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CONTINUOUS DEPENDENCE OF SOLUTIONS FOR INDEFINITE SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. We consider the superlinear elliptic problem

$$-\Delta u + m(x)u = a(x)u^p$$

in a bounded smooth domain under Neumann boundary conditions, where $m \in L^\sigma(\Omega)$, $\sigma \geq N/2$ and $a \in C(\overline{\Omega})$ is a sign changing function. Assuming that the associated first eigenvalue of the operator $-\Delta + m$ is zero, we use constrained minimization methods to study the existence of a positive solution when $\widehat{m}$ is a suitable perturbation of m.

1. Introduction

In this article, we consider the continuous dependence of positive solution for the semilinear elliptic problem

$$\begin{gathered} -\Delta u + m(x)u = a(x)u^p \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain, η is the outward unit normal, $1 < p < \frac{N+2}{N-2}$, $m \in L^\sigma(\Omega)$ for some $\sigma \geq N/2$, and $a \in C(\overline{\Omega})$ is a sign changing function, more specifically, a satisfies

(A1) $\Omega^+ := \{x \in \Omega : a(x) > 0\} \neq \emptyset$ and $\Omega^- := \{x \in \Omega : a(x) < 0\} \neq \emptyset$.

The existence of solutions for indefinite semilinear elliptic problems has been intensively studied in the literature; see for example [1, 3, 4, 5, 6, 11, 12] and references therein. As it has been established in several articles [1, 5, 12], the existence of solution for (1.1) depends on the interaction between the nonlinear term and the eigenfunctions corresponding to the associated eigenvalue problem

$$\begin{gathered} -\Delta u + m(x)u = \lambda u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Noting that the first eigenvalue of (1.2) is simple, isolated and that the associated eigenfunction does not change sign in Ω (see Section 3), we assume:

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(A2) the first eigenvalue for (1.2) is zero, $\lambda_1(-\Delta + m) = 0$,
(A3) $\int_\Omega a(x)\varphi_1^{p+1} < 0$,

where φ_1 is the first eigenfunction for (1.2) which is positive in Ω and normalized in $L^2(\Omega)$.

Our main interest in this article is to establish the existence of a positive solution for (1.1) with $\widehat{m}$ instead of m, when $\widehat{m} \in L^\sigma(\Omega)$ is a suitable perturbation of the function m. More specifically, throughout this paper we will consider $\{m_j\}_j \subset L^\sigma(\Omega)$ satisfying

(M1) $m_j \rightharpoonup m$ weakly in $L^\sigma(\Omega), \sigma \geq N/2$. If $\sigma = N/2$ we suppose in addition that $|m_j| \leq f$ a.e. in Ω for some $f \in L^{N/2}(\Omega)$.

It is worthwhile mentioning that the hypothesis (M1) has been motivated by the works of Tehrani [12] and Afrouzi [1] that studied the existence of positive solutions for (1.1) with m_j instead of m, supposing $m, \{m_j\}_j \subset C(\bar{\Omega})$ and $m_j \to m$ uniformly in $\bar{\Omega}$. We should emphasize that when the linear operator $-\Delta + m_j$ is coercive; i.e., when the associated first eigenvalue, $\lambda_1(-\Delta + m_j)$, is positive, it may be proved that Problem (1.1) with m_j instead of m always has a positive solution (see e.g. [4, 5]). The key point in this paper is that, under our hypotheses, the operator $-\Delta + m_j$ is not necessarily coercive, and the eigenvalue $\lambda_1(-\Delta + m_j)$ may even be strictly negative. Now we may state our main results:

Theorem 1.1. *Suppose* (A1)–(A3), (M1). *Then there exists $j_1 \in \mathbb{N}$ such that* (1.1) *with m_j instead of m possesses a positive solution w_j for all $j \geq j_1$.*

Theorem 1.2. *Suppose* (A1)–(A3), (M1). *Let $\{w_j\}_j$ be the sequence of solutions given by Theorem 1.1. Then $\{w_j\}_j$ has a subsequence which converges strongly in $H^1(\Omega)$ to a solution of* (1.1).

As a direct consequence of Theorem 1.1, the perturbed problem (1.1) with $\hat{m}$ instead of m has a positive solution if $\hat{m}$ is close to m in $L^\sigma(\Omega)$.

Corollary 1.3. *Assume* (A1)–(A3). *Then there exists $R > 0$ such that* (1.1) *with $\widehat{m}$ instead of m possesses a positive solution whenever $\|m - \widehat{m}\|_{L^\sigma(\Omega)} < R$.*

Our next result establishes the existence of a positive solution for (1.1) with $\widehat{m}$ instead of m, when the first eigenvalue $\lambda_1(-\Delta + \hat{m})$ is near to $\lambda_1(-\Delta + m)$, without supposing that $\hat{m}$ is close to m in $L^\sigma(\Omega)$:

Theorem 1.4. *Assume* (A1)–(A3). *Given $R > 0$ (or $f \in L^{N/2}(\Omega)$ when $\sigma = N/2$), then there exists $\mu > 0$ such that the Problem* (1.1) *with $\widehat{m}$ instead of m, possesses a positive solution for all $\widehat{m} \in L^\sigma(\Omega)$ satisfying*

(i) $\|m - \widehat{m}\|_{L^\sigma(\Omega)} \leq R$, *($|\widehat{m}| \leq f$ a.e in Ω if $\sigma = \frac{N}{2}$),*
(ii) $\int_\Omega |\nabla\varphi_1|^2 + \widehat{m}(x)\varphi_1^2 \leq \mu$,
(iii) $-\mu < \lambda_1(-\Delta + \widehat{m}) < 0$.

Afrouzi [1] proved Theorem 1.1, considering the Dirichlet boundary conditions, under the assumptions $m \in C(\overline{\Omega})$ and $\|m - m_j\|_{C(\overline{\Omega})} \to 0$. In [12], also considering the norm $\|\cdot\|_{C(\bar{\Omega})}$, Tehrani established Corollary 1.3 for $m \in C(\overline{\Omega})$ and Theorem 1.4 for $m \in C^{0,\alpha}(\overline{\Omega})$ supposing the additional hypothesis $\int_\Omega a(x)\varphi_{\widehat{m}}^{p+1} < 0$, where $\varphi_{\widehat{m}} > 0$ is the eigenfunction associated with the first eigenvalue $\lambda_1(-\Delta + \widehat{m})$. It is worthwhile mentioning that the condition (A3) is a necessary condition for the existence of positive solution for the Problem (1.1) in the Theorem 1.1 (see [12]).

Finally we observe that Beresticky, Capuzzo-Dolcetta and Nirenberg [5] proved the existence of a positive solution for (1.1) with $m-\tau$ instead of m, when $m \in C(\overline{\Omega})$ satisfies (A2) and $\tau \geq 0$ is sufficiently small. We observe that, as a consequence of Theorem 1.1, we may assert the existence of a positive solution for (1.1) with $m_j - \tau$ instead of m, under the hypothesis (M1) whenever $\tau \geq 0$ is sufficiently small (see Remark 4.4).

The method employed in this article is variational, more specifically, we use a constrained minimization method. In order to prove our main results, we establish continuity results for the first and the second eigenvalues and their associated eigenfunctions for (1.2) with m_j instead of m under the condition (M1), generalizing a version for the uniform convergence provided by [1]. We also verify an uniform equivalence result between the $H^1(\Omega)$-norm and a norm, depending on m_j, for the orthogonal space to the first eigenfunction of (1.2) with m_j instead of m.

The organization of this paper is as follows: in Section 2 we introduce some notation and preliminary results. In Section 3, based on the arguments used by Manes-Micheletti [10] and deFigueiredo [7] for Dirichlet boundary conditions, we present the technical properties and a convergence result, under the condition (M1), for the eigenvalues and eigenfunctions of (1.2) with m_j instead of m. The proofs of Theorems 1.1, 1.2 and 1.4 are presented in Section 4. In this section we also present an example that illustrate the application of Theorems 1.1 and 1.2 on a setting where the condition (M1) is satisfied. In the Appendix we establish the regularity results for the solutions of (1.1) and (1.2) and we prove Theorems 3.2 and 3.3 from Section 3.

2. Preliminaries

In this section we introduce the definitions and the technical results that will be used throughout the text. First we recall the strong unique continuation property (SUCP for short) due to Jerison and Kenig [9] which guarantees that the solutions and the first eigenfunction that we find are positive. Next, in the main result of this section, we establish the convergence $\int_\Omega m_j(x)v_j^2 \to \int_\Omega m(x)v_o^2$, under the hypothesis (M1), when $v_j \rightharpoonup v_o$ weakly in $H^1(\Omega)$.

We use $\|\cdot\|_s$ to denote the norm in $L^s(\Omega)$. If $E \subset \Omega$ is a proper subset, the norm in $L^s(E)$ is represented by $\|\cdot\|_{L^s(E)}$. $|A|$ denotes the Lebesgue's measure of $A \subset \mathbb{R}^N$. $B_\rho(x_o)$ denotes the open ball of radius $\rho > 0$ centered in x_o. The symbol $\rightharpoonup$ denotes the weakly convergence.

Definition 2.1. (i) We say that f possesses a zero of infinite order if there exists $x_o \in \Omega$ such that $\int_{|x-x_o|<\epsilon} f^2(x)dx = O(\epsilon^k)$ for every $k \in \mathbb{N}$.

(ii) Let $c : \Omega \to \mathbb{R}$ be a function. We say that the differential inequality

$$|\Delta f(x)| \leq |c(x)f(x)|, \quad x \in \Omega \tag{2.1}$$

has the SUCP in the Sobolev space $W^{2,q}_{\rm loc}(\Omega)$, if $f \equiv 0$ whenever f belongs to $W^{2,q}_{\rm loc}(\Omega)$, satisfies (2.1) almost everywhere in Ω, $f \in L^2_{\rm loc}(\Omega)$ and has a zero of infinite order.

Theorem 2.2 ([9])**.** *Let $\Omega \subset \mathbb{R}^N$ be a domain, $N \geq 3$, and $c \in L^{N/2}_{\rm loc}(\Omega)$. Then the differential inequality* (2.1) *has the SUCP in $W^{2,\frac{2N}{N+2}}_{\rm loc}(\Omega)$.*

To employ the SUCP we may use the following result.

Theorem 2.3 ([8]). *Let $\Omega \subset \mathbb{R}^N$, be a bounded domain, $N \geq 3$ and $c \in L^{N/2}_{\rm loc}(\Omega)$. Suppose that $u \in H^1_{\rm loc}(\Omega)$ is such that $\int_\Omega \nabla u \nabla v + c(x)uv = 0$, for all $v \in C^\infty_o(\Omega)$. If $u = 0$ on a set E of positive measure, then u possesses a zero of infinite order in almost everywhere point of E.*

Remark 2.4. If $u \geq 0$ is a weak solution for (1.1), after regularization (see Appendix A) $u \in W^{2,\frac{2N}{N+2}}(\Omega) \cap L^t(\Omega)$ for all $t \geq 1$. So, taking $c(x) := a(x)u^{p-1} - m(x)$,

$$|\Delta u(x)| = |c(x)u(x)|$$

for almost everywhere $x \in \Omega$, and $c \in L^{N/2}(\Omega)$. If $|\{x \in \Omega : u(x) = 0\}| > 0$, by Theorems 2.3 and 2.2, $u \equiv 0$ in Ω. Therefore, if $u \geq 0$ is a nontrivial solution for (1.1), then $u > 0$ almost everywhere in Ω. The same conclusion is valid if $\varphi \geq 0$ is the first eigenfunction for (1.2).

As a direct consequence of the Sobolev imbedding theorem, we have the following result.

Lemma 2.5. *Suppose* (A3) *is satisfied. Then there exists $\beta > 0$ such that*

$$\int_\Omega a(x)|\varphi_1 + w|^{p+1} < \frac{1}{2}\int_\Omega a(x)\varphi_1^{p+1}, \quad \forall \|w\|_{H^1(\Omega)} < \beta.$$

The next result is standard and it is also based on the Sobolev imbedding theorem.

Lemma 2.6 ([14]). *Assume $m \in L^{N/2}(\Omega)$ and $\{\omega_j\}_j \subset H^1(\Omega)$. If $\omega_j \rightharpoonup \omega_o$ weakly in $H^1(\Omega)$, then*

$$\int_\Omega |m(x)(\omega_j^2 - \omega_o^2)| \to 0.$$

Now we present a version of Lemma 2.6 when the sequence $\{m_j\}$ satisfies the hypothesis (M1).

Lemma 2.7. *Assume* (M1) *is satisfied. If $\omega_j \rightharpoonup \omega_o$ weakly in $H^1(\Omega)$, then*

$$\int_\Omega m_j(x)\omega_j^2 \to \int_\Omega m(x)\omega_o^2.$$

Proof. Given arbitrary subsequences of $\{m_j\}$ and $\{\omega_j\}$, (still denoted by $\{m_j\}$ and $\{\omega_j\}$), we may write

$$\int_\Omega m_j(x)\omega_j^2 - \int_\Omega m(x)\omega_o^2 = \int_\Omega m_j(x)(\omega_j^2 - \omega_o^2) + \int_\Omega \omega_o^2(m_j(x) - m(x)).$$

Since $m_j - m \rightharpoonup 0$ weakly in $L^\sigma(\Omega), \sigma \geq N/2$, and $\omega_o \in H^1(\Omega)$, we have

$$\int_\Omega \omega_o^2[m_j(x) - m(x)] \to 0.$$

Therefore, to prove Lemma 2.7, it suffices to verify that

$$\Big|\int_\Omega m_j(x)(\omega_j^2 - \omega_o^2)\Big| \to 0.$$

If $\sigma = N/2$, by (M1) and Lemma 2.6, we obtain

$$\int_\Omega |m_j(x)(\omega_j^2 - \omega_o^2)| \leq \int_\Omega |f(x)|\, |\omega_j^2 - \omega_o^2| \to 0.$$

On the other hand, if $\sigma > N/2$, taking a subsequence if necessary, we may suppose that $\|\omega_j^2 - \omega_o^2\|_{\frac{\sigma}{\sigma-1}} \to 0$ since $2\sigma/(\sigma-1) < 2^*$. Hence, by Hölder's inequality,

$$\int_\Omega |m_j(x)(\omega_j^2 - \omega_o^2)| \le \|m_j\|_r \|\omega_j^2 - \omega_o^2\|_{\frac{\sigma}{\sigma-1}} \to 0.$$

The proof of Lemma 2.7 is complete. □

Remark 2.8. Applying the same argument used in the proof of the above lemma, we have

$$\int_\Omega m_j(x)\omega_j z \to \int_\Omega m(x) w_o z, \quad \forall z \in H^1(\Omega).$$

Furthermore, as a direct consequence of Lemma 2.7, we have the following result.

Corollary 2.9. *Assume* (M1). *Then, for any given $\epsilon > 0$, there exists $j_\epsilon > 0$ such that*

$$|\int_\Omega [m_j(x) - m(x)]\omega^2| \le \epsilon \|\omega\|_{H^1(\Omega)}^2 \quad \forall \omega \in H^1(\Omega),\ j \ge j_\epsilon.$$

3. The eigenvalue problem

In this section we establish some basic properties for the eigenvalue problem (1.2). We start by observing that its first eigenvalue is simple, isolated and the associated eigenfunction does not change sign. After that we establish a continuity result for the first and the second eigenvalues and their corresponding eigenfunctions for (1.2) with m_j instead of m under the condition (M1). We finalize this section by proving the uniform equivalence between the standard $H^1(\Omega)$ norm and a norm associated with the sequence $\{m_j\}$ on the subspace of $H^1(\Omega)$ orthogonal to the first eigenfunction of (1.2) with m_j instead of m.

Setting $Q_m(u) := \int_\Omega |\nabla u|^2 + m(x)u^2$, we say that a function φ is associated with λ if $\int_\Omega \varphi^2 = 1$ and $Q_m(\varphi) = \lambda$. We define

$$\lambda_1 := \lambda_1(-\Delta + m) = \inf\{Q_m(u) : u \in H^1(\Omega),\ \|u\|_2 = 1\}. \tag{3.1}$$

Remark 3.1. If $\varphi \in H^1(\Omega)$ is a (normalized) solution for (1.2), then

$$\int_\Omega \nabla\varphi\nabla u + m(x)\varphi u = \lambda \int_\Omega \varphi u, \quad \forall u \in H^1(\Omega). \tag{3.2}$$

Using φ as test function in (3.2), we obtain

$$\lambda = \lambda \int_\Omega \varphi^2 = Q_m(\varphi) \ge \lambda_1. \tag{3.3}$$

Thus, any eigenvalue for (1.2) is greater than λ_1. Moreover, if there exists a non-trivial solution φ for (1.2) with λ_1 instead of λ, then λ_1 is the smallest eigenvalue for (1.2).

Theorem 3.2. *Suppose $m \in L^{N/2}(\Omega)$. Then the Problem* (1.2) *has its first eigenvalue given by* (3.1). *Moreover this first eigenvalue is simple and we may suppose that the associated eigenfunction, φ_1, is positive.*

To state our second result, we define

$$\lambda_2 := \lambda_2(-\Delta + m) = \inf\{Q_m(v) : v \in V\ ,\|v\|_2 = 1\}, \tag{3.4}$$

where $V := \{v \in H^1(\Omega) : \int_\Omega \varphi_1 v = 0\}$. The next theorem shows that λ_2 is the second eigenvalue for (1.2).

Theorem 3.3. *Suppose $m \in L^{N/2}(\Omega)$. Then the second eigenvalue for* (1.2) *is given by* (3.4). *Moreover $\lambda_1 < \lambda_2$ and any eigenfunction associated with λ_2 changes sign in Ω.*

The arguments used in the proofs of the above theorems are due to Manes and Micheletti [10](see also [7]). In order to verify the simplicity of the first eigenvalue, including the case $\sigma = N/2$, we apply Theorems 2.2 and 2.3 (see Remark 2.4). For the sake of completeness we present the proofs in the appendix.

Now let $\{m_j\}_j \subset L^{N/2}(\Omega)$ be the sequence satisfying (M1). Replacing m by m_j in Definition (3.1), we obtain the first eigenvalue $\lambda_1^j := \lambda_1(-\Delta + m_j)$ for (1.2) with m_j instead of m and its positive first eigenfunction $\varphi_{1,j}$. In a similar way, the second eigenvalue for (1.2) with m_j instead of m is given by

$$\lambda_2^j := \lambda_2(-\Delta + m_j) = \inf\{Q_{m_j}(v) : v \in V_j\ , \|v\|_2 = 1\},$$

where $V_j = \{v \in H^1(\Omega) : \int_\Omega \varphi_{1,j} v = 0\}$. We denote by $\varphi_{2,j}$ the eigenfunctions associated with λ_2^j. Hereafter we will always suppose that the given eigenfunctions are eigenfunctions normalized in $L^2(\Omega)$. Hence $Q_{m_j}(\varphi_{1,j}) = \lambda_1^j$ and $Q_{m_j}(\varphi_{2,j}) = \lambda_2^j$. The main result of this section is the following continuity result.

Theorem 3.4. *Assume* (M1) *is satisfied. Then*

(i) $\lim_{j\to\infty} \lambda_1^j = \lambda_1$ *and $\varphi_{1,j} \to \varphi_1$ strongly in $H^1(\Omega)$;*
(ii) $\lim_{j\to\infty} \lambda_2^j = \lambda_2$ *and $\varphi_{2,j} \to \varphi$ strongly in $H^1(\Omega)$, where φ is an eigenfunction associated with λ_2.*

Proof. First note that, by (M1), $\int_\Omega m_j(x)\varphi_1^2 \to \int_\Omega m(x)\varphi_1^2$. Therefore, in view of (3.1),

$$\limsup_{j\to\infty}\ \lambda_1^j \le \limsup_{j\to\infty} Q_{m_j}(\varphi_1) = \lambda_1. \tag{3.5}$$

Next we claim that there exists $M > 0$ such that $\|\varphi_{1,j}\|_{H^1(\Omega)} < M$, for all $j \in \mathbb{N}$. Indeed, otherwise we may suppose that $\|\nabla\varphi_{1,j}\|_2 \to \infty$. Defining $\omega_j := \frac{\varphi_{1,j}}{\|\nabla\varphi_{1,j}\|_2}$ and taking a subsequence if necessary, we have that $\omega_j \rightharpoonup 0$ weakly in $H^1(\Omega)$ since $\|\varphi_{1,j}\|_2 \equiv 1$. Applying Lemma 2.7, $\int_\Omega m_j(x)\omega_j^2 \to 0$. Thus, using the characterization for the first eigenvalue given by (3.1), we have

$$\frac{\lambda_1^j}{\|\nabla\varphi_{1,j}\|_2^2} = 1 + \int_\Omega m_j(x)\omega_j^2 \to 1.$$

Consequently $\lambda_1^j \to \infty$. However this contradicts (3.5). The claim is proved. Invoking the above claim, we may suppose $\varphi_{1,j} \rightharpoonup z$ weakly in $H^1(\Omega)$ and $\varphi_{1,j} \to z$ strongly in $L^2(\Omega)$. From Lemma 2.7,

$$\int_\Omega m_j(x)\varphi_{1,j}^2 \to \int_\Omega m(x)z^2. \tag{3.6}$$

Therefore, since $\liminf_{j\to\infty} \|\nabla\varphi_{1,j}\|_2^2 \ge \|\nabla z\|_2^2$,

$$\liminf_{j\to\infty} \lambda_1^j = \liminf_{j\to\infty} Q_{m_j}(\varphi_{1,j}) \ge Q_m(z) \ge \lambda_1. \tag{3.7}$$

By (3.5) and (3.7), $Q_m(z) = \lim_{j\to\infty} \lambda_1^j = \lambda_1$. Hence, from Theorem 3.2, $z = \varphi_1$. Moreover, by (3.6), we have

$$\int_\Omega |\nabla\varphi_{1,j}|^2 = \lambda_1^j - \int_\Omega m_j(x)\varphi_{1,j}^2 \to \lambda_1 - \int_\Omega m(x)\varphi_1^2 = \int_\Omega |\nabla\varphi_1|^2$$

and, consequently, $\varphi_{1,j} \to \varphi_1$ strongly in $H^1(\Omega)$. This concludes the proof of part (i).

Now we prove (ii): since $Q_{m_j}(\varphi_{2,j}) = \lambda_2^j$, $\varphi_{1,j} \to \varphi_1$ strongly in $H^1(\Omega)$ and $\{m_j\}_j$ satisfies (M1), we have

$$\int_\Omega \varphi_2\varphi_{1,j} \to \int_\Omega \varphi_2\varphi_1 = 0; \tag{3.8}$$

$$Q_{m_j}(\varphi_2) \to Q_m(\varphi_2) = \lambda_2; \tag{3.9}$$

$$\int_\Omega \nabla\varphi_{1,j}\nabla\varphi_2 + m_j(x)\varphi_{1,j}\varphi_2 \to \int_\Omega \nabla\varphi_1\nabla\varphi_2 + m(x)\varphi_1\varphi_2; \tag{3.10}$$

where φ_2 is an eigenfunction associated with λ_2. Defining $v_j := \varphi_2 - t_j\varphi_{1,j}$ with $t_j := \int_\Omega \varphi_2\varphi_{1,j}$ and using (3.8),

$$\|v_j\|_2^2 = \int_\Omega \varphi_2^2\,dx - t_j^2 \to 1. \tag{3.11}$$

Setting $w_j := \frac{v_j}{\|v_j\|_2}$, we obtain $\|w_j\|_{L^2(\Omega)} = 1$ and $\int_\Omega \varphi_{1,j}w_j = 0$, for all $j \in \mathbb{N}$. Hence $w_j \in V_j$ and, by (3.11) and the definition of λ_2^j,

$$\begin{aligned} Q_{m_j}(\varphi_{2,j}) &\le Q_{m_j}(w_j) \\ &= \frac{Q_{m_j}(\varphi_2) - 2t_j\int_\Omega(\nabla\varphi_2\nabla\varphi_{1,j} + m_j(x)\varphi_2\varphi_{1,j}) + t_j^2\lambda_1^j}{\|v_j\|_2^2}. \end{aligned}$$

Consequently, from (3.8)–(3.11) and the item (i),

$$\limsup_{j\to\infty} \lambda_2^j = \limsup_{j\to\infty} Q_{m_j}(\varphi_{2,j}) \le \limsup_{j\to\infty} Q_{m_j}(w_j) = \lambda_2. \tag{3.12}$$

Arguing as in the item (i), we may suppose that the sequence $\{\varphi_{2,j}\}_j$ is bounded in $H^1(\Omega)$, $\varphi_{2,j} \rightharpoonup \varphi$ weakly in $H^1(\Omega)$, with $\int_\Omega \varphi^2 = 1$ and $\varphi \in V$. We assert that φ is an eigenfunction associated with λ_2. Indeed, invoking Lemma 2.7 one more time,

$$\int_\Omega m_j(x)\varphi_{2,j}^2 \to \int_\Omega m(x)\varphi^2. \tag{3.13}$$

Therefore

$$\liminf_{j\to\infty} Q_{m_j}(\varphi_{2,j}) \ge Q_m(\varphi) \ge \lambda_2. \tag{3.14}$$

Hence, from (3.12),

$$\lambda_2^j \to \lambda_2, \quad Q_m(\varphi) = \lambda_2. \tag{3.15}$$

Since $\varphi \in V$ and $\int_\Omega \varphi^2 = 1$, we may apply Theorem 3.3 to conclude that φ is an eigenfunction associated with λ_2 as asserted. Finally, by (3.13) and (3.15),

$$\int_\Omega |\nabla\varphi_{2,j}|^2 = \lambda_2^j - \int_\Omega m_j\varphi_{2,j}^2 \to \lambda_2 - \int_\Omega m\varphi^2 = \int_\Omega |\nabla\varphi|^2.$$

Consequently $\varphi_{2,j} \to \varphi$ strongly in $H^1(\Omega)$. The proof of Theorem 3.4 is complete. □

Remark 3.5. (i) Afrouzi [1] proved Theorem 3.4 for the Dirichlet boundary conditions supposing $\{m_j\}_j \subset C(\overline{\Omega})$ and $m_j \to m$ uniformly in $\overline{\Omega}$.

(ii) Note that Theorem 3.4 holds if $m_j \to m$ strongly in $L^{N/2}(\Omega)$ since in this case the condition (M1) is satisfied.

Given $u \in H^1(\Omega)$, we may write $u = t_j\varphi_{1,j} + v_j$, where $t_j = \int_\Omega u\varphi_{1,j} \in \mathbb{R}$ and $v_j = u - t_j\varphi_{1,j} \in V_j$. Defining

$$\|v\|_{V_j}^2 = \int_\Omega |\nabla v|^2 + m_j(x)v^2 \tag{3.16}$$

and recalling that $\lambda_{2,j} \to \lambda_2 > 0$, it is not hard to see that $\|\cdot\|_{V_j}$ is an equivalent norm to $\|\cdot\|_{H^1(\Omega)}$ in V_j if j is large enough. Actually we have the following uniform equivalence result between the $H^1(\Omega)$-norm and the norm given by (3.16) on V_j.

Lemma 3.6. *Assume* (M1), (A2) *are satisfied. Then there exist $j_o \in \mathbb{N}$ and constants $A, B > 0$ such that*

$$A\|v\|_{H^1(\Omega)}^2 \le \|v\|_{V_j}^2 \le B\|v\|_{H^1(\Omega)}^2, \quad \forall v \in V_j,\ \forall j \ge j_o. \tag{3.17}$$

Proof. The existence of $B > 0$ follows from Hölder's inequality and the Sobolev Imbedding Theorem. Thus it suffices to show the existence of $A > 0$ in (3.17). Arguing by contradiction, we suppose that there exists a sequence $\{v_j\}_j$, with $v_j \in V_j$ for each j, such that $\|v_j\|_{H^1(\Omega)}^2 \equiv 1$ and $\|v_j\|_{V_j}^2 < 1/j$. Hence

$$0 = \lim_{j\to\infty} \int_\Omega |\nabla v_j|^2 + m_j(x)v_j^2 \ge \lim_{j\to\infty} \lambda_{2,j} \int_\Omega v_j^2.$$

By [(A2)] and Theorems 3.3 and 3.4, $\lambda_{2,j} \to \lambda_2 > 0$. Up to a subsequence, $v_j \rightharpoonup 0$ weakly in $H^1(\Omega)$. Applying Lemma 2.7, $\int_\Omega m_j(x)v_j^2 \to 0$. Consequently $\int_\Omega |\nabla v_j|^2 \to 0$ and $\|v_j\|_{H^1(\Omega)} \to 0$. This contradicts $\|v_j\|_{H^1(\Omega)} \equiv 1$. The lemma is proved. □

Remark 3.7. For further reference we remark that, since $\lambda_2 > 0$,

$$\|v\|_{V_o} := \Big(\int_\Omega |\nabla v|^2 + m(x)v^2\Big)^{1/2}$$

is a norm equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$ in $V_o := V$. Moreover,

$$\|u\| := \Big(t^2 + \|v\|_{V_o}^2\Big)^{1/2}, \quad u = t\varphi_1 + v,\ t \in \mathbb{R},\ v \in V_o,$$

is a norm equivalent to the standard norm in $H^1(\Omega)$.

4. Proofs of Theorems

In this section, using a constrained minimization method, we present the proof of Theorem 1.1. After that we prove Theorems 1.2 and 1.4. To prove Theorem 1.1, we define $J(u) = \int_\Omega a(x)|u|^{p+1}$ and set

$$S_g := \{u \in H^1(\Omega) : Q_g(u) = 1\},$$

for every $g \in L^{N/2}(\Omega)$, where $Q_g(u) = \int_\Omega |\nabla u|^2 + g(x)u^2$. Consider the maximization problem:

$$\alpha_g := \sup_{u\in S_g} J(u).$$

Lemma 4.1. *Given $g \in L^{N/2}(\Omega)$, then $S_g \neq \emptyset$. Furthermore, if* (A1) *holds, then $\alpha_g > 0$.*

Proof. Given an arbitrary subdomain $\hat{\Omega}$ of Ω, we let $\{\omega_j\}$ be a sequence in $H_0^1(\hat{\Omega})$ such that

$$\omega_j \rightharpoonup 0 \quad \text{weakly in } H_0^1(\hat{\Omega}) \text{ and } \int_\Omega |\nabla \omega_j|^2 = 1 \text{ for every } j.$$

Noting that $H_0^1(\hat{\Omega})$ is a closed subspace of $H^1(\Omega)$, we also have that $\omega_j \rightharpoonup 0$ weakly in $H^1(\Omega)$. Therefore, invoking Lemma 2.6 and the fact that $g \in L^{N/2}(\Omega)$, we obtain

$$Q_g(\omega_j) = \int_\Omega |\nabla \omega_j|^2 + \int_\Omega g(x)\omega_j{}^2 \to 1.$$

Hence, for j sufficiently large, we may find $t_g = t_g(j) > 0$ such that $Q_g(t_g\omega_j) = t_g^2 Q_g(\omega_j) = 1$. This implies that $S_g \neq \emptyset$.

When (A1) holds, taking $\hat{\Omega} \subset \Omega^+$, the above argument may be used to conclude that $\alpha_g > 0$. Indeed by construction $\omega_j \in H_0^1(\hat{\Omega})\setminus\{0\}$ and, consequently, $J(t_g w_j) = t_g^{p+1}\int_{\Omega^+} a(x)|w_j|^{p+1} > 0$. The proof of Lemma 4.1 is complete. □

Considering the sequence $\{m_j\}_j$, given by (M1), we set $S_j := S_{m_j} \neq \emptyset$ and $\alpha_j := \alpha_{m_j} > 0$. Moreover, as in Section 3, we set $m_o := m$. In the next results we verify the existence of a nonnegative function $u_j \in S_j$ such that $J(u_j) = \alpha_j$. After that we prove Theorem 1.1 by rescaling u_j and using Lagrange's Theorem and Theorems 2.2 and 2.3.

Applying Lemmas 2.5 and 3.6, we may find $\delta > 0$ such that, for every $j \geq j_o$, $\|\cdot\|_{V_j}$ satisfies (3.17) and

$$\int_\Omega a(x)|\varphi_{1,j} + v|^{p+1} < \frac{1}{2}\int_\Omega a(x)\varphi_1^{p+1} < 0, \quad \forall v \in V_j,\ \|v\|_{V_j}^2 < \delta. \tag{4.1}$$

Lemma 4.2. *Assume* (A1)–(A3), (M1). *Then there exists $j_1 \in \mathbb{N}$ such that, for every $j \geq j_1$, we may find $u = u_j \in S_j$, $u \geq 0$ and $J(u) = \alpha_j > 0$.*

Proof. Considering $j \geq j_o$, where j_o is given by (4.1), from (A2), (M1) and Theorem 3.4, we may find $j_1 \geq j_o$ such that $|\lambda_{1,j}| < \delta/2$ for every $j \geq j_1$. For the rest of this article, we fix $j \geq j_1$. Let $\{u_k\}_k \subset S_j$ be sequence such that

$$J(u_k) \to \alpha_j > 0 \quad \text{as } k \to \infty. \tag{4.2}$$

Using the decomposition $H^1(\Omega) = \mathbb{R}\varphi_{1,j} \oplus V_j$, we may write $u_k = t_k\varphi_{1,j} + v_k$, with $t_k = \int_\Omega u_k\varphi_{1,j} \in \mathbb{R}$ and $v_k = u_k - t_k\varphi_{1,j} \in V_j$. By Theorem 3.4 and Lemma 3.6, there is $C > 0$ such that,

$$\|u_k\|_{H^1(\Omega)}^2 \leq 2[t_k^2\|\varphi_{1,j}\|_{H^1(\Omega)}^2 + \|v_k\|_{H^1(\Omega)}^2] \leq C[t_k^2 + \|v_k\|_{V_j}^2]. \tag{4.3}$$

Furthermore,

$$1 = Q_{m_j}(u_k) = \|v_k\|_{V_j}^2 + \lambda_{1,j}\, t_k^2. \tag{4.4}$$

We assert that $\{u_k\}_k$ is a bounded sequence in $H^1(\Omega)$. Since $|\lambda_{1,j}| < \delta/2$, in view of (4.4) and (4.3), it suffices to show that $\{t_k\}_k \subset \mathbb{R}$ is bounded. Arguing by contradiction and taking a subsequence if necessary, we suppose that $|t_k| \to \infty$. Consequently, from (4.4),

$$\limsup_{k\to\infty} \|\frac{v_k}{t_k}\|_{V_j}^2 \leq \limsup_{k\to\infty}\{\frac{1}{t_k^2} + |\lambda_{1,j}|\} = |\lambda_{1,j}| \leq \frac{\delta}{2}.$$

Hence, using (4.1) one more time,

$$\lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} |t_k|^{p+1}\int_\Omega a(x)|\varphi_{1,j} + \frac{v_k}{t_k}|^{p+1} = -\infty.$$

This contradicts (4.2). The sequence $\{u_k\}_k \subset H^1(\Omega)$ is bounded as asserted. Using this assertion, we may suppose that $t_k \to t \in \mathbb{R}$, $v_k \rightharpoonup v$ weakly in V_j. Consequently $u_k \rightharpoonup u = t\varphi_{1,j} + v$ weakly in $H^1(\Omega)$ and

$$0 < \alpha_j = \lim_{k\to\infty} \int_\Omega a(x)|u_k|^{p+1} = \int_\Omega a(x)|u|^{p+1}. \tag{4.5}$$

It remains to show that $u \in S_j$. Applying Lemma 2.6, we have

$$1 = \liminf_{k\to\infty} Q_{m_j}(u_k) \geq \int |\nabla u|^2 + m_j(x)u^2 = Q_{m_j}(u).$$

Actually $Q_{m_j}(u) = 1$. Indeed, if $Q_{m_j}(u) = \|v\|_{V_j}^2 + \lambda_{1,j}t^2 \leq 0$, then $\|v\|_{V_j}^2 = 0$ for $t = 0$, or $\|v/t\|_{V_j}^2 \leq |\lambda_{1,j}| < \delta/2$ if $t \neq 0$. In both cases, by (4.1), $J(u) \leq 0$. This contradicts (4.5). Therefore we may suppose that $0 < Q_{m_j}(u) \leq 1$ and that there exists $\rho \geq 1$ such that $Q_{m_j}(\rho u) = 1$. Since $J(\rho u) = \rho^{p+1}J(u) = \rho^{p+1}\alpha_j$, we obtain $\rho = 1$, $Q_{m_j}(u) = 1$ and $u \in S_j$. Finally, observing that J and Q_{m_j} are even, we may suppose $u \geq 0$. The proof of Lemma 4.2 is complete. □

Proof of Theorem 1.1. By Lemmas 4.1, 4.2 and Lagrange's Theorem, there exist $j_1 \in \mathbb{N}$ and $u_j \in S_j, u_j \geq 0$, for every $j \geq j_1$, such that

$$\int_\Omega \nabla u_j \nabla w + m_j(x)u_j w = \frac{1}{\alpha_j}\int_\Omega a(x)|u_j|^{p-1}u_j w, \quad \forall w \in H^1(\Omega). \tag{4.6}$$

That is, u_j is a nonnegative weak solution for the problem

$$-\Delta u + m_j(x)u = \frac{1}{\alpha_j}a(x)u^p, \quad x \in \Omega,\ u \in H^1(\Omega),$$

under the Neumann boundary conditions. Applying Theorems 2.2 and 2.3, we obtain $u_j > 0$. Setting $\omega_j := (1/\alpha_j)^{1/(p-1)}u_j$, we have that $\omega_j > 0$ is a weak solution for (1.1) with m_j instead of m for all $j \geq j_1$. The proof of Theorem 1.1 is complete. □

Before proving Theorem 1.2, we consider the following lemma.

Lemma 4.3. *Assume* (A1)–(A3), (M1). *Then there exists $M > 0$ and $j_2 \in \mathbb{N}$, such that $\|u_j\|_{H^1(\Omega)} \leq M$ for each $u_j \in S_j$ satisfying $J(u_j) = \alpha_j$, for all $j \geq j_2$.*

Proof. Writing $u_j = s\varphi_1 + v \in \mathbb{R}\varphi_1 \oplus V$, if $s \neq 0$,

$$0 < \alpha_j = J(u_j) = |s|^{p+1}\int_\Omega a(x)|\varphi_1 + \frac{v}{s}|^{p+1}.$$

By Lemma 2.5, $\|v\|_{H^1(\Omega)} > \beta|s|$. From Remark 3.7, there exists $C > 1$ such that

$$\|u_j\|_{H^1(\Omega)} \leq \big(\frac{1}{\beta}\|\varphi_1\|_{H^1(\Omega)} + 1\big)\|v\|_{H^1(\Omega)} \leq C\|v\|_V. \tag{4.7}$$

Note that (4.7) is trivially true when $s = 0$. Hence, by Corollary 2.9, given $0 < \epsilon < 1/C$, there exists $j_2 \in \mathbb{N}$ such that

$$\begin{aligned} 1 = Q_{m_j}(u_j) &= \|v\|_V^2 + \int_\Omega [m_j(x) - m(x)]u_j^2 \\ &\geq \|v\|_V^2 - \epsilon\|u_j\|_{H^1(\Omega)}^2 \\ &\geq \big(\frac{1}{C} - \epsilon\big)\|u_j\|_{H^1(\Omega)}^2, \quad \forall j \geq j_2. \end{aligned}$$

The proof of Lemma 4.3 is complete. □

Proof of Theorem 1.2. Defining $u_j := (\alpha_j)^{1/(p-1)} w_j$, where $\{w_j\}_j$ is the sequence of solutions from Theorem 1.1, we obtain $u_j > 0$, $u_j \in S_j$ and $J(u_j) = \alpha_j$. Moreover u_j satisfies (4.6). From (M1), Lemma 4.3, Lemma 2.6, Corollary 2.9 and taking a subsequence if necessary, we may suppose that

$$u_j \rightharpoonup u \quad \text{weakly in } H^1(\Omega), \tag{4.8}$$

$$u_j \to u \quad \text{strongly in } L^\tau(\Omega),\ 1 \le \tau < 2^*, \tag{4.9}$$

$$\int_\Omega m_j(x) u_j^2 \to \int_\Omega m(x) u^2, \tag{4.10}$$

$$\int_\Omega m_j(x) u_j \phi \to \int_\Omega m(x) u \phi, \quad \forall \phi \in H^1(\Omega), \tag{4.11}$$

$$\int_\Omega [m_j(x) - m(x)] u_j^2 \to 0. \tag{4.12}$$

Applying the same arguments used in the proof of Lemma 4.2, we find a positive solution $u_o \in H^1(\Omega)$ for the maximization problem

$$0 < \alpha_o := \sup_{Q_m(u)=1} J(u) = \int_\Omega a(x)|u_o|^{p+1} < \infty.$$

We claim that $\alpha_j \to \alpha_o$. Indeed, by (M1), $Q_{m_j}(u_o) \to 1$. Thus there exist j_o and $\{\theta_j\}_j \subset (0,\infty)$ such that $Q_{m_j}(\theta_j u_o) \equiv 1$ for all $j \ge j_o$. Since $\theta_j^2 Q_{m_j}(u_o) \equiv 1$,

$$\lim_{j\to\infty} \theta_j = 1.$$

Using this and the fact that $\theta_j u_o \in S_j$,

$$\liminf_{j\to\infty} \alpha_j \ge \liminf_{j\to\infty} \int_\Omega a(x)|\theta_j u_o|^{p+1} = \liminf_{j\to\infty} |\theta_j|^{p+1} \alpha_o = \alpha_o > 0. \tag{4.13}$$

On the other hand, from (4.12),

$$Q_m(u_j) = Q_{m_j}(u_j) + \int_\Omega [m(x) - m_j(x)] u_j^2 = 1 + \int_\Omega [m(x) - m_j(x)] u_j^2 \to 1.$$

Thus, taking $\beta_j > 0$ such that $Q_m(\beta_j u_j) \equiv 1$, we have that $\beta_j \to 1$. Hence, from the definition of α_o,

$$\alpha_o \ge J(\beta_j u_j) = \beta_j^{p+1} \alpha_j,$$

$$\limsup_{j\to\infty} \alpha_j \le \limsup_{j\to\infty} \frac{\alpha_o}{\beta_j^{p+1}} = \alpha_o. \tag{4.14}$$

We conclude from (4.13) and (4.14) that $\alpha_j \to \alpha_o > 0$. The claim is proved.

By the above claim and (4.9),

$$\alpha_o = \lim_{j\to\infty} \alpha_j = \lim_{j\to\infty} \int_\Omega a(x)|u_j|^{p+1} = \int_\Omega a(x)|u|^{p+1}. \tag{4.15}$$

Hence $u \ne 0$. Since $u_j > 0$ satisfies (4.6) for each $j \ge j_1$, using (4.8), (4.9),(4.10) and (4.11), we obtain

$$\int_\Omega \nabla u \nabla \varphi + m(x) u \varphi = \frac{1}{\alpha_o} \int_\Omega a(x)|u|^{p-1} u \varphi \quad \forall\, \varphi \in H^1(\Omega).$$

In particular $w_o := \alpha_o^{-1/(p-1)} u$ is a solution for (1.1). A similar argument shows that

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega |\nabla u_j|^2 &= \lim_{j\to\infty}\Big\{\frac{1}{\alpha_j}\int_\Omega a(x)|u_j|^{p+1} - \int_\Omega m_j(x)u_j^2\Big\}\\ &= \frac{1}{\alpha_o}\int_\Omega a(x)|u|^{p+1} - \int_\Omega m(x)u^2 = \int_\Omega |\nabla u|^2.\end{aligned}$$

Consequently, we have that $u_j \to u$ strongly in $H^1(\Omega)$ and $Q_m(u) = 1$. Thus $u \in S_o = S_m$ and, by (4.15), $J(u) = \alpha_o$. As before by Theorems 2.2 and 2.3, $u > 0$. Moreover

$$w_j := \big(\frac{1}{\alpha_j}\big)^{\frac{1}{p-1}} u_j \to w_o \quad \text{in } H^1(\Omega).$$

The proof of Theorem 1.2 is complete. □

Remark 4.4. Note that under the hypotheses (A1)–(A3) and (M1), there exist $r_0 > 0$ and $j = j_0$ such that (1.1), with $m_{j_n} - \tau_n$ instead of m, possesses a positive solution whenever $\tau \in [0, r_0)$ and $j \geq j_0$. Indeed, arguing by contradiction, we find sequences $\{\tau_n\} \in [0,\infty)$ and $\{j_n\}$ such that (1.1), with $m_{j_n} - \tau_n$ instead of m, has no positive solution and $\tau_n \to 0$, $j_n \to \infty$ as $n \to \infty$. However, by Theorem 1.1, for every n sufficiently large, (1.1), with $m_{j_n} - \tau_n$ instead of m, possesses a positive solution since $m_{j_n} - \tau_n \rightharpoonup m$ weakly in $L^\sigma(\Omega)$ as $n \to \infty$. This contradiction implies that the result holds.

It is worthwhile mentioning that, by Theorem 3.4, $\lambda_1(-\Delta + m_j - \tau) \to \lambda_1(-\Delta + m - \tau) = -\tau < 0$ for every $\tau > 0$. We also note that the existence of positive solutions for (1.1), with $m - \tau$ instead of m, was considered in [5].

Proof of Theorem 1.4. Arguing by contradiction, we suppose that there exist $R > 0$ (or $f \in L^{N/2}(\Omega)$ if $\sigma = N/2$) and a sequence $\{m_j\}_j \subset L^r(\Omega)$ such that (i) $\|m - m_j\|_{L^\sigma(\Omega)} \leq R$ (or $|m_j| \leq f$ a.e. in Ω, if $\sigma = N/2$), (ii) $Q_{m_j}(\varphi_1) \leq 1/j$, (iii) $-1/j < \lambda_1(-\Delta + m_j) < 0$, and (1.1) with m_j instead of m has no solution for all $j \in \mathbb{N}$. Taking a subsequence if necessary, $m_j \rightharpoonup \widehat{m}$ weakly in $L^\sigma(\Omega)$. Then, from (A2), Theorem 3.4 and (ii) and (iii) above,

$$\lambda_1(-\Delta + \widehat{m}) = \lim_{j\to\infty} \lambda_1(-\Delta + m_j) = 0,$$

$$0 \leq \int_\Omega |\nabla\varphi_1|^2 + \widehat{m}(x)\varphi_1^2 = \lim_{j\to\infty}\int_\Omega |\nabla\varphi_1|^2 + m_j(x)\varphi_1^2 \leq 0.$$

Hence φ_1 is an eigenfunction for (1.2), with $\widehat{m}$ instead of m, associated with $\lambda_1(-\Delta + \widehat{m}) = 0$. Consequently

$$\int_\Omega \nabla\varphi_1\nabla\omega + \widehat{m}(x)\varphi_1\omega = 0 = \int_\Omega \nabla\varphi_1\nabla\omega + m(x)\varphi_1\omega, \quad \forall \omega \in H^1(\Omega),$$

and

$$\int_\Omega (\widehat{m}(x) - m(x))\varphi_1\omega = 0 \quad \forall \omega \in H^1(\Omega).$$

This implies that $(\widehat{m} - m)\varphi_1 = 0$ a.e. in Ω. Since $\varphi_1 > 0$, $\widehat{m} \equiv m$. Therefore, $m_j \rightharpoonup m$ weakly in $L^\sigma(\Omega)$ and $\{m_j\}_j$ satisfies the condition (M1). From (A1), (A2), (A3) and Theorem 1.1, (1.1) with m_j instead of m possesses a solution for j sufficiently large. This contradiction concludes the proof of Theorem 1.4. □

We conclude this article by presenting an application of Theorems 1.1 and 1.2 to the problem

$$\begin{gathered} -\Delta u+\varphi(x)\cos(jx_N)u=a(x)u^p, \quad x\in\Omega, \\ \frac{\partial u}{\partial\eta}=0 \quad x\in\partial\Omega, \end{gathered}$$

where $\varphi\in C_o^\infty(\Omega)$ is a nontrivial function, $1<p<(N+2)/(N-2)$ and $x_N\in\mathbb{R}$ is the last component of $x=(x_1,\dots,x_N)\in\Omega$.

Assuming that a changes sign and satisfies $\int_\Omega a(x)<0$, we may assert that this problem has a positive solution u_j for every $j\in\mathbb{N}$ sufficiently large. Moreover, up to a subsequence, $u_j\to u$ in $H^1(\Omega)$ where $u>0$ is a solution of (1.1) with 0 instead of m. Indeed, to derive such result, it suffices to observe that $m_j:=\varphi(x)\cos(jx_N)\in L^\infty(\Omega)$ and $m_j\rightharpoonup 0$ weakly in $L^s(\Omega)$, for all $s>1$. We note that m_j does not converge strongly in any $L^s(\Omega)$. Hence, in particular, the results from [1] and [12] may not be applied to this problem since $\{m_j\}_j$ does not converge uniformly to zero.

5. Appendix: Regularity of solutions

For the convenience of the reader, we present the argument necessary to establish the regularity of the solutions for (1.2) and (1.1). First we recall the following results.

Lemma 5.1 ([13]). *Suppose $\partial\Omega\in C^1$, $b\in L^{N/2}(\Omega)$ and $\alpha\in L^\infty(\Omega)$. If $u\in H^1(\Omega)$ is a weak solution of*

$$\begin{gathered} -\Delta u=b(x)u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial\eta}=\alpha(x)u \quad \text{on } \partial\Omega, \end{gathered}$$

then $u\in L^t(\Omega)$ for all $1\le t<\infty$.

Lemma 5.2 ([2]). *Suppose $\partial\Omega\in C^2$, $h\in L^s(\Omega)$ and $g\in H^{1,s}(\Omega)$ for some $s\in(1,\infty)$. If $u\in H^1(\Omega)$ is a weak solution of*

$$\begin{gathered} -\Delta u=h(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial\eta}=g(x) \quad \text{on } \partial\Omega, \end{gathered}$$

then $u\in W^{2,s}(\Omega)$ and $\|u\|_{W^{2,s}(\Omega)}\le C(\|h\|_{L^s(\Omega)}+\|g\|_{W^{1,s}(\partial\Omega)}+\|u\|_{L^s(\Omega)})$ for some $C>0$.

The above lemma due to Agmon, Douglis and Nirenberg is also cited in [13]. Now, as an application of the above mentioned results, we sate the following result.

Lemma 5.3. *Suppose $m\in L^\sigma(\Omega)$, $\sigma\ge N/2$, φ is an eigenfunction for (1.2) and u is a solution for (1.1). Then $\varphi,u\in L^t(\Omega)$, for $1\le t<\infty$. Moreover $\varphi,u\in C^{0,\gamma}(\overline{\Omega})$ provided $\sigma>\frac{N}{2}$. If $\sigma=\frac{N}{2}$ then $\varphi,u\in W^{2,s}(\Omega)$ for all $s\in[2N/(N+2),N/2)$.*

Proof. Defining the functions $b_1:=\lambda-m$ and $b_2:=au^{p-1}-m$, from Lemma 5.1 we obtain $\varphi,u\in L^t(\Omega)$ for all $1\le t<\infty$. Next, considering $\sigma>N/2$ and setting $h_1:=b_1\varphi$ and $h_2:=b_2u$, it is easy to see that $h_i\in L^s(\Omega)$ for all $s\in(N/2,\sigma)$, $i=$

$1, 2$. Indeed, for each $s \in (N/2, \sigma)$, choose $\alpha > \sigma/(\sigma - s)$ and take $\theta > 1$ such that $1/\theta + s/\sigma + 1/\alpha = 1$. Then, by Hölder inequality,

$$\|mu\|_s^s \leq |\Omega|^{1/\theta} \|m\|_\sigma^s \|u\|_{s\alpha}^s < \infty.$$

Since $au^p \in L^s(\Omega)$ and $s > N/2$, we may apply the Lemma 5.2 and the Sobolev Imbedding Theorem to conclude that

$$\varphi, u \in W^{2,s}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega}) \quad \text{for some } 0 < \gamma < 1.$$

In the case $\sigma = N/2$, given $s \in [2N/(N+2), N/2)$, we choose $\alpha > N/(N-2s)$ and take $\theta > 1$ such that $1/\theta + 2s/N + 1/\alpha = 1$. By Hölder inequality $mu \in L^s(\Omega)$. Applying Lemma 5.2, we conclude that $\varphi, u \in W^{2,s}(\Omega)$ for all $s \in [2N/(N+2), N/2)$. □

Remark 5.4. If $\sigma > N$, then $h_i \in L^s(\Omega)$ for all $s \in (N, \sigma)$. Repeating the above steps, $\varphi, u \in C^{1,\gamma}(\overline{\Omega})$, $i = 1, 2$.

Proofs of Theorems 3.2 and 3.3

As observed in Section 3, the arguments employed in the proofs of Theorems 3.2 and 3.3 are due to Manes and Micheletti [10]. In order to verify that the first eigenvalue is simple and that the first eigenfunction, φ_1, is positive (almost everywhere) in Ω, we apply Theorems 2.2 and 2.3 (see Remark 2.4).

Proof of Theorem 3.2. By Hölder's inequality and Sobolev Imbedding Theorem, there exists $C > 0$ such that

$$|Q_m(u)| \leq (1 + C\|m\|_{N/2})\|u\|_{H^1(\Omega)}^2. \tag{5.1}$$

As a direct consequence of (5.1), $\lambda_1 < \infty$. Now let $\{u_k\}_k \subset H^1(\Omega)$ be a sequence such that $\int u_k^2 = 1$ and $Q_m(u_k) \to \lambda_1$. We claim that $\{\|u_k\|_{H^1(\Omega)}\}_k$ is bounded. Otherwise, from (5.1), up to a subsequence, $\|\nabla u_k\|_2 \to \infty$. Defining $w_k := \frac{u_k}{\|\nabla u_k\|_2}$ and applying Sobolev Imbedding Theorem one more time, we may suppose $w_k \rightharpoonup 0$ weakly in $H^1(\Omega)$ since $\|u_k\|_2 \equiv 1$. By Lemma 2.6, $\int m(x) w_k^2 \to 0$. Thus, since

$$\frac{Q_m(u_k)}{\|\nabla u_k\|_2^2} = 1 + \int m(x) w_k^2, \tag{5.2}$$

$Q_m(u_k) \to \infty$. This contradicts $\lambda_1 < \infty$. Thus $\{u_k\}_k \subset H^1(\Omega)$ is a bounded sequence as claimed. Passing to a subsequence if necessary, $u_k \rightharpoonup \varphi_1$ weakly in $H^1(\Omega)$, $\|\varphi_1\|_2 = 1$ and, by Lemma 2.6,

$$\int m(x) u_k^2 \to \int m(x) \varphi_1^2.$$

Then, since $\liminf_{k\to\infty} \|\nabla u_k\|_2^2 \geq \|\nabla \varphi\|_2^2$,

$$\lambda_1 = \liminf_{k\to\infty} Q_m(u_k) \geq Q_m(\varphi_1).$$

This implies that $Q_m(\varphi_1) = \lambda_1$. Now we show that φ_1 solves (1.2) with λ_1 instead of λ. Given $u \in H^1(\Omega)$, for every $t \in \mathbb{R}\backslash\{0\}$ sufficiently small,

$$\frac{Q_m(\varphi_1 + tu)}{\int (\varphi_1 + tu)^2} = \frac{\int |\nabla(\varphi_1 + tu)|^2 + m(x)(\varphi_1 + tu)^2}{\int (\varphi_1 + tu)^2} \geq \lambda_1.$$

Consequently,

$$2t\int \nabla u\nabla\varphi_1 + m(x)u\varphi_1 + t^2Q_m(u) \geq 2t\lambda_1\int u\varphi_1 + t^2\lambda_1\int u^2. \tag{5.3}$$

Dividing (5.3) by $2t$ and taking $t\to 0$ ($t\to 0^+$ and $t\to 0^-$), we obtain

$$\int \nabla u\nabla\varphi_1 + m(x)u\varphi_1 = \lambda_1\int u\varphi_1, \quad \forall u\in H^1(\Omega). \tag{5.4}$$

This shows that φ_1 is a solution for (1.2) with λ_1 instead of λ. As observed in Remark 3.1, this also implies that λ_1 is the first eigenvalue for (1.2). Notice that we may suppose $\varphi_1\geq 0$ since $Q_m(|u|)=Q_m(u)$. By Remark 2.4, φ_1 is positive in Ω.

Finally we verify that λ_1 is a simple eigenvalue and that any associated eigenfunction $v\in H^1(\Omega)\backslash\{0\}$ does not change sign. A simple arithmetic shows that: for every $A,B,X,Y\in\mathbb{R}$ with $X,Y>0$, we have either

$$\frac{A+B}{X+Y}=\frac{A}{X}=\frac{B}{Y}$$

or

$$\min\big\{\frac{A}{X},\frac{B}{Y}\big\}<\frac{A+B}{X+Y}<\max\big\{\frac{A}{X},\frac{B}{Y}\big\}.$$

Since

$$\lambda_1=\frac{Q_m(v)}{\int v^2}=\frac{Q_m(v^+)+Q_m(v^-)}{\int(v^+)^2+\int(v^-)^2},$$

if $v^+:=\max\{v,0\}\geq 0$ and $v^-:=\max\{-v,0\}\geq 0$ are nontrivial functions, we have

$$(I)\quad \lambda_1=\frac{Q_m(v^+)}{\int(v^+)^2}=\frac{Q_m(v^-)}{\int(v^-)^2}$$

or

$$(II)\quad \lambda_1>\min\big\{\frac{Q_m(v^+)}{\int(v^+)^2},\frac{Q_m(v^-)}{\int(v^-)^2}\big\}.$$

In case I, we also have that v^+ and v^- are (after normalization) eigenfunctions associated with λ_1. Applying Theorems 2.2 and 2.3, with $c(x):=m(x)-\lambda_1$, we have simultaneously $v^+>0$ and $v^->0$ in Ω, which it is not possible. On the other hand, case II contradicts the definition of λ_1. This shows that v does not change sign in Ω.

Now, given any $\alpha\in\mathbb{R}$, by the above claim the function $\varphi_1-\alpha v$ does not change sign in Ω. Defining

$$A:=\{\alpha\in\mathbb{R}:\varphi_1\geq\alpha v \text{ a.e.}\},\quad B:=\{\alpha\in\mathbb{R}:\varphi_1\leq\alpha v\text{ a.e.}\},$$

it is not difficult to verify that A and B are closed subsets of $\mathbb{R}$. Since A and B are nonempty and $A\cup B=\mathbb{R}$, there is $\alpha_o\in A\cap B$ and, consequently, $\varphi_1=\alpha_o v$ a.e. in Ω. Therefore, λ_1 is a simple eigenvalue. The verification that λ_1 is isolated will be made in the proof of Theorem 3.3. The proof of Theorem 3.2 is complete. □

Proof of Theorem 3.3. By definition 3.4, $\lambda_2\geq\lambda_1$. Similar argument to the one used in the proof of Theorem 3.2 implies that there exists $\varphi_2\in V$ associated with λ_2. In order to show that $\lambda_1<\lambda_2$ we suppose by contradiction that $\lambda_1=\lambda_2$. Then, by Theorem 3.2, there exists $\alpha\in\mathbb{R}\backslash\{0\}$ such that $\varphi_1=\alpha\varphi_2$. But this contradicts $\int\varphi_1\varphi_2=0$. Moreover it is clear, from the definition of V, that φ_2 changes sign in Ω.

Now we show that φ_2 is a solution for (1.2): given $u \in H^1(\Omega)$, we write $u = t\varphi_1 + v$, with $t = \int \varphi_1 u \ \in \mathbb{R}$ and $v = u - t\varphi_1 \ \in V$. Arguing as in the verification of (5.4),

$$\int \nabla v \nabla \varphi_2 + m(x) v \varphi_2 = \lambda_2 \int v \varphi_2, \quad \forall v \in V. \tag{5.5}$$

Noting that $\int \varphi_1(\varphi_2 + sv) = 0$ for each $s \in \mathbb{R}$ and $v \in V$ and, in view of (5.4) and (5.5)), we obtain

$$\begin{aligned}\int \nabla u \nabla \varphi_2 + m(x) u \varphi_2 &= t \int \nabla \varphi_1 \nabla \varphi_2 + m(x) \varphi_1 \varphi_2 + \int \nabla v \nabla \varphi_2 + m(x) v \varphi_2 \\ &= \lambda_2 \int u \varphi_2, \quad \forall u \in H^1(\Omega).\end{aligned}$$

Thus φ_2 is an eigenfunction for (1.2). Our final task is to verify that (1.2) has no solution if $\lambda \in (\lambda_1, \lambda_2)$. That is, λ_1 is isolated. Otherwise, take φ a normalized eigenfunction for (1.2) with $\lambda \in (\lambda_1, \lambda_2)$. Hence

$$\lambda \int \varphi \varphi_1 = \int \nabla \varphi_1 \nabla \varphi + m(x) \varphi_1 \varphi = \lambda_1 \int \varphi \varphi_1.$$

Since $(\lambda_1 - \lambda) \neq 0$, $\int \varphi \varphi_1 = 0$. Thus $\varphi \in V$. However this contradicts the definition of λ_2, since $\lambda = \lambda \int \varphi^2 = Q_m(\varphi) < \lambda_2$. The proof of Theorem 3.3 is complete. $\square$

References

[1] G. A. Afrouzi; *Existence of positive solutions on indefinite superlinear elliptic equations*, Applied Math. Computation **157** (2004) 841–848.

[2] S. Agmon, A. Douglis, L. Nirenberg; *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Applied Math. **XII**, 623–727, (1959).

[3] S. Alama, M. Del Pino; *Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking*, Ann. Inst. H. Poincaré Anal. Non Lineaire **13** (1996) 95-115.

[4] S. Alama, G. Tarantello; *On semilinear elliptic equations with indefintite nonlinearities*, Calc. Var. **1** (1993) 439-475.

[5] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg; *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA 2 No 4 (1995) 553–572.

[6] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg; *Superlinear indefinite elliptic problems and nonlinear Liouville theorems*, Topol. Methods Nonlinear Anal. **4** (1994) 59-78.

[7] D. deFigueiredo; *Differential equations*, Lecture Notes in Mathematics 957. Springer-Verlag, Berlin-New York, 34-87, 1982.

[8] D. deFigueiredo, J. P. Gossez; *Strict monotonicity of the eigenvalues and unique continuation*, Comm. Partial Differential Equations 17 no. 1-2 (1992) 339–346 .

[9] D. Jerison, C. E. Kenig; *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Annals of Mathematics **121** (1985) 463-494.

[10] M. Manes, A. M. Micheletti; *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine*, Bolletino U.M.I 7 (1973) 285-301.

[11] M. Ramos, S. Terracini, C. Troestler; *Superlinear indefinite elliptic problems and Pohozaev type identities*, J. Funct. Anal. **159** (1998) 596-628.

[12] H. Tehrani; *On indefinite superlinear elliptic equations*, Cal. Var. **4** (1996) 139-153.

[13] Xu-Jia Wang; *Neumann Problems of Semilinear Elliptic Equations Involving Critical Sobolev Exponent*, J. Diff. Eq., **93**, 283-310, (1991).

[14] M. Willem; *Minimax Theorems*, Birkhäuser, Basel, 1996.

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