\tilde A_2-\lambda)(t-s)}f_2(s)ds,\quad t\in\Delta_2,\\
u_3(\lambda;t)=e^{i(\tilde A_3-\lambda)(t-a_3)}h_\lambda-i\int_{a_3}^{t} e^{i(\tilde A_3-\lambda)(t-s)}f_3(s)ds,\quad t\in\Delta_3,
\end{gathered}
$$

where $-i\int_{-\infty}^{a_1} e^{i(\tilde A_1-\lambda)(a_1-s)}f_1(s)ds=WV(g_\lambda,h_\lambda)$ and since

$$
\begin{aligned}
\Big\|-i\int_{-\infty}^{a_1} e^{i(\tilde A_1-\lambda)(a_1-s)}f_1(s)ds\Big\|_{H_1}^2
&\le \Big(\int_{-\infty}^{a_1} e^{\lambda_i(a_1-s)}\|f_1(s)\|_{H_1}ds\Big)^2\\
&\le \Big(\int_{-\infty}^{a_1} e^{2\lambda_i(a_1-s)}ds\Big)\Big(\int_{-\infty}^{a_1}\|f_1(s)\|_{H_1}^2ds\Big)\\
&\le \frac{1}{2|\lambda_i|}\|f_1\|_{\mathfrak{L}(1)}^2<\infty,
\end{aligned}
$$

we have

$$
(g_\lambda,h_\lambda)=V^{-1}W^*\Big(-i\int_{-\infty}^{a_1} e^{i(\tilde A_1-\lambda)(a_1-s)}f_1(s)ds\Big)\in H_2\oplus H_3.
$$

First, we prove that $u(\lambda;t)\in\mathfrak{L}$. In this case,

$$
\begin{aligned}
\|u_1(\lambda;t)\|_{\mathfrak{L}(1)}^2&=\int_{-\infty}^{a_1}\Big\|-i\int_{-\infty}^{t} e^{i(\tilde A_1-\lambda)(t-s)}f_1(s)ds\Big\|_{H_1}^2dt\\
&\le \int_{-\infty}^{a_1}\Big(\int_{-\infty}^{t} e^{\lambda_i(t-s)}ds\Big)\Big(\int_{-\infty}^{t} e^{\lambda_i(t-s)}\|f_1(s)\|_{H_1}^2ds\Big)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|\lambda_i|} \int_{-\infty}^{a_1} \int_{-\infty}^{t} e^{\lambda_i (t-s)} \|f_1(s)\|_{H_1}^2 \, ds \, dt \\
&= \frac{1}{|\lambda_i|} \int_{-\infty}^{a_1} \Big(\int_{s}^{a_1} e^{\lambda_i (t-s)} \|f_1(s)\|_{H_1}^2 dt \Big) ds \\
&= \frac{1}{|\lambda_i|} \int_{-\infty}^{a_1} \Big(\int_{s}^{a_1} e^{\lambda_i (t-s)} dt \Big) \|f_1(s)\|_{H_1}^2 ds \\
&= \frac{1}{|\lambda_i|^2} \int_{-\infty}^{a_1} (1 - e^{\lambda_i (a_1 - s)}) \|f_1(s)\|_{H_1}^2 ds \\
&\le \frac{1}{|\lambda_i|^2} \|f_1\|_{\mathfrak{L}(1)}^2 < \infty,
\end{aligned}
$$

$$
\|e^{i(\tilde{A}_2 - \lambda)(t-a_2)} g_\lambda\|_{\mathfrak{L}(2)}^2 \le \int_{a_2}^{\infty} e^{2\lambda_i (t-a_2)} dt \|g_\lambda\|_{H_2}^2 = \frac{1}{2|\lambda_i|} \|g_\lambda\|_{H_2}^2 < \infty
$$

and

$$
\begin{aligned}
&\Big\| -i \int_{a_2}^{t} e^{i(\tilde{A}_2 - \lambda)(t-s)} f_2(s) ds \Big\|_{\mathfrak{L}(2)}^2 \\
&\le \int_{a_2}^{\infty} \Big(\int_{a_2}^{t} e^{\lambda_i (t-s)} \|f_2(s)\|_{H_2} ds \Big)^2 dt \\
&\le \int_{a_2}^{\infty} \Big(\int_{a_2}^{t} e^{\lambda_i (t-s)} ds \Big) \Big(\int_{a_2}^{t} e^{\lambda_i (t-s)} \|f_2(s)\|_{H_2}^2 ds \Big) dt \\
&= \int_{a_2}^{\infty} \Big(\frac{1}{\lambda_i} (1 - e^{\lambda_i (t-a_2)}) \Big) \Big(\int_{a_2}^{t} e^{\lambda_i (t-s)} \|f_2(s)\|_{H_2}^2 ds \Big) dt \\
&\le \frac{1}{|\lambda_i|} \int_{a_2}^{\infty} \Big(\int_{a_2}^{t} e^{\lambda_i (t-a_2)} \|f_2(s)\|_{H_2}^2 ds \Big) dt \\
&= \frac{1}{|\lambda_i|} \int_{a_2}^{\infty} \Big(\int_{s}^{\infty} e^{\lambda_i (t-s)} \|f_2(s)\|_{H_2}^2 dt \Big) ds \\
&= \frac{1}{|\lambda_i|} \int_{a_2}^{\infty} \Big(\int_{s}^{a_2} e^{\lambda_i (t-s)} dt \Big) \|f_2(s)\|_{H_2}^2 ds \\
&= \frac{1}{|\lambda_i|^2} \|f_2\|_{\mathfrak{L}(2)}^2 < \infty.
\end{aligned}
$$

In a similar way, it can be shown that

$$
\|e^{i(\tilde{A}_3 - \lambda)(t-a_3)} h_\lambda\|_{\mathfrak{L}(3)} < \infty, \quad \| -i \int_{a_3}^{t} e^{i(\tilde{A}_3 - \lambda)(t-s)} f_3(s) ds\|_{\mathfrak{L}(3)} < \infty.
$$

The above calculations show that $u_1(\lambda; \cdot) \in \mathfrak{L}(1)$, $u_2(\lambda; \cdot) \in \mathfrak{L}(2)$ and that $u_3(\lambda; \cdot) \in \mathfrak{L}(3)$; i.e., $u(\lambda; \cdot) = (u_1(\lambda; \cdot), u_2(\lambda; \cdot), u_3(\lambda, \cdot)) \in \mathfrak{L}$ for $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im} \lambda < 0$. On the other hand it can be verified that the function $u(\lambda; \cdot)$ satisfies the equation and boundary condition in (3.1). □

Now, we will study continuous spectrum $\sigma_c(L_W)$ of the selfadjoint extension L_W.

Theorem 3.3. *The continuous spectrum of any selfadjoint extension L_W is*

$$
\sigma_c(L_W) = \mathbb{R}.
$$

Proof. For $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda > 0$, norm of the resolvent operator $R_\lambda(L_W)$ of the L_W is of the form

$$\begin{aligned}\|R_\lambda(L_W)f(t)\|^2_{\mathfrak{L}} &= \big\|e^{i(\tilde{A}_1-\lambda)(t-a_1)}f_\lambda + i\int_t^{a_1} e^{i(\tilde{A}_1-\lambda)(t-s)}f_1(s)ds\big\|^2_{\mathfrak{L}(1)}\\ &\quad+\big\|i\int_t^\infty e^{i(\tilde{A}_2-\lambda)(t-s)}f_2(s)ds\big\|^2_{\mathfrak{L}(2)}\\ &\quad+\big\|i\int_t^\infty e^{i(\tilde{A}_3-\lambda)(t-s)}f_3(s)ds\big\|^2_{\mathfrak{L}(3)},\end{aligned}$$

where $f = (f_1, f_2, f_3) \in \mathfrak{L}$,

$$f_\lambda = WV\Big(i\int_{a_2}^\infty e^{i(\tilde{A}_2-\lambda)(a_2-s)}f_2(s)ds,\ i\int_{a_3}^\infty e^{i(\tilde{A}_3-\lambda)(a_3-s)}f_3(s)ds\Big).$$

Then, it is clear that for any $f = (f_1, f_2, f_3) \in \mathfrak{L}$,

$$\|R_\lambda(L_W)f(t)\|^2_{\mathfrak{L}} \ge \big\|i\int_t^\infty e^{i(\tilde{A}_2-\lambda)(t-s)}f_2(s)ds\big\|^2_{\mathfrak{L}(2)}.$$

The vector functions $f^*(\lambda;t)$ which is of the form $f^*(\lambda;t) = (0, e^{-i(\bar{\lambda}-\tilde{A}_2)t}f, 0)$, $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda > 0$, $f \in H_2$ belong to $\mathfrak{L}$. Indeed,

$$\begin{aligned}\|f^*(\lambda;t)\|^2_{\mathfrak{L}} &= \int_{a_2}^\infty \|e^{-i(\bar{\lambda}-\tilde{A}_2)t}f\|^2_{H_2}dt = \int_{a_2}^\infty e^{-2\lambda_i t}dt\|f\|^2_{H_2}\\ &= \frac{1}{2\lambda_i}e^{-2\lambda_i a_2}\|f\|^2_{H_2} < \infty.\end{aligned}$$

For such functions $f^*(\lambda;\cdot)$, we have

$$\begin{aligned}\|R_\lambda(L_W)f^*(\lambda;t)\|^2_{\mathfrak{L}(2)} &\ge \big\|i\int_t^\infty e^{-i(\lambda-\tilde{A}_2)(t-s)}e^{-i(\bar{\lambda}-\tilde{A}_2)s}fds\big\|^2_{\mathfrak{L}}\\ &= \|\int_t^\infty e^{-i\lambda t}e^{-2\lambda_i s}e^{i\tilde{A}_2 t}fds\big\|^2_{\mathfrak{L}(2)}\\ &= \big\|e^{-i\lambda t}e^{i\tilde{A}_2 t}\int_t^\infty e^{-2\lambda_i s}fds\big\|^2_{\mathfrak{L}(2)}\\ &= \big\|e^{-i\lambda t}\int_t^\infty e^{-2\lambda_i s}ds\big\|^2_{\mathfrak{L}(2)}\|f\|^2_{H_2}\\ &= \frac{1}{4\lambda_i^2}\int_{a_2}^\infty e^{-2\lambda_i t}dt\|f\|^2_{H_2}\\ &= \frac{1}{8\lambda_i^3}e^{-2\lambda_i a_2}\|f\|^2_{H_2}.\end{aligned}$$

From this we obtain

$$\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{\mathfrak{L}} \ge \frac{e^{-\lambda_i a_2}}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}}\|f\|_{H_2} = \frac{1}{2\lambda_i}\|f^*(\lambda;\cdot)\|_{\mathfrak{L}};$$

i.e., for $\lambda_i = \operatorname{Im}\lambda > 0$ and $f \ne 0$,

$$\frac{\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{\mathfrak{L}}}{\|f^*(\lambda;\cdot)\|_{\mathfrak{L}}} \ge \frac{1}{2\lambda_i}.$$

is valid. On the other hand, it is clear that

$$\|R_\lambda(L_W)\| \ge \frac{\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{\mathfrak{L}}}{\|f^*(\lambda;\cdot)\|_{\mathfrak{L}}}, \quad f \ne 0.$$

Consequently, we have

$$\|R_\lambda(L_W)\| \geq \frac{1}{2\lambda_i} \quad \text{for } \lambda \in \mathbb{C}, \quad \lambda_i = \operatorname{Im}\lambda > 0.$$

The last relation implies the validity of assertion. □

Example 3.4. By the previous theorem, the spectrum of the boundary-value problem

$$\begin{gathered} i\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad |t| > 1,\ x \in [0,1], \\ u(1,x) = e^{i\varphi}u(-1,x), \quad \varphi \in [0,2\pi), \\ u'_x(t,0) = u'_x(t,1) = 0, \quad |t| > 1, \end{gathered}$$

in the space $L_2((-\infty,-1)\times(0,1)) \oplus L_2((1,+\infty)\times(0,1))$ is continuous and coincides with $\mathbb{R}$. This corresponds to the case when $H_1 = H_2 = \mathbb{C}$ and $H_3 = \{0\}$.

References

[1] E. Bairamov, R. Öztürk Mert, Z. I. Ismailov; *Selfadjoint Extension of Singular Differential Operator*, J.Math.Chemistry, 50, p.1100-1110, 2012.

[2] N. Dunford, J. T. Schwartz; *Linear Operators I; II, Second ed.*, Interscience, New York, 1958; 1963.

[3] W. N. Everitt, A. Zettl; *Differential Operators Generated by a Countable Number of Quasi-Differential Expressions on the Real Line*, Proc. London Math. Soc., 64, p.524-544, 1992.

[4] V. I. Gorbachuk, M. L. Gorbachuk; *Boundary Value Problems for a First Order Differential Operator with Operator Coefficients and Expansion in the Eigenfunctions of that Equation*, Dokl. Akad. Nauk SSSR, 208, p.1268-1271, 1973.

[5] V. I. Gorbachuk, M. L. Gorbachuk; *Boundary value problems for operator-differential equations*, First ed., Kluwer Academic Publisher, Dordrecht, 1991.

[6] Z. I. Ismailov, R. Öztürk Mert; *Selfadjoint Extension of Singular Multipoint Differential Operator of First Order*, Electr. J. Differential Equat., Vol. 2013, No. 129, p.1-11, 2013.

[7] M. A. Naimark; *Linear Differential Operators*, Ungar, New York, 1968.

[8] F. S. Rofe-Beketov, A. M. Kholkin; *Spectral Analysis of Differential Operators*, World Scientific Monograph Series in Mathematics, v.7, 2005.

[9] J. von Neumann; *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann., 102, p.49-131, 1929-1930.

[10] A. Zettl; *Sturm-Liouville Theory*, First ed., Amer. Math. Soc., Math. Survey and Monographs vol. 121, USA, 2005.

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