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EXISTENCE AND UNIQUENESS OF FIXED POINTS FOR MIXED MONOTONE OPERATORS WITH PERTURBATIONS

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Abstract. In this article, we study a class of mixed monotone operators with perturbations. Using a monotone iterative technique and the properties of cones, we show the existence and uniqueness for fixed points for such operators. As applications, we prove the existence and uniqueness of positive solutions for nonlinear integral equations of second-order on time scales. In particular, we do not assume the existence of upper-lower solutions or compactness or continuity conditions.

1. Introduction

Mixed monotone operators were introduced by Guo and Lakshmikantham in [12]. Their study has wide applications in the applied sciences such as engineering, biological chemistry technology, nuclear physics and in mathematics (see [13, 14, 33] and references therein). Various existence (and uniqueness) theorems of fixed points for mixed monotone operators have been discussed extensively, see for example [5, 8, 10, 16, 23, 28, 29, 34, 35]. Bhaskar and Lakshmikantham [5] established some coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces and discussed the existence and uniqueness of a solution for a periodic boundary value problem. Instead of using a direct proof as in [5], Drici, McRae and Devi [10] employed the notion of a reflection operator, and investigated fixed point theorems for mixed monotone operators by weakening the requirements in the contractive assumption and strengthening the metric space utilized with a partial order. These theorems are generalizations of the results of [5]. Moreover, in [16], Harjani, López and Sadarangani generalized the main results of [5] using the altering distance functions.

On the other hand, in recent years, there is much attention paid to various existence and uniqueness theorems of fixed points for monotone operators with perturbation. We would like to mention the results of Li [19], Li, Liang and Xiao [20, 21], Liu, Zhang and Wang [24], and Zhai and Anderson [30]. Li [19] proved the existence, uniqueness and iteration of the positive fixed points for operator $A = B + C$, where B is a positive linear operator with the spectral radius $r(B) < 1$, and C is a φ-concave increasing operator. Li, Liang and Xiao [20] obtained the

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existence and uniqueness of positive fixed points for operator $C = A + B$, where A is a decreasing operator and B is sublinear. Furthermore, Li, Liang and Xiao [21] used partial ordering methods, cone theory and iterative technique to investigate the existence and uniqueness of positive solutions of operator equation $A(x, x)+Bx = x$ in a real ordered Banach space E, where A is a mixed monotone operator with convexity and concavity, and B is affine. Liu, Zhang and Wang [24] discussed the existence and uniqueness of positive solutions of operator equation $A(x, x)+Bx = x$ on ordered Banach spaces, where A is a mixed monotone operator, and B is a sublinear operator. Without any compactness and continuity of the operators, some new fixed point theorems were obtained. Very recently, Zhai and Anderson [30] considered the existence and uniqueness of positive solutions to the following operator equation on ordered Banach spaces

$$Ax + Bx + Cx = x,$$

where A is an increasing α-concave operator, B is an increasing sub-homogeneous operator and C is a homogeneous operator.

However, we note that the upper-lower solutions conditions play a fundamental role in the main results of [5, 10, 16, 34, 19, 20, 21, 24, 25], as we know, which are not easy to verify for some concrete nonlinear equations. Thus, how to remove these conditions is an important and interesting question, much effort has been devoted to this topic. In [31], without demanding the existence of upper and lower solutions conditions, Zhai and Cao proved the existence, uniqueness and monotone iterative techniques of fixed points for τ-φ-concave operators. Moreover, Zhai, Yang and Zhang [32] studied a class of nonlinear operator equations $x = Ax + x_0$ on ordered Banach spaces, where A is a monotone generalized concave operator. In particular, the authors did not suppose the existence of upper-lower solutions conditions.

In this article, E is a real Banach space with norm $\|\cdot\|$, P is a cone in E, θ is the zero element in E. A partially ordered relation in E is given by $x \le y$ if and only if $y - x \in P$. $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is nondecreasing in x and non-increasing in y; i.e., u_i, v_i $(i = 1, 2) \in P$, $u_1 \le u_2$, $v_1 \ge v_2$ implies $A(u_1, v_1) \le A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Recall that a cone P is said to be solid if the interior P° is nonempty and we denote $x \gg \theta$ if $x \in P^\circ$. P is said to be normal if there exists a positive constant N, such that $\theta \le x \le y \Longrightarrow \|x\| \le N\|y\|$, the smallest N is called the normal constant of P. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. If $P^\circ \ne \emptyset$ and $h \in P^\circ$, it is clear that $P_h = P^\circ$.

An operator $B : E \to E$ is called a sublinear operator if $B(sx) \le sBx$, for $x \in P$ and $s \ge 1$.

All the concepts discussed above can be found in [12, 13, 14]. For more results about mixed monotone operators and other related concepts, the reader is referred to [8, 10, 16, 25, 29, 34, 34] and the references therein.

In 2010, Zhao [35] introduced the the following h-concave-convex operator.

Definition 1.1. Let $A : P_h \times P_h \to P_h$ and $h \in P \setminus \{\theta\}$. If there exists an $\eta(u, v, t) > 0$ such that

$$A(tu, t^{-1}v) \geq t(1 + \eta(u, v, t))A(u, v), \quad \forall u, v \in P_h \text{ and } 0 < t < 1,$$

Then A is called an h-concave-convex operator.

Without assumptions on the coupled upper-lower solutions, Zhao [35] proved the following theorem.

Theorem 1.2. *Suppose P is a normal cone of E, $h \in P \setminus \{\theta\}$, $A : P_h \times P_h \to P_h$ is a mixed monotone and h-concave-convex operator. Assume that one of the following conditions is satisfied*

- (A1) *for any $t \in (0, 1)$, $\eta(u, v, t)$ is non-increasing with respect to $u \in P_h$, non-decreasing with respect to $v \in P_h$;*
- (A2) *for any $t \in (0, 1)$, $\eta(u, v, t)$ is non-decreasing with respect to $u \in P_h$, non-increasing with respect to $v \in P_h$, and there exist $x_0, y_0 \in P_h$, $x_0 \leq y_0$ such that $\limsup_{t\to 0^+} \eta(x_0, y_0, t) = +\infty$.*

Then A has exactly one fixed point x^* in P_h. Moreover, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots,$$

for any initial $x_0, y_0 \in P_h$, we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Recently, Zhai and Zhang [29] proved a new existence-uniqueness result of positive fixed points for mixed monotone operators. They obtained the following result.

Theorem 1.3. *Suppose P is a normal cone of E, and $A : P \times P \to P$ is a mixed monotone operator. Assume that the following conditions are satisfied*

- (B1) *there exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$;*
- (B2) *for any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that*

$$A(tu, t^{-1}v) \geq \varphi(t)A(u, v).$$

Then operator A has a unique fixed point x^* in P_h. Moreover, for any initial x_0, $y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

In this article, we use the partial ordering theory and monotone iterative technique to obtain the existence and uniqueness of solutions of the following operator equation

$$A(x, x) + Bx = x, \quad x \in E, \tag{1.1}$$

where A is a mixed monotone operator, B is sublinear, and E is a real ordered Banach space, we do not require the operator discussed in this paper to have upper-lower solutions. In addition, we show some applications to nonlinear integral equation and second-order boundary-value problem on time scales.

2. Abstract theorems

Our main results are the following theorems.

Theorem 2.1. *Let P be a normal cone in E, and $A : P\times P \to P$ a mixed monotone operator. Let $B : E \to E$ be sublinear. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t), \varphi(t, x, y)$ on an interval (a, b) such that*

- (H1) *$\tau : (a, b) \to (0, 1)$ is a surjection;*
- (H2) *$\varphi(t, x, y) > \tau(t)$ for all $t \in (a, b)$, $x, y \in P$;*
- (H3) *$A\big(\tau(t)x, \frac{1}{\tau(t)}y\big) \geq \varphi(t, x, y)A(x, y)$ for all $t \in (a, b)$, $x, y \in P$; item[(H4)] $(I - B)^{-1} : E \to E$ exists and is an increasing operator.*

For any $t \in (a, b)$, $\varphi(t, x, y)$ is nondecreasing in x for fixed y and non-increasing in y for fixed x. In addition, suppose that there exist $h \in P \setminus \{\theta\}$ and $t_0 \in (a, b)$ such that

$$\frac{\tau(t_0)}{\varphi(t_0, h, h)}h \leq (I - B)^{-1}A(h, h) \leq \frac{1}{\tau(t_0)}h. \tag{2.1}$$

Then

- (i) *there are u_0, $v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$, $u_0 \leq (I - B)^{-1}A(u_0, v_0) \leq (I - B)^{-1}A(v_0, u_0) \leq v_0$;*
- (ii) *equation (1.1) has a unique solution x^* in $[u_0, v_0]$;*
- (iii) *for any initial x_0, $y_0 \in P_h$, constructing successively the sequences*

$$x_n = (I - B)^{-1}A(x_{n-1}, y_{n-1}), \quad y_n = (I - B)^{-1}A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Theorem 2.2. *Let P be a normal cone in E, and $A : P\times P \to P$ a mixed monotone operator. Let $B : E \to E$ be sublinear. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t), \varphi(t, x, y)$ on interval (a, b) such that the properties* (H1)–(H4) *in Theorem 2.1 are satisfied. Furthermore, for any $t \in (a, b)$, $\varphi(t, x, y)$ is non-increasing in x for fixed y, and nondecreasing in y for fixed x. In addition, suppose that there exist $h \in P \setminus \{\theta\}$ and $t_0 \in (a, b)$ such that*

$$\tau(t_0)h \leq (I - B)^{-1}A(h, h) \leq \frac{\varphi\big(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h\big)}{\tau(t_0)}h. \tag{2.2}$$

Then the conclusions (i), (ii), (iii) in Theorem 2.1 hold.

Proof of Theorem 2.1. For convenience, we denote $C = (I - B)^{-1}A$. By the fact that operator B is sublinear, we have $B\theta \geq \theta$, which together with (H4) imply

$$\theta \leq (I - B)^{-1}\theta \leq (I - B)^{-1}x, \quad x \in P.$$

Consequently, $(I-B)^{-1}$ is a positive operator. Hence, we have that $C : P\times P \to P$. According to (H4), we know that C is mixed monotone.

Since B is sublinear, we know that for any $x \in P$ and $\beta \in (0, 1)$, we obtain

$$(I - B)(\beta x) \leq \beta(I - B)x.$$

Thus

$$(I - B)(\beta(I - B)^{-1}x) \leq \beta(I - B)(I - B)^{-1}x = \beta x;$$

i.e., $(I - B)(\beta(I - B)^{-1}x) \leq \beta x$. Therefore, we have

$$\beta(I - B)^{-1}x \leq (I - B)^{-1}(\beta x). \tag{2.3}$$

For any $t \in (a,b)$, it follows from (H1)-(H4) and (2.3) that

$$\begin{aligned} C\big(\tau(t)x, \frac{1}{\tau(t)}y\big) &= (I-B)^{-1}A\big(\tau(t)x, \frac{1}{\tau(t)}y\big) \\ &\geq (I-B)^{-1}\varphi(t,x,y)A(x,y) \\ &\geq \varphi(t,x,y)(I-B)^{-1}A(x,y) \\ &= \varphi(t,x,y)C(x,y). \end{aligned} \tag{2.4}$$

Since $\tau(t_0) < \varphi(t_0,h,h)$, we can take a positive integer k such that

$$\big(\frac{\varphi(t_0,h,h)}{\tau(t_0)}\big)^k \geq \frac{1}{\tau(t_0)}. \tag{2.5}$$

Let $u_0 = [\tau(t_0)]^k h,\ v_0 = \frac{1}{[\tau(t_0)]^k}h$, and construct successively the sequences

$$u_n = C(u_{n-1}, v_{n-1}), \quad v_n = C(v_{n-1}, u_{n-1}), \quad n = 1,2,\dots.$$

It is clear that $u_0, v_0 \in P_h$ and $u_0 < v_0,\ u_1 = C(u_0,v_0) \leq C(v_0,u_0) = v_1$. In general, we obtain $u_n \leq v_n,\ n = 1,2,\dots$. Note that $\varphi(t,x,y) > \tau(t)$ for all $t \in (a,b)$, $x,y \in P$. Combining (2.1) with (2.4), we have

$$\begin{aligned} u_1 &= C(u_0,v_0) = C\Big([\tau(t_0)]^k h, \frac{h}{[\tau(t_0)]^k}\Big) \\ &= C\Big(\tau(t_0)[\tau(t_0)]^{k-1}h, \frac{1}{\tau(t_0)}\frac{h}{[\tau(t_0)]^{k-1}}\Big) \\ &\geq \varphi\Big(t_0, [\tau(t_0)]^{k-1}h, \frac{h}{[\tau(t_0)]^{k-1}}\Big)C\Big([\tau(t_0)]^{k-1}h, \frac{h}{[\tau(t_0)]^{k-1}}\Big) \\ &= \varphi\Big(t_0, [\tau(t_0)]^{k-1}h, \frac{h}{[\tau(t_0)]^{k-1}}\Big)C\Big(\tau(t_0)[\tau(t_0)]^{k-2}h, \frac{1}{\tau(t_0)}\frac{h}{[\tau(t_0)]^{k-2}}\Big) \\ &\geq \varphi\Big(t_0, [\tau(t_0)]^{k-1}h, \frac{h}{[\tau(t_0)]^{k-1}}\Big)\varphi\Big(t_0, [\tau(t_0)]^{k-2}h, \frac{h}{[\tau(t_0)]^{k-2}}\Big) \\ &\quad \times C\Big([\tau(t_0)]^{k-2}h, \frac{h}{[\tau(t_0)]^{k-2}}\Big) \\ &\geq \dots \geq \varphi\Big(t_0, [\tau(t_0)]^{k-1}h, \frac{h}{[\tau(t_0)]^{k-1}}\Big)\varphi\Big(t_0, [\tau(t_0)]^{k-2}h, \frac{h}{[\tau(t_0)]^{k-2}}\Big) \\ &\quad \times \dots \varphi(t_0,h,h)C(h,h) \\ &\geq [\tau(t_0)]^{k-1}\varphi(t_0,h,h)C(h,h) \\ &\geq [\tau(t_0)]^k h = u_0. \end{aligned}$$

From (2.4), we have

$$C\big(\frac{x}{\tau(t)}, \tau(t)y\big) \leq \frac{1}{\varphi\big(t, \frac{x}{\tau(t)}, \tau(t)y\big)}C(x,y), \quad \forall t \in (a,b), \quad x,y \in P. \tag{2.6}$$

Note that $\varphi(t,x,y)$ is nondecreasing in x and nonincreasing in y, it follows from (2.1), (2.5) and (2.6) that

$$\begin{aligned} v_1 &= C(v_0,u_0) = C\Big(\frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h\Big) \\ &= C\Big(\frac{1}{\tau(t_0)}\frac{h}{[\tau(t_0)]^{k-1}}, \tau(t_0)[\tau(t_0)]^{k-1}h\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h\big)} C\Big(\frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1}h\Big)\\
&= \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h\big)} C\Big(\frac{1}{\tau(t_0)}\frac{h}{[\tau(t_0)]^{k-2}}, \tau(t_0)[\tau(t_0)]^{k-2}h\Big)\\
&\le \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h\big)} \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h\big)}\\
&\quad \times C\Big(\frac{h}{[\tau(t_0)]^{k-2}}, [\tau(t_0)]^{k-2}h\Big) \le \dots\\
&\le \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h\big)} \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h\big)}\\
&\quad \times \dots \frac{1}{\varphi\big(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h\big)} C(h,h)\\
&< \frac{1}{[\varphi(t_0,h,h)]^k}\frac{h}{\tau(t_0)}\\
&\le \frac{1}{[\tau(t_0)]^k} h = v_0.
\end{aligned}
$$

Thus, we obtain

$$u_0 \le u_1 \le v_1 \le v_0. \tag{2.7}$$

By induction, it is easy to obtain that

$$u_0 \le u_1 \le \dots \le u_n \le \dots \le v_n \le \dots \le v_1 \le v_0.$$

Take any $r \in (0, [\tau(t_0)]^{2k})$, then $r \in (0,1)$ and $u_0 \ge r v_0$. So we can know that

$$u_n \ge u_0 \ge r v_0 \ge r v_n, \quad n = 1, 2, \dots.$$

Let

$$r_n = \sup\{r > 0 | u_n \ge r v_n\}, \quad n = 1, 2, \dots.$$

Thus, we have $u_n \ge r_n v_n$, $n = 1, 2, \dots$, and then

$$u_{n+1} \ge u_n \ge r_n v_n \ge r_n v_{n+1}, \quad n = 1, 2, \dots.$$

Therefore, $r_{n+1} \ge r_n$; i.e.,

$$0 < r_0 \le r_1 \le \dots \le r_n \le \dots \le 1.$$

Set $r^* = \lim_{n\to\infty} r_n$, we will show that $r^* = 1$. In fact, if $0 < r^* < 1$, by (H1), there exists $t_1 \in (a,b)$ such that $\tau(t_1) = r^*$. Consider the following two cases:

Case i: There exists an integer N such that $r_N = r^*$. In this case, we have $r_n = r^*$ and $u_n \ge r^* v_n$ for all $n \ge N$ hold. Hence

$$
\begin{aligned}
u_{n+1} = C(u_n, v_n) &\ge C\Big(r^* v_n, \frac{1}{r^*} u_n\Big)\\
&= C\Big(\tau(t_1) v_n, \frac{1}{\tau(t_1)} u_n\Big)\\
&\ge \varphi(t_1, v_n, u_n) C(v_n, u_n)\\
&\ge \varphi(t_1, u_0, v_0) C(v_n, u_n) = \varphi(t_1, u_0, v_0) v_{n+1}, \quad n \ge N.
\end{aligned}
$$

By the definition of r_n, we have

$$r_{n+1} = r^* \ge \varphi(t_1, u_0, v_0) > \tau(t_1) = r^*, \quad n \ge N,$$

which is a contradiction.

Case ii: For all integers n, $r_n < r^*$. Then we obtain $0 < \frac{r_n}{r^*} < 1$. By (H1), there exist $z_n \in (a,b)$ such that $\tau(z_n) = \frac{r_n}{r^*}$. Hence

$$\begin{aligned} u_{n+1} &= C(u_n, v_n) \geq C\Big(r_n v_n, \frac{1}{r_n} u_n\Big) \\ &= C\Big(\frac{r_n}{r^*} r^* v_n, \frac{1}{\frac{r_n}{r^*} r^*} u_n\Big) = C\Big(\tau(z_n) r^* v_n, \frac{1}{\tau(z_n) r^*} u_n\Big) \\ &\geq \varphi\Big(z_n, r^* v_n, \frac{1}{r^*} u_n\Big) C\Big(r^* v_n, \frac{1}{r^*} u_n\Big) \\ &\geq \varphi\Big(z_n, r^* u_0, \frac{1}{r^*} v_0\Big) C\Big(\tau(t_1) v_n, \frac{1}{\tau(t_1)} u_n\Big) \\ &\geq \varphi\Big(z_n, r^* u_0, \frac{1}{r^*} v_0\Big) \varphi(t_1, v_n, u_n) C(v_n, u_n) \\ &\geq \varphi\Big(z_n, r^* u_0, \frac{1}{r^*} v_0\Big) \varphi(t_1, u_0, v_0) v_{n+1}. \end{aligned}$$

By the definition of r_n, we have

$$r_{n+1} \geq \varphi\big(z_n, r^* u_0, \frac{1}{r^*} v_0\big) \varphi(t_1, u_0, v_0) > \tau(z_n) \varphi(t_1, u_0, v_0) = \frac{r_n}{r^*} \varphi(t_1, u_0, v_0).$$

Let $n \to \infty$, we have

$$r^* \geq \varphi(t_1, u_0, v_0) > \tau(t_1) = r^*,$$

which is also a contradiction. Thus, $\lim_{n\to\infty} r_n = 1$.

Furthermore, as in the proof of [20, Theorem 2.1], there exists $x^* \in [u_0, v_0]$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = x^*$, and x^* is the fixed point of operator C.

In the following, we prove that x^* is the unique fixed point of C in P_h. In fact, suppose that $x_* \in P_h$ is another fixed point of operator C. Let

$$c_1 = \sup\big\{0 < c \leq 1 | c x_* \leq x^* \leq \frac{1}{c} x_*\big\}.$$

Clearly, $0 < c_1 \leq 1$ and $c_1 x_* \leq x^* \leq \frac{1}{c_1} x_*$. If $0 < c_1 < 1$, according to (H1), there exists $t_2 \in (a,b)$ such that $\tau(t_2) = c_1$. Then

$$\begin{aligned} x^* &= C(x^*, x^*) \geq C\Big(c_1 x_*, \frac{1}{c_1} x_*\Big) \\ &= C\Big(\tau(t_2) x_*, \frac{1}{\tau(t_2)} x_*\Big) \\ &\geq \varphi(t_2, x_*, x_*) C(x_*, x_*) = \varphi(t_2, x_*, x_*) x_*, \end{aligned}$$

and

$$\begin{aligned} x^* &= C(x^*, x^*) \leq C\Big(\frac{1}{c_1} x_*, c_1 x_*\Big) \\ &= C\Big(\frac{1}{\tau(t_2)} x_*, \tau(t_2) x_*\Big) \\ &\leq \frac{1}{\varphi\big(t_2, \frac{x_*}{\tau(t_2)}, \tau(t_2) x_*\big)} C(x_*, x_*) \\ &= \frac{1}{\varphi\big(t_2, \frac{x_*}{\tau(t_2)}, \tau(t_2) x_*\big)} x_*. \end{aligned}$$

Since
$$\varphi(t_2,x_*,x_*)\le \varphi\Big(t_2,\frac{x_*}{\tau(t_2)},\tau(t_2)x_*\Big),$$
we have
$$\varphi(t_2,x_*,x_*)x_*\le x^*\le \frac{1}{\varphi(t_2,x_*,x_*)}x_*.$$
Hence, $c_1\ge \varphi(t_2,x_*,x_*)>\tau(t_2)=c_1$, which is a contradiction. Thus we have $c_1=1$; i.e., $x_*=x^*$. Therefore, C has a unique fixed point x^* in P_h. Note that $[u_0,v_0]\subset P_h$, so we know that x^* is the unique fixed point of C in $[u_0,v_0]$. For any initial $x_0,\ y_0\in P_h$, we can choose a small number $\overline{e}\in(0,1)$ such that
$$\overline{e}h\le x_0\le \frac{1}{\overline{e}}h,\quad \overline{e}h\le y_0\le \frac{1}{\overline{e}}h.$$
From (H1), there is $t_3\in(a,b)$ such that $\tau(t_3)=\overline{e}$, thus
$$\tau(t_3)h\le x_0\le \frac{1}{\tau(t_3)}h,\qquad \tau(t_3)h\le y_0\le \frac{1}{\tau(t_3)}h.$$
We can choose a sufficiently large positive integer q such that
$$\Big(\frac{\varphi(t_3,h,h)}{\tau(t_3)}\Big)^q\ge \frac{1}{\tau(t_3)}.$$
Take $\hat u_0=[\tau(t_3)]^q h$, $\hat v_0=\frac{1}{[\tau(t_3)]^q}h$. We can find that
$$\hat u_0\le x_0\le \hat v_0,\quad \hat u_0\le y_0\le \hat v_0.$$
Constructing successively the sequences
$$\begin{gathered}
x_n=C(x_{n-1},y_{n-1}),\quad y_n=C(y_{n-1},x_{n-1}),\quad n=1,2,\dots,\\
\hat u_n=C(\hat u_{n-1},\hat v_{n-1}),\quad \hat v_n=C(\hat v_{n-1},\hat u_{n-1}),\quad n=1,2,\dots.
\end{gathered}$$
By using the mixed monotone properties of operator C, we have
$$\hat u_n\le x_n\le \hat v_n,\quad \hat u_n\le y_n\le \hat v_n,\quad n=1,2,\dots.$$
Similarly to the above proof, we can know that there exists $y^*\in P_h$ such that
$$C(y^*,y^*)=y^*,\quad \lim_{n\to\infty}\hat u_n=\lim_{n\to\infty}\hat v_n=y^*.$$
By the uniqueness of fixed points of operator C in P_h, we have $y^*=x^*$. Taking into account that P is normal, we deduce that $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=x^*$. This completes the proof. $\square$

Proof of Theorem 2.2. As in the proof of Theorem 2.1, it suffices to verify that (2.7) holds. For any $t\in(a,b)$, note that $\varphi(t,x,y)$ is non-increasing in x and nondecreasing in y, it follows from (2.2), (2.4) and (2.5) that
$$\begin{aligned}
&u_1\\
&=C(u_0,v_0)=C\Big([\tau(t_0)]^k h,\frac{h}{[\tau(t_0)]^k}\Big)\\
&\ge \varphi\Big(t_0,[\tau(t_0)]^{k-1}h,\frac{h}{[\tau(t_0)]^{k-1}}\Big)\varphi\Big(t_0,[\tau(t_0)]^{k-2}h,\frac{h}{[\tau(t_0)]^{k-2}}\Big)\dots\varphi(t_0,h,h)C(h,h)\\
&\ge [\varphi(t_0,h,h)]^k\tau(t_0)h\\
&\ge [\tau(t_0)]^k h=u_0.
\end{aligned}$$

Note that $\varphi(t,x,y) > \tau(t)$ for all $t \in (a,b)$, $x, y \in P$. Combining (2.2) with (2.6), we obtain

$$
\begin{aligned}
v_1 = C(v_0, u_0) &= C\Big(\frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h\Big) \\
&\le \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h\big)} \frac{1}{\varphi\big(t_0, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h\big)} \cdots \\
&\quad \times \frac{1}{\varphi\big(t_0, \frac{h}{\tau(t_0)}, \tau(t_0) h\big)} C(h,h) \\
&< \frac{1}{[\tau(t_0)]^{k-1}} \frac{1}{\varphi\big(t_0, \frac{h}{\tau(t_0)}, \tau(t_0) h\big)} C(h,h) \\
&\le \frac{1}{[\tau(t_0)]^k} h = v_0.
\end{aligned}
$$

Thus, we know that (2.7) holds. The rest proof is similar to that of Theorem 2.1, we omit it here. □

Remark 2.3. Compared with Theorem 1.2, the main contribution in this paper to weaken the restriction on operator A; i.e., the condition $A : P_h \times P_h \to P_h$ in Theorem 1.2 is replaced by (2.1) in Theorem 2.1 and (2.2) in Theorem 2.2. We also remove the condition "there exist $x_0, y_0 \in P_h$, $x_0 \le y_0$ such that $\limsup_{t\to 0^+} \eta(x_0, y_0, t) = +\infty$" in Theorem 1.2. In Theorems 1.3, and 2.2 we consider more general operators; i.e., concave and convex mixed monotone operators with perturbations.

By the proof of [16, Corollary 2.5], we can obtain the following corollary.

Corollary 2.4. *Let P be a normal cone in E, and $A : P \times P \to P$ a mixed monotone operator. Let B be a linear operator in E, such that*

- (C1) *$\|B\| < 1$, there exists some number $b \ge 0$ such that $B + bI \ge 0$;*
- (C2) *A satisfies the conditions of Theorem 2.1 or Theorem 2.2.*

Then equation (1.1) *has a unique solution x^* in $[u_0, v_0]$.*

3. An application to integral equations

We consider nonlinear integral equation

$$
\int_{a_1}^{a_2} G(t,s)[f(x(s)) + g(x(s))]ds = [1 + G_1(t)]x(t) - G_2(t)x(t+\tau), \quad t \in \mathbb{R}, \tag{3.1}
$$

where a_1, a_2, τ are constants.

Let $E = C(\mathbb{R})$ denote the real Banach space of all bounded and continuous functions on $\mathbb{R}$ with the supremum norm. Define a cone

$$
P = \{x \in E : x(t) \ge 0, \forall t \in \mathbb{R}\}.
$$

Theorem 3.1. *Assume that*

- (D1) *Let $G_1, G_2 \in E$, $G(t,s)$ be uniformly continuous on $\mathbb{R} \times [a_1, a_2]$, $f(x)$ be increasing, $g(x)$ be decreasing and $f(x) \ge 0$, $g(x) \ge 0$ for $x \ge 0$;*
  item[(D2)] there exist $g_1, g_2 \in [0, +\infty)$ such that $0 \le G_1(t) \le g_1$, $0 \le G_2(t) \le g_2$, and $g_1 + g_2 < 1$, where $t \in \mathbb{R}$;

(D3) *there exist* $\tau(t), \varphi(t, x_1, x_2)$ *on an interval* $t \in \mathbb{R}$ *such that* $\tau : \mathbb{R} \to (0, 1)$ *is a surjection and* $\varphi(t, x_1, x_2) > \tau(t)$ *for all* $t \in \mathbb{R}$, $x_1, x_2 \in P$ *which satisfy*

$$\begin{aligned}&\int_{a_1}^{a_2} G(t,s)\big[f(\tau(\mu)x_1(s)) + g\big(\frac{1}{\tau(\mu)}x_2(s)\big)\big]ds\\ &\geq \varphi(\mu, x_1, x_2)\int_{a_1}^{a_2} G(t,s)[f(x_1(s)) + g(x_2(s))]ds, \quad \forall \mu \in \mathbb{R},\ x_1, x_2 \in P;\end{aligned}$$

(D4) *for fixed* $t \in \mathbb{R}$, $\varphi(t, x_1, x_2)$ *is non-increasing in* x_1 *and non-decreasing in* x_2*;*

(D5) *there exist* $e \in P \setminus \{0\}$ *and* $t_0 \in \mathbb{R}$ *such that*

$$\begin{aligned}\tau(t_0)e(t) &\leq \int_{a_1}^{a_2} G(t,s)[f(e(s)) + g(e(s))]ds + G_2(t)e(t+\tau) - G_1(t)e(t)\\ &\leq \frac{\varphi\big(t_0, \frac{e(t)}{\tau(t_0)}, \tau(t_0)e(t)\big)}{\tau(t_0)}e(t), \quad \forall t \in \mathbb{R}.\end{aligned}$$

Then (3.1) *has a unique positive solution* x^* *in* P_e*.*

Proof. We rewrite (3.1) as

$$x(t) = \int_{a_1}^{a_2} G(t,s)[f(x(s)) + g(x(s))]ds + G_2(t)x(t+\tau) - G_1(t)x(t), \quad t \in \mathbb{R}.$$

Define

$$\begin{gathered}A(x_1, x_2)(t) = \int_{a_1}^{a_2} G(t,s)[f(x_1(s)) + g(x_2(s))]ds, \quad t \in \mathbb{R},\\ Bx(t) = G_2(t)x(t+\tau) - G_1(t)x(t), \quad t \in \mathbb{R}.\end{gathered}$$

According to (D1), we have that $A : P \times P \to P$ is a mixed monotone operator.

For the linear operator B, we have $\|B\| \leq g_1 + g_2 < 1$, and $B + bI \geq 0$ for $b \geq g_1$.

On the other hand, for any $\mu \in \mathbb{R}$ and $x_1,\ x_2 \in P$, according to (D3), we obtain

$$\begin{aligned}A\big(\tau(\mu)x_1, \frac{1}{\tau(\mu)}x_2\big) &= \int_{a_1}^{a_2} G(t,s)\big[f(\tau(\mu)x_1(s)) + g\big(\frac{1}{\tau(\mu)}x_2(s)\big)\big]ds\\ &\geq \varphi(\mu, x_1, x_2)\int_{a_1}^{a_2} G(t,s)[f(x_1(s)) + g(x_2(s))]ds\\ &= \varphi(\mu, x_1, x_2)A(x_1, x_2).\end{aligned}$$

That is,

$$A\Big(\tau(\mu)x_1, \frac{1}{\tau(\mu)}x_2\Big) \geq \varphi(\mu, x_1, x_2)A(x_1, x_2), \quad \text{for } \mu \in \mathbb{R},\ x_1, x_2 \in P.$$

In addition, from (D5), for any $t \in \mathbb{R}$, we have

$$\begin{aligned}\tau(t_0)e(t) &\leq A(e, e) + B(e)\\ &= \int_{a_1}^{a_2} G(t,s)[f(e(s)) + g(e(s))]ds + G_2(t)e(t+\tau) - G_1(t)e(t)\\ &\leq \frac{\varphi\big(t_0, \frac{e(t)}{\tau(t_0)}, \tau(t_0)e(t)\big)}{\tau(t_0)}e(t), \quad \forall t \in \mathbb{R}.\end{aligned}$$

Consequently, all conditions in Corollary 2.4 are satisfied; therefore, (3.1) has a unique positive solution in P_e. ∎

4. An application to a boundary-value problem on time scales

In this section, we will apply Theorem 2.2 to the following second-order boundary value problem with Sturm-Liouville boundary conditions on time scales

$$(py^{\Delta})^{\nabla}(t) + [f(t, y(t)) + g(t, y(t))] = 0, \quad t \in (a, b]_{\mathbb{T}}, \tag{4.1}$$

$$\alpha y(a) - \beta(py^{\Delta})(a) = 0, \quad \gamma y^{\sigma}(b) + \delta(py^{\Delta})(b) = 0, \tag{4.2}$$

where

$$p : [a, \sigma(b)]_{\mathbb{T}} \to (0, +\infty), \quad p \in C[a, \sigma(b)]_{\mathbb{T}}, \tag{4.3}$$

$$\beta, \delta \in (0, +\infty), \quad \alpha, \gamma \in [0, +\infty), \quad \beta\gamma + \alpha\delta + \alpha\gamma \int_a^{\sigma(b)} \frac{\Delta\tau}{p(\tau)} > 0. \tag{4.4}$$

Some definitions and theorems on time scales can be found in [4, 6, 7, 17] which are excellent references for the calculus on time scales. The study of dynamic equations on time scales goes back to its founder Hilger [17], and is a new area of still fairly theoretical exploration in mathematics. In recent years, there has been a great deal of research work on the existence of positive solutions of second-order boundary value problems on time scales, we refer the reader to [1, 2, 3, 4, 6, 7, 9, 11, 15, 18, 22, 26, 27] for some recent results. Such investigations can provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. We would like to mention some results of Anderson and Wong [2], Jankowski [18] and Wang, Wu and Wu [27], which motivated us to consider problem (4.1) and (4.2).

Anderson and Wong [2] studied the second-order time scale semipositone boundary value problem

$$(py^{\Delta})^{\nabla}(t) + \lambda f(t, u(t)) = 0, \quad t \in (a, b]_{\mathbb{T}}$$

with Sturm-Liouville boundary conditions (4.2).

The methods of lower and upper solutions have been applied extensively in proving the existence results for dynamic equations on time scales. Jankowski [18] investigated second order dynamic equations with deviating arguments on time scales of the form

$$\begin{gathered} -x^{\Delta\Delta}(t) = f(t, x(t), x(\alpha(t))) \equiv (Fx)(t), \quad t \in [0, T]_{\mathbb{T}}, \\ x(0) = k_1 \in \mathbb{R}, \quad x(T) = k_2 \in \mathbb{R}. \end{gathered}$$

They formulated sufficient conditions, under which such problems have a minimal and a maximal solution in a corresponding region bounded by upper-lower solutions.

Wang, Wu and Wu [27] considered a method of generalized quasilinearization, with even-order k ($k \geq 2$) convergence, for the problem

$$\begin{gathered} -(p(t)x^{\Delta})^{\nabla} + q(t)x^{\sigma} = f(t, x^{\sigma}) + g(t, x^{\sigma}), \quad t \in [a, b]_{\mathbb{T}}, \\ \tau_1 x(\rho(a)) - \tau_2 x^{\Delta}(\rho(a)) = 0, \quad x(\sigma(b)) - \tau_3 x(\eta) = 0. \end{gathered}$$

The main contribution in [27] is to relaxed the monotone conditions on $f^{(i)}(t, x)$ and $g^{(i)}(t, x)$ for $1 < i < k$, including a more general concept of upper and lower solution in mathematical biology, so that the high-order convergence of the iterations is ensured for a larger class of nonlinear functions on time scales.

We would also like to mention the results of Sang [26]. There, we considered the existence of positive solutions and established the corresponding iterative schemes for the m-point boundary value problem on time scales

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \quad t \in [0, 1] \subset \mathbb{T}, \tag{4.5}$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad m \geq 3. \tag{4.6}$$

By considering the "heights" of the nonlinear term f on some bounded sets and applying monotone iterative techniques on a Banach space, we did not only obtain the existence of positive solutions for problem (4.5) and (4.6), but also established the iterative schemes for approximating the solutions. In essence, we combined the method of lower and upper solutions with the cone expansion and compression fixed point theorem of norm type.

We note that the upper and lower solutions conditions are both required in [18, 27], and the researchers mentioned above [18, 26, 27] only studied the existence of positive solutions. Therefore, it is natural to discuss the uniqueness and iteration of positive solutions to problem (4.1) and (4.2).

To obtain our main result, we will employ several lemmas. These lemmas are based on the linear equation

$$-(py^{\Delta})^{\nabla}(t) = u(t), \quad t \in (a, b]_{\mathbb{T}}, \tag{4.7}$$

with boundary conditions (4.2). Define the constant

$$d := \beta\gamma + \alpha\delta + \alpha\gamma \int_a^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}. \tag{4.8}$$

Lemma 4.1 ([2])**.** *Assume* (4.3) *and* (4.4)*. Then problem* (4.7)-(4.2) *has a unique solution y such that*

$$y(t) = \int_a^b G(t, s) u(s) \nabla s, \quad t \in [a, \sigma(b)]_{\mathbb{T}},$$

where the Green function is

$$G(t, s) = \frac{1}{d} \begin{cases} \left(\beta + \alpha \int_a^s \frac{\Delta\tau}{p(\tau)}\right)\left(\delta + \gamma \int_t^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right) & a \leq s \leq t \leq \sigma(b), \\ \left(\beta + \alpha \int_a^t \frac{\Delta\tau}{p(\tau)}\right)\left(\delta + \gamma \int_s^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right) & a \leq t \leq s \leq \sigma(b), \end{cases} \tag{4.9}$$

for all $t, s \in [a, \sigma(b)]_{\mathbb{T}}$, where d is given by (4.8)*.*

Lemma 4.2 ([2])**.** *Assume* (4.3) *and* (4.4)*. Then the Green function* (4.9) *satisfies*

$$g(t) G(s, s) \leq G(t, s) \leq G(s, s), \quad t, s \in [a, \sigma(b)]_{\mathbb{T}},$$

where

$$g(t) = \min_{t \in [a, \sigma(b)]_{\mathbb{T}}} \Big\{ \frac{\delta + \gamma \int_t^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}}{\delta + \gamma \int_a^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}}, \frac{\beta + \alpha \int_a^t \frac{\Delta\tau}{p(\tau)}}{\beta + \alpha \int_a^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}} \Big\} \in [0, 1]. \tag{4.10}$$

Let the Banach space $\mathbb{B} = C[a, \sigma(b)]_{\mathbb{T}}$ be equipped with norm

$$\|u\| = \sup_{t \in [a, \sigma(b)]_{\mathbb{T}}} |u(t)|.$$

In this space define a cone

$$K = \{u \in \mathbb{B} : \min_{t \in [a, \sigma(b)]_{\mathbb{T}}} u(t) \geq 0\}.$$

Our main result is the following theorem.

Theorem 4.3. *Assume that*

- (E1) $f, g : (a, b]_{\mathbb{T}} \times [0, +\infty) \to [0, +\infty)$ *are continuous;*
- (E2) *for fixed* t, $f(t, u)$ *is nondecreasing in* u, $g(t, u)$ *is non-increasing in* u*;*
- (E3) *there exist* $\tau(t), \varphi(t, y_1, y_2)$ *on the interval* $(a, b)_{\mathbb{T}}$ *such that* $\tau : (a, b)_{\mathbb{T}} \to (0, 1)$ *is a surjection and* $\varphi(t, y_1, y_2) > \tau(t)$ *for all* $t \in (a, b)_{\mathbb{T}}$, $y_1, y_2 \in K$ *which satisfy*

$$\begin{aligned}
&\int_a^b G(t,s)\big[f(s,\tau(\lambda)y_1(s)) + g\big(s, \frac{1}{\tau(\lambda)} y_2(s)\big)\big]\nabla s \\
&\geq \varphi(\lambda, y_1, y_2) \int_a^b G(t,s)[f(s, y_1(s)) + g(s, y_2(s))]\nabla s, \quad \forall \lambda \in (a,b)_{\mathbb{T}},\ y_1, y_2 \in K;
\end{aligned}$$

- (E4) *for any* $t \in (a, b)_{\mathbb{T}}$, $\varphi(t, y_1, y_2)$ *is non-increasing in* y_1 *for fixed* y_2 *and nondecreasing in* y_2 *for fixed* y_1*;*
- (E5) *there exist* $h \in K \setminus \{0\}$ *and* $t_0 \in (a, b)_{\mathbb{T}}$ *such that*

$$\tau(t_0)h(t) \leq \int_a^b G(t,s)[f(h(s)) + g(h(s))]\nabla s \leq \frac{\varphi\big(t_0, \frac{h(t)}{\tau(t_0)}, \tau(t_0)h(t)\big)}{\tau(t_0)} h(t),$$

  for all $t \in [a, \sigma(b)]_{\mathbb{T}}$.

Then (4.1)-(4.2) *has a unique positive solution* x^* *in* K_h*. Moreover, for any initial condition* $x_0, y_0 \in K_h$*, constructing successively the sequences*

$$\begin{aligned}
x_n(t) &= \int_a^b G(t,s)[f(s, x_{n-1}(s)) + g(s, y_{n-1}(s))]\nabla s, \quad t \in [a, \sigma(b)]_{\mathbb{T}},\ n = 1, 2, \dots, \\
y_n(t) &= \int_a^b G(t,s)[f(s, y_{n-1}(s)) + g(s, x_{n-1}(s))]\nabla s, \quad t \in [a, \sigma(b)]_{\mathbb{T}},\ n = 1, 2, \dots,
\end{aligned}$$

we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof. Define

$$F(y_1, y_2)(t) = \int_a^b G(t,s)[f(s, y_1(s)) + g(s, y_2(s))]\nabla s, \quad t \in [a, \sigma(b)]_{\mathbb{T}}.$$

By Lemma 4.3, we can know that problem (4.1)-(4.2) is equivalent to the fixed point equation

$$y(t) = F(y, y)(t), \quad t \in [a, \sigma(b)]_{\mathbb{T}}.$$

By (E2), it is easy to check that $F : K \times K \to K$ is mixed monotone.

For any $\lambda \in (a, b)_{\mathbb{T}}$ and $y_1, y_2 \in K$, from (E3) it follows that

$$\begin{aligned}
F\big(\tau(\lambda)y_1, \frac{1}{\tau(\lambda)} y_2\big) &= \int_a^b G(t,s)\big[f(s,\tau(\lambda)y_1(s)) + g\big(s, \frac{1}{\tau(\lambda)} y_2(s)\big)\big]\nabla s \\
&\geq \varphi(\lambda, y_1, y_2) \int_a^b G(t,s)[f(s, y_1(s)) + g(s, y_2(s))]\nabla s \\
&= \varphi(\lambda, y_1, y_2) F(y_1, y_2);
\end{aligned}$$

i.e.,

$$F\big(\tau(\lambda)y_1, \frac{1}{\tau(\lambda)} y_2\big) \geq \varphi(\lambda, y_1, y_2) F(y_1, y_2), \quad \text{for } \lambda \in (a,b)_{\mathbb{T}},\ y_1, y_2 \in K.$$

Moreover, from (E5), we obtain

$$\begin{aligned} \tau(t_0)h(t) \le F(h,h) &= \int_a^b G(t,s)[f(s,h(s)) + g(s,h(s))]\nabla s \\ &\le \frac{\varphi\big(t_0, \frac{h(t)}{\tau(t_0)}, \tau(t_0)h(t)\big)}{\tau(t_0)} h(t), \quad \forall t \in [a,\sigma(b)]_{\mathbb{T}}. \end{aligned}$$

Note that all the conditions in Theorem 2.2 hold, which implies the the conclusions of Theorem 4.3. □

We conclude this article with the following example.

Example 4.4. Let $\mathbb{T} = \{2^k\}_{k\in\mathbb{Z}} \cup \{0\}$, where $\mathbb{Z}$ denotes the set of integers. Consider the following problem on time scales $\mathbb{T}$:

$$(y^{\Delta})^{\nabla}(t) + [f(y(t)) + g(y(t))] = 0, \quad t \in (0,1]_{\mathbb{T}}, \tag{4.11}$$

$$y(0) - (y^{\Delta})(0) = 0, \quad y^{\sigma}(1) + (y^{\Delta})(1) = 0, \tag{4.12}$$

where $f(y) = 2 + y^{1/2}$, $g(y) = 1/(7+y)$. It is easy to check that $f, g : [0,+\infty) \to [0,+\infty)$ are continuous, and f is non-decreasing in y, g is non-increasing in y. For any $\lambda \in (0,1)$, $y_1, y_2 \in K$, we have

$$\begin{aligned} &\int_0^1 G(t,s)\Big[2 + (\lambda y_1(s))^{1/2} + \frac{1}{7 + \frac{1}{\lambda} y_2(s)}\Big]\nabla s \\ &\ge \lambda \int_0^1 G(t,s)\big[2\lambda^{-1} + \lambda^{-\frac12} y_1^{1/2}(s) + \frac{1}{7 + y_2(s)}\big]\nabla s \\ &\ge \lambda \frac{2\lambda^{-1} + \lambda^{-\frac12} y_1^{1/2}(s) + \frac{1}{7+y_2(s)}}{2 + y_1^{1/2}(s) + \frac{1}{7+y_2(s)}} \int_0^1 G(t,s)\big[2 + y_1^{1/2}(s) + \frac{1}{7+y_2(s)}\big]\nabla s. \end{aligned} \tag{4.13}$$

In (4.13), we note that

$$\lambda < \varphi(\lambda, y_1, y_2) = \lambda \frac{2\lambda^{-1} + \lambda^{-\frac12} y_1^{1/2} + \frac{1}{7+y_2}}{2 + y_1^{1/2} + \frac{1}{7+y_2}} < 1.$$

For any $\lambda \in (0,1)$, by means of some calculations, we can obtain that φ is non-increasing in y_1 for fixed y_2 and nondecreasing in y_2 for fixed y_1.

In the following, it suffices to verify that the condition (E5) of Theorem 4.3 is satisfied. Some direct calculations show that $g(t) = \frac{1}{1+\sigma(1)} = \frac{1}{3}$, and

$$\begin{aligned} \int_0^1 G(s,s)\nabla s &= 4 + \sum_{n=0}^{\infty} 2^{-1-2n} - \sum_{n=0}^{\infty} 2^{-3-3n} - \sum_{n=0}^{\infty} 2^{-2n} + \sum_{n=0}^{\infty} 2^{-1-3n} \\ &= \frac{79}{21} > 0. \end{aligned}$$

From Lemma 4.2 it follows that that

$$\frac{79}{63} \le \int_0^1 G(t,s)\nabla s \le \frac{79}{21}, \quad t \in [0,\sigma(1)]_{\mathbb{T}}.$$

Since $f(1) + g(1) = 25/8$, we choose $h = 1$, $t_0 = 2^{-9}$, it is easy to check that

$$0.01 < \frac{79}{63} \cdot \frac{25}{8} \le \int_0^1 G(t,s)[f(1) + g(1)]\nabla s$$

$$\le \frac{79}{21}\cdot\frac{25}{8} < \frac{2(2^{-9})^{-1} + (2^{-9})^{-1/2}(2^9)^{1/2} + \frac{1}{7+2^{-9}}}{2 + (2^{-9})^{-1/2} + \frac{1}{7+2^{-9}}}.$$

This implies that (E5) of Theorem 4.3 holds. The proof is complete.

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