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OSCILLATION AND NONOSCILLATION FOR SECOND-ORDER NONLINEAR NEUTRAL FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this article, we investigate the oscillation and nonoscillation of second order nonlinear neutral dynamic equations with retarded and advanced arguments by means of the theory of upper and lower solutions for related dynamic equations along with some additional estimates on positive solutions. We also apply the Kranoselskii's fixed point theorem to obtain nonoscillation results. Some interesting examples are given to illustrate the versatility of our results.

1. Introduction

Following Hilger's pioneering work [11], a rapidly expanding body of literature has sought to unify, extend and generalize ideas from continuous and discrete calculus to arbitrary time-scale calculus, where a time scale is simply an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ with the topology and ordering inherited form $\mathbb{R}$. For some basic facts on time scale calculus and dynamic equations on time scales, one may consult the excellent texts [6, 7] by Bohner and Peterson.

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on time scales. We refer the readers to the monographs [4, 14], the papers [1-3,5,8-10,12-13,15-16] and the references therein. Particularly, in 2010 Higgins [9] discussed the oscillation of the second-order delay dynamic equation

$$[p(t)x^{\Delta}(t)]^{\Delta} + f(t, x^{\sigma}(t), x(\tau_1(t)), \dots, x(\tau_n(t))) = 0.$$

In this article, we shall consider the second-order nonlinear neutral functional dynamic equation with retarded and advanced arguments

$$[p(t)z^{\Delta}(t)]^{\Delta} + f(t, x^{\sigma}(t), x(\tau_1(t)), \dots, x(\tau_n(t)), x(\xi_1(t)), \dots, x(\xi_m(t))) = 0 \quad (1.1)$$

on a time scale $\mathbb{T}$, where $n, m \in \mathbb{N}$, $z(t) = x(t) + r(t)x(g(t))$, $f \in C(\mathbb{T} \times \mathbb{R}^{n+m+1}, \mathbb{R})$, and $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ satisfies

$$\int_{t_0}^{\infty} \frac{1}{p(s)} \Delta s = \infty.$$

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We shall also need the following hypothesis:

- (A1) There exists constant $0 \le r_0 < 1$ such that $r \in C_{rd}(\mathbb{T}, [0, r_0])$.
- (A2) $g \in C_{rd}(\mathbb{T}, \mathbb{T})$, $g(t) \le t$, $\lim_{t\to\infty} g(t) = \infty$.
- (A3) $\tau_1, \tau_2, \dots, \tau_n \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau_i(t) \le t, \lim_{t\to\infty} \tau_i(t) = \infty$, $i = 1, 2, \dots, n$.
- (A4) $\xi_1, \xi_2, \dots, \xi_m \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\xi_j(t) \ge \sigma(t)$, $j = 1, 2, \dots, m$.
- (A5) $x_i f(t, x_0, x_1, \dots, x_{n+m}) > 0$ if $x_i x_j > 0$, $0 \le i, j \le n + m$, $i \ne j$, and for each $t \in \mathbb{T}$, $f(t, x_0, x_1, \dots, x_{n+m})$ is nondecreasing in x_i, $0 \le i \le n + m$.

Since we are interested in the oscillatory behavior of (1.1), we assume throughout that the time scale $\mathbb{T}$ under consideration satisfies $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

2. Preliminaries

To prove our main results in a straightforward manner, we establish some fundamental results in this section. Now we introduce the auxiliary functions

$$P(t, a) = \int_a^t \frac{\Delta s}{p(s)}, \quad \eta_i(t, a) = \frac{P(\tau_i(t), a)}{P(\sigma(t), a)}, \quad \nu_j(t, a) = \frac{P(\xi_j(t), a)}{P(\sigma(t), a)}, \tag{2.1}$$

where $1 \le i \le n$, $1 \le j \le m$, $a \in [t_0, \infty)_{\mathbb{T}}$. First of all, we give the following lemma.

Lemma 2.1. *If $x(t)$ is an eventually positive solution of* (1.1)*, then there exists some $T \ge t_0$ such that*

- (i) *for all $t \ge T$, $z(t) > 0, z^\Delta(t) > 0, x(t) \ge (1 - r(t))z(t)$;*
- (ii) *for each $1 \le i \le n$ and for $t \ge \tau_i(t) \ge T$, we have*
$$z(\tau_i(t)) \ge \eta_i(t, T) z^\sigma(t);$$
- (iii) *if $p(t)$ is nondecreasing, then for each $1 \le i \le n$ and for $\sigma(t) \ge t \ge \tau_i(t) \ge T$, we have*
$$\frac{z(\tau_i(t))}{z^\sigma(t)} \ge \frac{\tau_i(t) - \tau_i(T)}{\sigma(t) - \tau_i(T)}.$$

Proof. (i) Suppose that $x(t)$ is an eventually positive solution of (1.1). In view of the conditions (A1)–(A4), there exists $T \in \mathbb{T}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(\tau_i(t)) > 0$, $1 \le i \le n$, $x(\xi_j(t)) > 0$, $1 \le j \le m$, $t \in [T, \infty)_{\mathbb{T}}$. It follows that $z(t) = x(t) + r(t)x(g(t)) > 0$ and from (1.1) that $[p(t)z^\Delta(t)]^\Delta < 0$ on $[T, \infty)_{\mathbb{T}}$, which means that $p(t)z^\Delta(t)$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$.

Next, we claim that $z^\Delta(t)$ is eventually positive. Otherwise, if there exists $t_1 \in \mathbb{T}$ with $t_1 \ge T$ such that $z^\Delta(t_1) < 0$, then

$$z^\Delta(t) \le \frac{p(t_1)z^\Delta(t_1)}{p(t)} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.2}$$

Integrating (2.2) from t_1 to t ($t \ge t_1$), we obtain that $\lim_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Hence, $z^\Delta(t) > 0$ on $[T, \infty)_{\mathbb{T}}$.

Since $x(t)$ is a solution of (1.1) satisfying $x(t) > 0$ and $z^\Delta(t) > 0$, we see that

$$x(t) = z(t) - r(t)x(g(t)) \ge z(t) - r(t)z(g(t)) \ge z(t) - r(t)z(t) = (1 - r(t))z(t).$$

(ii) For $t \geq \tau_i(t) \geq T \geq t_0$ and $1 \leq i \leq n$, we have

$$z^\sigma(t) - z(\tau_i(t)) = \int_{\tau_i(t)}^{\sigma(t)} \frac{1}{p(s)} p(s) z^\Delta(s)\Delta s \leq p(\tau_i(t)) z^\Delta(\tau_i(t)) \int_{\tau_i(t)}^{\sigma(t)} \frac{1}{p(s)} \Delta s.$$

Dividing both sides of the above inequality by $z(\tau_i(t))$, we obtain

$$\frac{z^\sigma(t)}{z(\tau_i(t))} \leq 1 + \frac{p(\tau_i(t)) z^\Delta(\tau_i)}{z(\tau_i(t))} P(\sigma(t), \tau_i(t)). \tag{2.3}$$

Likewise, we also have

$$z(\tau_i(t)) - z(T) = \int_T^{\tau_i(t)} \frac{1}{p(s)} p(s) z^\Delta(s)\Delta s \geq p(\tau_i(t)) z^\Delta(\tau_i(t)) \int_T^{\tau_i(t)} \frac{1}{p(s)} \Delta s,$$

and hence

$$\frac{p(\tau_i(t)) z^\Delta(\tau_i(t))}{z(\tau_i(t))} \leq \frac{1}{P(\tau_i(t), T)}. \tag{2.4}$$

Therefore, (2.3) and (2.4) imply

$$\frac{z^\sigma(t)}{z(\tau_i(t))} \leq 1 + \frac{p(\tau_i(t)) z^\Delta(\tau_i)}{z(\tau_i(t))} P(\sigma(t), \tau_i(t)) \leq \frac{P(\sigma(t), T)}{P(\tau_i(t), T)}.$$

This gives the desired result

$$z(\tau_i(t)) \geq \eta_i(t, T) z^\sigma(t), \quad 1 \leq i \leq n.$$

(iii) It is based on similar arguments developed in [5] and the corresponding proof can be found in [1, Lemma 2.4]. The proof is complete. □

In addition to the above lemma, we need a method for studying separated boundary value problems (SBVP) to prove our main results. Namely, we will define functions called upper and lower solutions that, not only imply the existence of a solution of a SBVP, but also provide bounds on the location of the solution. Consider the SBVP

$$-(p(t) z^\Delta(t))^\Delta + q(t) z^\sigma = h(t, z^\sigma), \quad t \in [a, b]^{\kappa^2}, \tag{2.5}$$

$$z(a) = A, \quad z(b) = B, \tag{2.6}$$

where functions $h \in C([a, b]^{\kappa^2} \times \mathbb{R}, \mathbb{R})$, $p, q \in C([a, b]^{\kappa^2})$ with $p(t) > 0$ and $q(t) \geq 0$ on $[a, b]^{\kappa^2}$. We define the set

$$\mathbb{D}_1 := \{z \in \mathbb{X} : z^\Delta \text{ is continuous and } (pz^\Delta)^\Delta \text{ is rd-continuous on } [a, b]^{\kappa^2}\},$$

where the Banach space $\mathbb{X} = C([a, b]_{\mathbb{T}})$ is equipped with the norm $\| \cdot \|$ defined by

$$\|z\| := \max_{t \in [a, b]_{\mathbb{T}}} |z(t)| \quad \text{for all } z \in \mathbb{X}.$$

A function z is called a solution of the equation $-(p(t) z^\Delta(t))^\Delta + q(t) z^\sigma = 0$ on $[a, b]^{\kappa^2}$ if $z \in \mathbb{D}_1$ and solves this equation for all $t \in [a, b]^{\kappa^2}$. Next, we define for any $u, v \in \mathbb{D}_1$ the sector $[u, v]_1$ by

$$[u, v]_1 := \{w \in \mathbb{D}_1 \mid u \leq w \leq v\}.$$

Definition 2.2 ([7, Definition 6.1]). We call $\alpha \in \mathbb{D}_1$ a lower solution of the SBVP (2.5)-(2.6) on $[a,b]_{\mathbb{T}}$ provided

$$-(p(t)\alpha^{\Delta}(t))^{\Delta} + q(t)\alpha^{\sigma}(t) \le h(t,\alpha^{\sigma}(t)) \quad \text{for all } t \in [a,b]^{\kappa^2},$$
$$\alpha(a) \le A, \quad \alpha(b) \le B.$$

Similarly, $\beta \in \mathbb{D}_1$ is called an upper solution of the SBVP (2.5)-(2.6) on $[a,b]$ provided

$$-(p(t)\beta^{\Delta}(t))^{\Delta} + q(t)\beta^{\sigma}(t) \ge h(t,\beta^{\sigma}(t)) \quad \text{for all } t \in [a,b]^{\kappa^2},$$
$$\beta(a) \ge A, \quad \beta(b) \ge B.$$

The following theorem is an extension of [7, Theorem 6.5] to $[a,\infty)_{\mathbb{T}}$.

Theorem 2.3 ([10, Theorem 1.5]). *Assume that there exists a lower solution α and an upper solution β of (2.5) with $\alpha(t) \le \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. Then*

$$-\left(p(t)z^{\Delta}(t)\right)^{\Delta} + q(t)z^{\sigma} = h(t,z^{\sigma}) \tag{2.7}$$

has a solution z with $z(a) = A$ and $z \in [\alpha,\beta]_1$ on $[a,\infty)_{\mathbb{T}}$.

Our next preliminary result is a generalization of [9, Theorem 2.4]. When $r(t) = 0$, [9, Theorem 2.4] is still a special case of the following Theorem.

Theorem 2.4. *Let $h(t,x)$ be a continuous function of the variables $t > t_0$ and $|x| < \infty$. Assume that for all $t > 0$ and $x \ne 0$, $xh(t,x) > 0$, and for each fixed t, $h(t,x)$ is nondecreasing in x for $x > 0$. Then a necessary condition for equation*

$$(p(t)z^{\Delta}(t))^{\Delta} + h(t,x^{\sigma}(t)) = 0, \quad t \ge t_0 > 0 \tag{2.8}$$

to have a bounded nonoscillatory solution is that

$$\int^{\infty} P(t,a)h(t,c)\Delta t < \infty$$

for any fixed $a \in [t_0,\infty)_{\mathbb{T}}$ and for some constant $c > 0$.

Proof. Suppose $x(t)$ is a bounded eventually positive solution of (2.8). Then, there exists $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ for $t \ge T$. As $h(t,x) > 0$ for all $x > 0$, $(p(t)z^{\Delta}(t))^{\Delta}$ is eventually negative. Hence, $p(t)z^{\Delta}(t)$ is decreasing and according to Lemma 2.1, $\lim_{t\to\infty} p(t)z^{\Delta}(t) = L$ with $0 \le L < \infty$. Integrating (2.8) from s to T_1, we obtain

$$p(T_1)z^{\Delta}(T_1) - p(s)z^{\Delta}(s) + \int_s^{T_1} h(\theta,x^{\sigma}(\theta))\Delta\theta = 0.$$

It follows that

$$z^{\Delta}(s) \ge \frac{1}{p(s)}\int_s^{\infty} h(\theta,x^{\sigma}(\theta))\Delta\theta.$$

Integrating again for $T \le t_1 < t$ and by change of integration order [12, Lemma 1]

$$\int_s^t \Big[\int_\eta^t f_1(\eta,\xi)\Delta\xi\Big]\Delta\eta = \int_s^t \Big[\int_s^{\sigma(\xi)} f_1(\eta,\xi)\Delta\eta\Big]\Delta\xi,$$

we obtain

$$z(t) - z(t_1) \ge \int_{t_1}^t \frac{1}{p(s)}\int_s^{\infty} h(\theta,x^{\sigma}(\theta))\Delta\theta\Delta s \ge \int_{t_1}^t \int_s^t \frac{h(\theta,x^{\sigma}(\theta))}{p(s)}\Delta\theta\Delta s$$

$$= \int_{t_1}^{t} \int_{t_1}^{\sigma(\theta)} \frac{h(\theta, x^\sigma(\theta))}{p(s)} \Delta s \Delta\theta \geq \int_{t_1}^{t} P(\theta, t_1) h(\theta, x^\sigma(\theta)) \Delta\theta.$$

Consequently, for $t \geq t_1 \geq T$, we obtain

$$z(t) > \int_{t_1}^{t} P(\theta, t_1) f(\theta, x^\sigma(\theta)) \Delta\theta.$$

Since $x(t) \leq M$ for some $M > 0$ and $\int_{t_1}^{t} P(\theta, t_1) f(\theta, x^\sigma(\theta)) \Delta\theta$ is increasing function of t, for $r \in [0, 1)$, we have

$$\begin{aligned}\int_{t_1}^{\infty} P(\theta, t_1) f(\theta, (1 - r(\sigma(\theta))) z^\sigma(\theta)) \Delta\theta &\leq \int_{t_1}^{\infty} P(\theta, t_1) f(\theta, x^\sigma(\theta)) \Delta\theta \\ &< \lim_{t\to\infty} z(t) \leq 2M < \infty.\end{aligned}$$

By the monotonicity of f, we have

$$\int_{a}^{\infty} P(\theta, a) f(\theta, (1 - r(\sigma(T))) z^\sigma(T)) \Delta\theta < \infty.$$

Letting $c = (1 - r(\sigma(T))) z^\sigma(T)$, we obtain the desired result. $\square$

We end this section with time scale version of Arzerel-Ascoli theorem (see [16, Lemma 4]) and Kranoselskii's fixed point theorem (see [16]). These will be used in the proof of Theorem 3.3.

For $T_0, T_1 \in \mathbb{T}$, let $[T_0, \infty)_\mathbb{T} := \{t \in \mathbb{T} : t \geq T_0\}$ and $[T_0, T_1]_\mathbb{T} := \{t \in \mathbb{T} : T_0 \leq t \leq T_1\}$. Further, let $C([T_0, \infty)_\mathbb{T}, \mathbb{R})$ denote all continuous functions mapping $[T_0, \infty)_\mathbb{T}$ into $\mathbb{R}$,

$$BC[T_0, \infty)_\mathbb{T} := \big\{x \in C([T_0, \infty)_\mathbb{T}, \mathbb{R}) : \sup_{t\in[T_0,\infty)_\mathbb{T}} |x(t)| < \infty\big\}. \tag{2.9}$$

which endowed with the norm $\|x\| = \sup_{t\in[T_0,\infty)_\mathbb{T}} |x(t)|$, $(BC[T_0, \infty)_\mathbb{T}, \|\cdot\|)$ is a Banach space. Let $X \subseteq BC[T_0, \infty)_\mathbb{T}$, we say X is uniformly Cauchy if for any given $\varepsilon > 0$, there exists a $T_1 \in [T_0, \infty)_\mathbb{T}$ such that for any $x \in X$,

$$|x(t_1 - x(t_2)| < \varepsilon \quad \text{for all } \ t_1, t_2 \in [T_1, \infty)_\mathbb{T}.$$

The set X is said to be equi-continuous on $[a, b]_\mathbb{T}$ if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x \in X$ and $t_1, t_2 \in [a, b]_\mathbb{T}$ with $|t_1 - t_2| < \delta$, $|x(t_1) - x(t_2)| < \varepsilon$.

Lemma 2.5 ([16, Lemma 4])**.** *Suppose that $X \subseteq BC[T_0, \infty)_\mathbb{T}$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[T_0, T_1]_\mathbb{T}$ for any $T_1 \in [T_0, \infty)_\mathbb{T}$. Then X is relatively compact.*

Lemma 2.6 (Kranoselskii's fixed point theorem)**.** *Suppose that X is a Banach space and Ω is a bounded, convex and closed subset of X. Suppose further that there exist two operators $U, S : \Omega \to X$ such that*

(i) *$Ux + Sy \in \Omega$ for all $x, y \in \Omega$;*
(ii) *U is a contraction mapping;*
(iii) *S is completely continuous.*

Then $U + S$ has a fixed point in Ω.

3. Main results

Theorem 3.1. *Assume that conditions* (A1)–(A5) *hold, then any bounded solution* $x(t)$ *of* (1.1) *is oscillatory when*

$$\Big|\int^{\infty} P(t,a)f(t,\alpha w^{\sigma}(t),\alpha w^{\tau_1}\eta_1(t,a),\dots,\alpha w^{\tau_n}\eta_n(t,a),\alpha w^{\xi_1},\dots,\alpha w^{\xi_m})\Delta t\Big| = \infty, \tag{3.1}$$

or

$$\Big|\int^{\infty} P(t,a)f(t,\alpha,\alpha\eta_1(t,a),\dots,\alpha\eta_n(t,a),\alpha,\dots,\alpha)\Delta t\Big| = \infty \tag{3.2}$$

for all $\alpha \neq 0$, *where* $w(t) = 1 - r(t)$, $w^{\tau_i} = w(\tau_i(t))$, $1 \le i \le n$, $w^{\xi_j} = w(\xi_j(t))$, $1 \le j \le m$, $\eta_i(t,a)$, $1 \le i \le n$, *is given in* (2.1).

Proof. First, we point out that (3.1) is equivalent to (3.2). In fact, by assumptions (A1), (A5), and the monotonicity of f, we have

$$\begin{aligned}
&\Big|\int_a^{\infty} P(t,a)f(t,\alpha m_1,\alpha m_1\eta_1(t,a),\dots,\alpha m_1\eta_n(t,a),\alpha m_1,\dots,\alpha m_1)\Delta t\Big| \\
&\le \Big|\int_a^{\infty} P(t,a)f(t,\alpha w^{\sigma}(t),\alpha w(\tau_1(t))\eta_1(t,a),\dots,\alpha w(\tau_n(t))\eta_n(t,a),\alpha w^{\xi_1}, \\
&\quad \dots,\alpha w^{\xi_m})\Delta t\Big| \\
&\le \Big|\int_a^{\infty} P(t,a)f(t,\alpha m_2,\alpha m_2\eta_1(t,a),\dots,\alpha m_2\eta_n(t,a),\alpha m_2,\dots,\alpha m_2)\Delta t\Big|,
\end{aligned}$$

where $w(t) = 1 - r(t)$, $0 < m_1 = \inf_{t\in[a,\infty)_{\mathbb{T}}} w(t)$ and $0 < m_2 = \sup_{t\in[a,\infty)_{\mathbb{T}}} w(t)$.

Assume not and let $u(t)$ be a bounded nonoscillatory solution of (1.1) which we may assume satisfies

$$u(t) > 0, \quad u(\tau_i(t)) > 0, \quad u(\xi_j(t)) > 0 \quad t \ge T \ge t_0, \quad 1 \le i \le n, \quad 1 \le j \le m.$$

For convenience, let $z_1(t) = u(t) + r(t)u(g(t))$. Consequently,

$$\begin{aligned}
&[p(t)(u(t) + r(t)u(g(t)))^{\Delta}]^{\Delta} = [p(t)z_1^{\Delta}(t)]^{\Delta} \\
&= -f(t,u^{\sigma}(t),u(\tau_1(t)),\dots,u(\tau_n(t)),u(\xi_1(t)),\dots,u(\xi_m(t))) < 0
\end{aligned}$$

for all $t \ge T$ and so $p(t)z_1^{\Delta}(t)$ is decreasing for $t \ge T$.

By Lemma 2.1, we have

$$u(t) \ge (1 - r(t))z_1(t), \quad z_1(\tau_i(t)) \ge \eta_i(t,T)z_1^{\sigma}(t), \quad t \ge \tau_i(t) \ge T.$$

Define the function

$$F(t,\theta_1) := f(t,w^{\sigma}(t)\theta_1,w^{\tau_1}\eta_1\theta_1,\dots,w^{\tau_n}\eta_n\theta_1,w^{\xi_1}\theta_1,\dots,w^{\xi_m}\theta_1),$$

where $w(t) = 1 - r(t)$, $w^{\tau_i} = w(\tau_i(t))$, $\eta_i = \eta_i(t,T)$ $1 \le i \le n$, $w^{\xi_j} = w(\xi_j(t))$, $1 \le j \le m$. Then by the monotonicity of f, for $t \ge T$, we have

$$\begin{aligned}
0 &= [p(t)z_1^{\Delta}(t)]^{\Delta} + f(t,u^{\sigma}(t),u(\tau_1(t)),\dots,u(\tau_n(t)),u(\xi_1(t)),\dots,u(\xi_m(t))) \\
&\ge [p(t)z_1^{\Delta}(t)]^{\Delta} + F(t,z_1^{\sigma}(t)).
\end{aligned}$$

Applying Theorem 2.3 with $\alpha(t) \equiv z_1(T) \le z_1(t) \equiv \beta(t)$, we obtain the existence of a solution $y(t)$ of

$$(p(t)y^{\Delta}(t))^{\Delta} + F(t,y^{\sigma}(t)) = 0, \quad y(T) = z_1(T)$$

with $z_1(T) \le y(t) \le z_1(t)$ on $[T,\infty)_{\mathbb{T}}$. However, by Theorem 2.4, it follows that

$$\int^{\infty} P(t,a)F(t,c)\Delta t < \infty$$

for any fixed $a \in [t_0,\infty)_{\mathbb{T}}$ and some $c > 0$, which contradicts (3.2). $\square$

Theorem 3.2. *Assume that conditions $(A1)-(A5)$ hold and for each i, $1 \le i \le n$, there exists $\rho_i > 0$ such that*

$$\liminf_{t\to\infty} \eta_i(t,a) \ge \rho_i \quad \textit{for } a \in \mathbb{T}. \tag{3.3}$$

If $x(t)$ is a bounded nonoscillatory solution of (1.1)*, then for some $\alpha \ne 0$, we have*

$$\int_a^{\infty} P(\sigma(t),a)f(t,\alpha,\alpha,\dots,\alpha)\Delta t < \infty. \tag{3.4}$$

Proof. By (2.1) and (3.3), we have

$$1 \le \frac{P(\sigma(t),a)}{P(t,a)} \le \frac{P(\sigma(t),a)}{P(\tau_i(t),a)} = \frac{1}{\eta_i(t,a)} \le \frac{1}{\rho_i}$$

for sufficiently large $t \in \mathbb{T}$. So we conclude that $P(\sigma(t),a)/P(t,a)$ is bounded on $\mathbb{T}$. Then for any $\beta \ne 0$,

$$\int_a^{\infty} P(\sigma(t),a)f(t,\beta,\beta,\dots,\beta)\Delta t < \infty$$

if and only if

$$\int_a^{\infty} P(t,a)f(t,\beta,\beta,\dots,\beta)\Delta t < \infty.$$

Furthermore, observe that by (3.3), given any $\varepsilon > 0$ with $\varepsilon < \frac{1}{2}\min\{\rho_i : 1 \le i \le n\}$, there exists $T_i \ge t_0$ such that $1 \ge \eta_i(t,a) \ge \rho_i - \varepsilon =: \hat{\rho}_i > 0$ for $t \ge T_i$ and $1 \le i \le n$.

Assume (1.1) has a bounded nonoscillatory solution. Then by Theorem 3.1, we have

$$\Big|\int_a^{\infty} P(t,a)f(t,\alpha,\alpha\eta_1(t,a),\dots,\alpha\eta_n(t,a),\alpha,\dots,\alpha)\Delta t\Big| < \infty$$

for all $\alpha \ne 0$. Let $\hat{\rho} := \min\{\rho_i : 1 \le i \le n\}$. Consequently, $\alpha\hat{\rho} \le \alpha\eta_i(t,a) \le \alpha$ for all $1 \le i \le n$, and so by the monotonicity of f, we obtain

$$\Big|\int_a^{\infty} P(t,a)f(t,\alpha\hat{\rho},\alpha\hat{\rho},\dots,\alpha\hat{\rho})\Delta t\Big| < \infty.$$

We obtain (3.4) as desired with $\nu = \alpha\hat{\rho}$. $\square$

Note that in Theorem 3.2, we do not assume that $P(\sigma(t),a)/P(t,a)$ is bounded on $\mathbb{T}$; thus we improve [9, Theorem 3.2]. The previous result says the condition (3.3) is sufficient in order to replace the auxiliary functions $\eta_i(t,a)$, $1 \le i \le n$ with upper bounds. Our next result gives a sufficient condition for (1.1) to have bounded nonoscillatory solutions.

Theorem 3.3. *Assume that conditions* (A1)-(A5) *hold. If for some $\alpha \ne 0$,*

$$\int_a^{\infty} P(\sigma(t),a)f(t,\alpha,\alpha,\dots,\alpha)\Delta t < \infty, \tag{3.5}$$

then (1.1) *has a bounded nonoscillatory solution.*

Proof. Assume that (3.5) holds. According to (A5), without loss of generality, we may assume that $\alpha > 0$. Since $p(t) > 0$ and $f(t, \alpha, \alpha, \dots, \alpha) > 0$ on $\mathbb{T}$, by the change of integration order [12, Lemma 1], we obtain

$$\begin{aligned}
&\int_a^\infty \int_s^\infty \frac{f(t, \alpha, \alpha, \dots, \alpha)}{p(s)} \Delta t \Delta s \\
&= \lim_{u \to +\infty} \int_a^u \int_s^u \frac{f(t, \alpha, \alpha, \dots, \alpha)}{p(s)} \Delta t \Delta s \\
&= \lim_{u \to +\infty} \int_a^u \int_a^{\sigma(t)} \frac{f(t, \alpha, \alpha, \dots, \alpha)}{p(s)} \Delta s \Delta t \\
&= \int_a^\infty \Big[\int_a^{\sigma(t)} \frac{1}{p(s)} \Delta s \Big] f(t, \alpha, \alpha, \dots, \alpha) \Delta t \\
&= \int_a^\infty P(\sigma(t), a) f(t, \alpha, \alpha, \dots, \alpha) \Delta t < \infty.
\end{aligned} \tag{3.6}$$

By (A1) and (3.6), we can choose $T_0 \in \mathbb{T}$ large enough such that

$$\int_{T_0}^\infty \int_s^\infty \frac{f(t, \alpha, \alpha, \dots, \alpha)}{p(s)} \Delta t \Delta s \le \frac{(1 - r_0)\alpha}{2}. \tag{3.7}$$

According to (A2) and (A3), we see that there exists $T_1 \in \mathbb{T}$ with $T_1 > T_0$ such that $g(t) \ge T_0$ and $\tau_i(t) \ge T_0$, $1 \le i \le n$, for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define the Banach space $BC[t_0, \infty)_{\mathbb{T}}$ as in (2.9), and let

$$\Omega = \big\{ x = x(t) \in BC[t_0, \infty)_{\mathbb{T}} : \frac{(1 - r_0)\alpha}{2} \le x(t) \le \alpha \big\}. \tag{3.8}$$

It is easy to verify that Ω is a bounded, convex and closed subset of $BC[t_0, \infty)_{\mathbb{T}}$. For the sake of convenience, set

$$F(t) := f(t, x^\sigma(t), x(\tau_1(t)), \dots, x(\tau_n(t)), x(\xi_1(t)), \dots, x(\xi_m(t))).$$

By (A5), for any $x \in \Omega$ and $t \in [T_0, \infty)_{\mathbb{T}}$, we have

$$F(t) \le f(t, \alpha, \alpha, \dots, \alpha).$$

Now we define two operators U and $S : \Omega \to BC[t_0, \infty)_{\mathbb{T}}$ as follows

$$(Ux)(t) = \begin{cases} -r(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \\ (Ux)(T_1), & t \in [T_0, T_1]_{\mathbb{T}}, \end{cases}$$

and

$$(Sx)(t) = \begin{cases} \alpha - \int_t^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s, & t \in [T_1, \infty)_{\mathbb{T}}, \\ \alpha - \int_{T_1}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s, & t \in [T_0, T_1]_{\mathbb{T}}. \end{cases}$$

Next, we will show that U and S satisfy the conditions in Lemma 2.6.

(i) We first prove that $Ux + Sy \in \Omega$ for any $x, y \in \Omega$. Note that for any $x, y \in \Omega$, $\alpha/2 \le x, y \le \alpha$ by (3.8). For any $x, y \in \Omega$ and $t \in [T_1, \infty)_{\mathbb{T}}$, by (3.7), we have

$$\begin{aligned}
(Ux)(t) + (Sy)(t) &= \alpha - r(t)x(g(t)) - \int_t^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s \\
&\ge (1 - r_0)\alpha - \frac{(1 - r_0)\alpha}{2} \ge \frac{(1 - r_0)\alpha}{2},
\end{aligned}$$

and

$$(Ux)(t) + (Sy)(t) = \alpha - r(t)x(g(t)) - \int_t^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s \le \alpha.$$

Similarly, we can show that $\alpha/2 \le Ux + Sy \le \alpha$ for any $x, y \in \Omega$ and $t \in [T_0, T_1]_{\mathbb{T}}$.

(ii) It is not difficult to check that U is a contraction mapping.

(iii) We will prove that S is a completely continuous mapping. It is easy to check that S maps Ω into Ω.

Again, for the sake of convenience, let

$$F_l(t) := f(t, x_l^\sigma(t), x_l(\tau_1(t)), \dots, x_l(\tau_n(t)), x_l(\xi_1(t)), \dots, x_l(\xi_m(t))).$$

Next, we show that the continuity of S. Let $x_l \in \Omega$ and $\|x_l - x\| \to 0$ as $l \to \infty$, then $x \in \Omega$ and $x_l \to x$ as $l \to \infty$. By the monotonicity and continuity of f, we have

$$|F_l(t) - F(t)| \to 0, \quad \text{as } l \to \infty,$$
$$|F_l(t) - F(t)| \le 2f(t, \alpha, \dots, \alpha).$$

For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$|(Sx_l)(t) - (Sx)(t)| \le \int_t^\infty \int_s^\infty \frac{|F_l(u) - F(u)|}{p(s)} \Delta u \Delta s,$$

and for $t \in [T_0, T_1]_{\mathbb{T}}$,

$$|(Sx_l)(t) - (Sx)(t)| \le \int_{T_1}^\infty \int_s^\infty \frac{|F_l(u) - F(u)|}{p(s)} \Delta u \Delta s.$$

Employing Lebegues's dominated convergence theorem [7, Chapter 5], we obtain

$$\|(Sx_l) - (Sx)\| \to 0 \quad \text{as } l \to \infty.$$

Thus S is continuous.

Third, we show that $S\Omega$ is relatively compact. According to Lemma 2.5, it suffices to show that $S\Omega$ is bounded, uniformly Cauchy and equi-continuous. The boundedness is obvious. For any $x \in \Omega$, by (3.6), we have

$$\int_{T_1}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s < \infty.$$

Then for any given $\varepsilon > 0$, there exists $T_2 \in [T_1, \infty)_{\mathbb{T}}$ large enough such that

$$\int_{T_2}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta < \varepsilon/2.$$

Hence, for any $x \in \Omega$ and $t_1, t_2 \in [T_2, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} |(Sx)(t_1) - (Sx)(t_2)| &= \Big| \int_{t_1}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s - \int_{t_2}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s \Big| \\ &\le 2 \int_{T_2}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s < \varepsilon. \end{aligned}$$

So $S\Omega$ is uniformly Cauchy.

Finally, we will prove that $S\Omega$ is equi-continuous. For $T_1 \le t_1 < t_2 \le T_2 + 1$, we have

$$|(Sx)(t_1) - (Sx)(t_2)| = \Big| \int_{t_1}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s - \int_{t_2}^\infty \int_s^\infty \frac{F(u)}{p(s)} \Delta u \Delta s \Big|$$

$$\leq \int_{t_1}^{t_2} \int_{s}^{\infty} \frac{F(u)}{p(s)} \Delta u \Delta s.$$

For $T_0 \leq t_1 < T_1 \leq t_2 \leq T_2 + 1$, we have

$$|(Sx)(t_1) - (Sx)(t_2)| \leq \int_{T_1}^{t_2} \int_{s}^{\infty} \frac{F(u)}{p(s)} \Delta u \Delta s.$$

For $t_1, t_2 \in [T_0, T_1]_{\mathbb{T}}$, $|(Sx)(t_1) - (Sx)(t_2)| = 0$.

Then there exists $0 < \delta < 1$ such that $|(Sx)(t_1) - (Sx)(t_2)| < \varepsilon$ if $t_1, t_2 \in [T_0, T_2 + 1)$ and $|t_2 - t_1| < \delta$. This means that $S\Omega$ is equi-continuous. It follows from Lemma 2.5 that $S\Omega$ is relatively compact, and then S is completely continuous.

By Lemma 2.6, there exists $x \in \Omega$ such that $(U + S)x = x$, which indicates that $x(t)$ is a solution of (1.1). In particular, for $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = \alpha - r(t)x(g(t)) - \int_{t}^{\infty} \int_{s}^{\infty} \frac{F(u)}{p(s)} \Delta u \Delta s.$$

Let $t \to \infty$, we obtain the desired result. $\square$

To extend Theorems 3.1 and 3.2 to unbounded solutions, we introduce the class Φ of functions ϕ such that $\phi(u)$ is a nondecreasing continuous function of u satisfying $u\phi(u) > 0$ ($u \neq 0$) with

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty.$$

Definition 3.4. We say that $f(t, u, v_1, \dots, v_n)$ satisfies condition (C) provided for some $\phi \in \Phi$ there exists $c \neq 0$ such that for all $t \geq T$, $\eta_i = \eta_i(t, T)$, $1 \leq i \leq n$,

$$\inf_{|u| \to \infty} \frac{f(t, u, \eta_1 u, \dots, \eta_n u, u, \dots, u)}{\phi(u/m_1)} \geq k|f(t, c, \eta_1 c, \dots, \eta_n c, c, \dots, c)|,$$

for some positive constant k and $m_1 = \inf_{t \in [T_0, \infty)_{\mathbb{T}}}\{1 - r(t)\}$.

We continue with a generalization of [9, Theorem 3.4].

Theorem 3.5. *Suppose $\phi \in \Phi$. Let $h(t, x)$ be a continuous function of the variables $t \geq t_0$ and $|x| < \infty$ such that for all $t > 0$ $xh(t, x) > 0$, $x \neq 0$ and satisfies with respect to $\phi(x)$ the following conditions: there is a $c \neq 0$ such that*

$$\inf_{|x| \to \infty} \frac{h(t, x)}{\phi(x/m_1)} \geq k|h(t, c)| \quad \text{for } r(t) \in [0, 1), \tag{3.9}$$

for some positive constant k, $m_1 = \inf_{t \in [T_0, \infty)_{\mathbb{T}}}\{1 - r(t)\}$ and for all $t \geq T$, and that

$$\lim_{|x| \to \infty} \Big| \int^{\infty} \frac{1}{\phi(u)} du \Big| < \infty. \tag{3.10}$$

If

$$\int^{\infty} P(t, a)h(t, c)\Delta t = \infty \tag{3.11}$$

holds for all $c \neq 0$, then (2.8) is oscillatory.

In addition, if $P(\sigma(t), a)/P(t, a)$ is bounded on $\mathbb{T}$, then (3.11) is also a necessary condition for (2.8) to be oscillatory.

Proof. Assume (3.11) holds and let $x(t)$ be an eventually positive solution of (2.8). It follows from Theorem 2.4 that $x(t)$ cannot be bounded. By Lemma 2.1 we see that $\lim_{t\to\infty} x(t) = \infty$. Also, as in the proof of Theorem 2.4,

$$\int_T^t z^\Delta(s)\Delta s \ge \int_T^t P(\theta,T)h(\theta,x^\sigma(\theta))\Delta\theta \tag{3.12}$$

for sufficiently large T.

Next we define the continuously differentiable real-valued function

$$G(u) := \int_{u_0}^u \frac{ds}{\phi(s)}\,.$$

Observe that $G'(u) = 1/\phi(u)$. For $r(t) \in [0,1)$, by the Pötzsche Chain Rule [6, Theorem 1.90],

$$\begin{aligned}(G(z(t)))^\Delta &= \Big(\int_0^1 \frac{dh}{\phi(z_h(t))}\Big)z^\Delta(t) \ge \Big(\int_0^1 \frac{dh}{\phi(z^\sigma(t))}\Big)z^\Delta(t)\\ &= \frac{z^\Delta(t)}{\phi(z^\sigma(t))} \ge \frac{z^\Delta(t)}{\phi(x^\sigma(t)/m_1)},\end{aligned}$$

where $z_h(t) := z(t) + h\mu(t)z^\Delta(t) \le z^\sigma(t)$. Now multiplying (3.12) by $[\phi(z^\sigma(s))]^{-1}$, we obtain

$$\int_T^t \frac{z^\Delta(s)}{\phi(z^\sigma(s))}\Delta s \ge \int_T^t P(\theta,T)\frac{h(\theta,x^\sigma(\theta))}{\phi(x^\sigma(\theta)/m_1)}\Delta\theta \ge \int_T^t kP(\theta,T)h(\theta,c)\Delta\theta$$

for sufficiently large T by (3.9), where $c := x(T) > 0$. Since $\lim_{t\to\infty} x(t) = \infty$, we have

$$\lim_{t\to\infty} G(z(t)) = \lim_{t\to\infty}\int_T^{z(t)} \frac{du}{\phi(u)} = \int_T^\infty \frac{du}{\phi(u)} < \infty.$$

Therefore,

$$\int_T^t (G(z(s)))^\Delta\Delta s \ge \int_T^t \frac{z^\Delta(s)}{\phi(z^\sigma(s))}\Delta s \ge \int_T^t kP(\theta,T)h(\theta,c)\Delta\theta.$$

However, letting $t \to \infty$ in above inequality, the left side is bounded whereas the right side is unbounded by assumption (3.11). This contradiction shows that (3.11) is sufficient condition for all solutions of (2.8) to be oscillatory.

Conversely, suppose that (2.8) is oscillatory and $\big|\int^\infty P(t,a)h(t,c)\Delta t\big| < \infty$ for some $c \ne 0$. Since $\frac{P(\sigma(t),a)}{P(t,a)}$ is bounded on $\mathbb{T}$, we see that for all $c \ne 0$,

$$\Big|\int^\infty P(t,a)h(t,c)\Delta t\Big| < \infty \quad \text{if and only if} \quad \Big|\int^\infty P(\sigma(t),a)h(t,c)\Delta t\Big| < \infty.$$

It follows from Theorem 3.3 that (2.8) has a bounded nonoscillatory solution. This contradiction shows that (3.11) is necessary. $\square$

Now we give our last result.

Theorem 3.6. *Assume that conditions* (A1)–(A5) *hold and f satisfies condition* (C). *Then* (1.1) *is oscillatory when*

$$\Big|\int^\infty P(t,a)f(t,\alpha,\alpha\eta_1(t,a),\dots,\alpha\eta_n(t,a),\alpha,\dots,\alpha)\Delta t\Big| = \infty \tag{3.13}$$

holds for all $\alpha \ne 0$. In addition, if inequality (3.3) *holds, then* (3.13) *is also necessary.*

Proof. Assume that (3.13) holds for all $\alpha \neq 0$ and let $u(t)$ be a nonoscillatory solution of (1.1) which we may assume satisfies

$$u(t), u(\tau_i(t)), z_1^{\Delta}(t) := (u(t) + r(t)u(g(t)))^{\Delta} > 0, \quad [p(t)z_1^{\Delta}(t)]^{\Delta} \leq 0,$$

for $t \geq T \geq t_0$, $1 \leq i \leq n$. For $r(t) \in [0,1)$, according to Lemma 2.1, we have

$$u(t) \geq (1 - r(t))z_1(t), \quad z_1(\tau_i(t)) \geq \eta_i(t,T)z_1^{\sigma}(t), \quad t \geq \tau_i(t) \geq T.$$

By the monotonicity of f, for $t \geq T$, we have

$$\begin{aligned}
0 &= [p(t)z_1^{\Delta}(t)]^{\Delta} + f(t, u^{\sigma}(t), u(\tau_1(t)), \dots, u(\tau_n(t)), u(\xi_1(t)), \dots, u(\xi_m(t))) \\
&\geq [p(t)z_1^{\Delta}(t)]^{\Delta} + f(t, w^{\sigma}z_1^{\sigma}, w^{\tau_1}\eta_1 z_1^{\sigma}(t), \dots, w^{\tau_n}\eta_n z_1^{\sigma}(t), w^{\xi_1}z_1^{\sigma}(t), \dots, w^{\xi_m}z_1^{\sigma}(t)) \\
&\geq [p(t)z_1^{\Delta}(t)]^{\Delta} + f(t, m_1 z_1^{\sigma}, m_1\eta_1 z_1^{\sigma}(t), \dots, m_1\eta_n z_1^{\sigma}(t), m_1 z_1^{\sigma}(t), \dots, m_1 z_1^{\sigma}(t)),
\end{aligned}$$

where $w(t) = 1 - r(t)$, $w^{\tau_i} = w(\tau_i(t)), \eta_i = \eta_i(t,T)$ $1 \leq i \leq n$, $w^{\xi_j} = w(\xi_j(t))$, $1 \leq j \leq m$.

As in the proof of Theorem 3.1, we obtain the existence of a solution $y(t)$ of

$$(p(t)y^{\Delta}(t))^{\Delta} + F(t, y^{\sigma}(t)) = 0, \quad y(T) = z_1(T)$$

with $z_1(T) \leq y(t) \leq z_1(t)$ on $[T,\infty)_{\mathbb{T}}$. By Theorem 3.5, it immediately follows that

$$\Big|\int_a^{\infty} P(t,a)f(t,c,c\eta_1(t,a),\dots,c\eta_n(t,a),c,\dots,c)\Delta t\Big| < \infty$$

for some $c \neq 0$, which contradicts (3.13).

Conversely, assume that (3.3) holds and (3.13) does not hold for some $\alpha \neq 0$; i.e.,

$$\Big|\int^{\infty} P(t,a)f(t,\alpha,\alpha\eta_1(t,a),\dots,\alpha\eta_n(t,a),\alpha,\dots,\alpha)\Delta t\Big| < \infty. \tag{3.14}$$

Noting that (3.3) implies that $P(\sigma(t),a)/P(t,a)$ is bounded on $\mathbb{T}$, (3.14) holds if and only if

$$\Big|\int^{\infty} P(\sigma(t),a)f(t,\alpha,\alpha\eta_1(t,a),\dots,\alpha\eta_n(t,a),\alpha,\dots,\alpha)\Delta t\Big| < \infty.$$

It follows that for any $\varepsilon > 0$ with $\varepsilon < \frac{1}{2}\min\{\rho_i \mid 1 \leq i \leq n\}$, there exists $T_i \geq t_0$ such that $\eta_i(t,a) \geq \rho_i - \varepsilon =: \hat{\rho}_i$ for $t \geq T_i$ and $1 \leq i \leq n$. Let $\hat{\rho} := \min\{\rho_i \mid 1 \leq i \leq n\}$. It follows that $\alpha\eta_i(t,a) \geq \alpha\hat{\rho}$ for $t \geq T_i$. Then by the monotonicity of f and the fact that $\eta_i(t,a) \leq 1$ for $t \geq T_i$, we have

$$\Big|\int_a^{\infty} P(\sigma(t),a)f(t,\alpha\hat{\rho},\alpha\hat{\rho},\dots,\alpha\hat{\rho})\Delta t\Big| < \infty,$$

which gives (3.5). Therefore, by Theorem 3.3, Equation (1.1) has a bounded nonoscillatory solution. This contradiction shows that (3.13) is necessary. □

4. Examples

We would like to illustrate our main results by three examples. We begin with the following example that extends some results of Higgins [9] and [10, Example 3.2].

Example 4.1. Let $\Phi_*(u) = |u|^{*-1}u$ and $P(\sigma(t),a)/P(t,a)$ is bounded on $\mathbb{T}$. Consider the dynamic equation

$$\begin{aligned}&[p(t)z^{\Delta}(t)]^{\Delta} + q_0(t)\Phi_\gamma(x^\sigma(t))\\ &+ \sum_{i=1}^{2} q_i(t)\Phi_{\gamma_i}(x(\tau_i(t)) + s(t)\Phi_{\gamma_3}(x(\xi(t)) + |\Phi_\gamma(x^\sigma(t))| = 0,\end{aligned} \tag{4.1}$$

where $z(t) = x(t) + r(t)x(g(t))$. We assume that $q_0(t) \geq 1$, $q_1(t)$, $q_2(t)$ and $s(t)$ are rd-continuous and eventually positive on $[t,\infty)_{\mathbb{T}}$, $\gamma > 1, \gamma_i$, $i = 1,2,3$, are positive real numbers and $p(t), g(t), r(t)$, $\tau_1(t), \tau_2(t), \xi(t)$ satisfy conditions (A1)–(A4). It is not difficult to check that $f(t,x_0,x_1,x_2,x_3) = q_0(t)\Phi_\gamma(x_0) + \sum_{i=1}^{2} q_i(t)\Phi_{\gamma_i}(x_i) + s(t)\Phi_{\gamma_3}(x_3) + |\Phi_\gamma(x^\sigma(t))|$ satisfies condition (A5). By Theorem 3.1, all bounded solution of (4.1) are oscillatory in case

$$\Big|\int^{\infty} P(t,a)(q_0(t)\Phi_\gamma(\alpha) + \sum_{i=1}^{2} q_i(t)\Phi_{\gamma_i}(\alpha\eta_i(t,a)) + s(t)\Phi_{\gamma_3}(\alpha) + |\Phi_\gamma(\alpha)|)\Delta t\Big| = \infty$$

for all $\alpha \neq 0$. Now suppose $\gamma_i > 1$, $i = 1,2,3$ and let $\phi(u) = u^{\gamma_0}$, where $1 < \gamma_0 \leq min\{\gamma,\gamma_1,\gamma_2,\gamma_3\}$. It is easy to show that $f(t,x_0,x_1,x_2,x_3)$ satisfies condition (C). Moreover, if one of $\gamma,\gamma_1,\gamma_2,\gamma_2$ is larger than 1, $f(t,x_0,x_1,x_2,x_3)$ still satisfies condition (C). Therefore, according to Theorem 3.6, we conclude that all solution of (4.1) are oscillatory in case

$$\Big|\int^{\infty} P(t,a)(q_0(t)\Phi_\gamma(\alpha) + \sum_{i=1}^{2} q_i(t)\Phi_{\gamma_i}(\alpha\eta_i(t,a)) + s(t)\Phi_{\gamma_3}(\alpha) + \alpha^2\eta_1(t,a))\Delta t\Big| = \infty \tag{4.2}$$

for all $\alpha \neq 0$. Furthermore, if $\eta_i(t,a) \geq \rho_i > 0$, $i = 1,2$, (4.2) becomes a sufficient and necessary condition.

In [9, 10], it is assumed that $f(t,x_0,x_1,x_2,x_3) = -f(t,-x_0,-x_1,-x_2,-x_3)$ if $x_i > 0$, $i = 0,1,2,3$. However, we only need f satisfies condition (A5) which relaxes the hypothesis in [9, 10].

Example 4.2. Assume $P(\sigma(t),a)/P(t,a)$ is bounded on $\mathbb{T}$. Consider the dynamic equation

$$[p(t)(x(t) + r(t)x(g(t)))^{\Delta}]^{\Delta} + q_0(t)x^\sigma(t) + q_1(t)x(\tau_1(t)) + q_2(t)x(\xi_1(t)) = 0. \tag{4.3}$$

We assume that $p^{\Delta}(t) \geq 0$ and $q_0(t), q_1(t), q_2(t)$ are rd-continuous and eventually positive on $[t_0,\infty)_{\mathbb{T}}$, and $g(t), r(t) \geq 0$, $\tau_1(t), \xi_1(t)$ satisfy conditions (A1)–(A4). If we set

$$Q(t) := q_0(t) + \eta_1(t,a)q_1(t) + q_2(t)$$

for $t \geq T \geq t_0$, where $\eta_1(t,a) = \frac{\tau_1(t)-\tau_1(a)}{\sigma(t)-\tau_1(a)} \sim \frac{\tau_1(t)}{\sigma(t)}$, then (4.3) is oscillatory in case

$$[p(t)z(t)^{\Delta}]^{\Delta} + \lambda Q(t)z^\sigma(t) = 0 \tag{4.4}$$

is oscillatory for some $0 < \lambda < 1$, where $z(t) = x(t)+r(t)x(g(t))$. If not, we suppose that u is a nonoscillatory solution of (4.3) with $u > 0, t \geq T$. By Lemma 2.1, we have

$$z(t) > 0, \quad z(\tau_1(t)) > 0, \quad z(\xi_1(t)) > 0, \quad z^{\Delta} > 0, \quad [p(t)z(t)^{\Delta}]^{\Delta} < 0, \quad t \geq T,$$

and then

$$[p(t)z_1(t)^{\Delta}]^{\Delta} + (q_0(t) + \eta_1(t,a)q_1(t) + q_2(t))z_1(\sigma(t)) \leq 0, \quad t \geq T, \tag{4.5}$$

where $z_1(t) = u(t) + r(t)u(g(t))$. Let $y(t) := \frac{p(t)z_1^\Delta}{z_1(t)}$, using (4.5) we see that $z_1(t)$ satisfies the Riccati dynamic inequality

$$z_1(t)^\Delta + Q(t) + \frac{z_1(t)^2}{p(t) + \mu(t)z_1(t)} \le 0.$$

This implies that the equation

$$[p(t)z(t)^\Delta]^\Delta + Q(t)z^\sigma(t) = 0$$

is nonoscillatory and so by the Sturm comparison theorem [3, Theorem 6.1], (4.4) is also nonoscillatory. This contradiction shows that (4.3) is oscillatory.

We can apply a variety of oscillatory criteria, such as an extension of the well-know Leighton-Wintner criteria [3, Theorem 8.1], a generalization of the Hinton-Lewis criteria [3, Theorems 8.2-8.3] and Hille-Nehari criteria [13, Theorem 3.1], to obtain oscillatory results of (4.4). In particular, when $r(t) = 0$, [10, Example 3.1] is a special case of Example 4.2.

Example 4.3. Let $\Phi_*(u) = |u|^{*-1}u$. Consider the dynamic equation

$$[x(t) + r(t)x(g(t))]^{\Delta\Delta} + q_0(t)\Phi_\gamma(x^\sigma(t)) + \sum_{i=1}^n q_i(t)\Phi_{\gamma_i}(x(\tau_i(t)) = 0, \tag{4.6}$$

where $q_0(t), q_1(t), \dots, q_n(t)$ are continuous and eventually positive on $[t_0,\infty)_{\mathbb{T}}$, $\gamma > 1, \gamma_1 > \gamma_2 > \dots > \gamma_m > \gamma > \gamma_{m+1} > \dots > \gamma_n > 0$ and $g(t), r(t), \tau_1(t), \dots, \tau_n(t)$ satisfy conditions (A1)–(A3).

We need the following two lemmas to prove Corollary 4.6.

Lemma 4.4 ([1, Lemma 2.2])**.** *For any given n-tuple $\{\gamma_1, \dots, \gamma_n\}$ satisfying*

$$\gamma_1 > \gamma_2 > \dots > \gamma_m > \gamma > \gamma_{m+1} > \dots > \gamma_n > 0,$$

there corresponds an n-tuple $\{\xi_1, \dots, \xi_n\}$ such that

$$\sum_{i=1}^n \gamma_i\xi_i = \gamma, \quad \sum_{i=1}^n \xi_i = 1, \quad 0 < \xi_1, \dots, \xi_n < 1. \tag{4.7}$$

When $n = 2$, it turns out that

$$\xi_1 = \frac{\gamma - \gamma_2}{\gamma_1 - \gamma_2}, \quad \xi_2 = \frac{\gamma_1 - \gamma}{\gamma_1 - \gamma_2}.$$

Next we have the Arithmetic-Geometric Mean Inequality, see [1].

Lemma 4.5. *If $\xi_1, \dots, \xi_n > 0$ satisfy $\sum_{i=1}^n \xi_i = 1$, and $u_1, \dots, u_n \ge 0$, then*

$$\sum_{i=1}^n u_i\xi_i \ge \prod_{i=1}^n u_i^{\xi_i}. \tag{4.8}$$

By the property of the convex function $\varphi(t) = -\ln t$, it is easy to get (4.8).

Corollary 4.6. *Assume that $\mu(t)/t$ is bounded, then all solutions of (4.6) are oscillatory provided*

$$\int^\infty \Big[tq_0(t) + \sum_{i=1}^n q_i(t)(\tau_i(t))^{\gamma_i}t^{1-\gamma_i}\Big]\Delta t = \infty, \tag{4.9}$$

$$\xi \prod_{i=1}^{n}[q_i(t)]^{\xi_i}[\eta_i(t,a)]^{\gamma_i\xi_i} \geq \sum_{i=1}^{n} q_i(t)|\Phi_{\gamma_i}(c)|, \tag{4.10}$$

for some $c \neq 0$*, where* $\xi = \prod_{i=1}^{n}(\frac{1}{\xi_i})^{\xi_i}$*,* $\sum_{i=1}^{n}\xi_i = 1$*,* $0 < \xi_1, \dots, \xi_n < 1$*, as in Lemma 4.4.*

Proof. Assume (4.9)–(4.10) hold. Define $\phi(u) = \Phi_\gamma(u) = |u|^{\gamma-1}u$. Then it is easy to check that

$$u\phi(u) > 0, \quad \int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty, \quad \text{for } u \neq 0.$$

Let $f(t,u,v_1,\dots v_n) = q_0(t)\Phi_\gamma(u) + \sum_{i=1}^{n} q_i(t)\Phi_{\gamma_i}(v_i)$ and $c \neq 0$ to be determined. By (A5) and $u\phi(u) > 0$, without loss of generality, we may assume $u > 0$. Note that $0 < \eta_i(t,a) \leq 1$, $1 \leq i \leq n$, and $\Phi_*(u) = |u|^{*-1}u$ is an increasing function. Then according to Lemmas 4.4 and 4.5, we have

$$\begin{aligned}
\frac{f(t,u,\eta_1(t)u,\dots,\eta_n(t)u)}{\phi(u)} &= \frac{q_0(t)\Phi_\gamma(u) + \sum_{i=1}^{n} q_i(t)\Phi_{\gamma_i}(\eta_i(t,a)u)}{\Phi_\gamma(u)} \\
&= q_0(t) + \sum_{i=1}^{n} q_i(t)\Phi_{\gamma_i}(\eta_i(t,a))\Phi_{\gamma_i-\gamma}(u) \\
&\geq q_0(t) + \prod_{i=1}^{n}(\frac{1}{\xi_i})^{\xi_i}(q_i(t))^{\xi_i}(\eta_i(t,a))^{\gamma_i\xi_i}\Phi_{(\gamma_i-\gamma)\xi_i}(u) \\
&= q_0(t) + \xi\prod_{i=1}^{n}[q_i(t)]^{\xi_i}[\eta_i(t,a)]^{\gamma_i\xi_i} \\
&\geq kq_0(t)|\Phi_\gamma(c)| + k\sum_{i=1}^{n} q_i(t)|\Phi_{\gamma_i}(c)| \\
&\geq kq_0(t)|\Phi_\gamma(c)| + k\sum_{i=1}^{n} q_i(t)|\Phi_{\gamma_i}(\eta_i(t)c)| \\
&\geq k|f(t,c,\ \eta_1(t,a)c,\dots,\eta_n(t,a)c)|
\end{aligned}$$

for $k\Phi_\gamma(c) \leq 1$, $0 < k \leq 1$ and all $t \geq t_0$. Thus $f(t,u,v_1,\dots,v_n)$ satisfies condition (C).

According to Lemma 2.1 and that $\mu(t)/t$ is bounded, we have

$$\eta_i(t,a) = \frac{\tau_i(t) - \tau_i(a)}{\sigma(t) - \tau_i(a)} \sim \frac{\tau_i(t)}{\sigma(t)} \sim \frac{\tau_i(t)}{t}, \quad \text{as } t \to \infty,\ 1 \leq i \leq n.$$

For any $\alpha \neq 0$, we have

$$\begin{aligned}
&|f(t,\alpha,\alpha\eta_1(t,a),\dots,\alpha\eta_n(t,a))| \\
&= |q_0(t)\Phi_\gamma(\alpha) + \sum_{i=1}^{n} q_i(t)\Phi_{\gamma_i}(\alpha\eta_i(t,a))| \\
&= q_0(t)\Phi_\gamma(|\alpha|) + \sum_{i=1}^{n} q_i(t)\Phi_{\gamma_i}(|\alpha|\eta_i(t,a)) \\
&\geq \delta[q_0(t) + \sum_{i=1}^{n} q_i(t)\Phi_{\gamma_i}(\eta_i(t,a))] \geq 0,
\end{aligned}$$

where $\delta = \min\{\Phi_\gamma(|\alpha|), \Phi_{\gamma_1}(|\alpha|), \dots, \Phi_{\gamma_n}(|\alpha|)\}$. It immediately follows that

$$\begin{aligned} &\Big|\int^\infty tf(t,\alpha,\alpha\eta_1(t),\dots,\alpha\eta_n(t))\Delta t\Big| \\ &\geq \int^\infty \Big[\delta[tq_0(t)+\sum_{i=1}^n tq_i(t)\Phi_{\gamma_i}(\eta_i(t,a))]\Big]\Delta t = \infty. \end{aligned}$$

Hence, by Theorem 3.6, Equation (4.6) is oscillatory. $\square$

In particular, for $\mathbb{T}=\mathbb{R}, \mathbb{T}=\mathbb{Z}$, $p_i(t)=c_i$ is constant and $\tau_i(t)=a_it+b_i, a_i>0$, $1\leq i\leq n$. It is easy to check that (4.10) holds as $c=1$. By (4.9), (4.6) is oscillatory in case

$$\int^\infty tq_0(t)\Delta t = \infty.$$

Remark 4.7. If $q_0(t)\geq q_i(t), i=1,2,\dots,n$, it is easy to check that Condition (C) holds and (4.10) can be removed.

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