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UNIQUE CONTINUATION PRINCIPLE FOR HIGH ORDER EQUATIONS OF KORTEWEG-DE VRIES TYPE

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Abstract. In this article we consider the problem of unique continuation for high-order equations of Korteweg-de Vries type which include the kdV hierarchy. It is proved that if the difference w of two solutions of an equation of this form has certain exponential decay for $x > 0$ at two different times, then w is identically zero.

1. Introduction

This article concerns a unique continuation principle for the equation

$$\partial_t u + (-1)^{k+1}\partial_x^n u + P(u, \partial_x u, \dots, \partial_x^p u) = 0\,, \quad u = u(x,t), \quad x, t \in \mathbb{R}, \tag{1.1}$$

where $n = 2k+1$, $k = 1, 2\dots$, and P is a polynomial in $u, \partial_x u, \dots, \partial_x^p u$, with $p \le n-1$. In particular, we will focus our attention to the case in which P has the form

$$\begin{gathered} P(u, \partial_x u, \dots, \partial_x^{n-2} u) = \sum_{d=2}^{k+1} \sum_{|m|=2(k+1-d)+1} a_{d,m}\, \partial_x^{m_1} u \dots \partial_x^{m_d} u \equiv \sum_{d=2}^{k+1} A_d(z), \\ z = (u, \partial_x u, \dots, \partial_x^{n-2} u), \end{gathered} \tag{1.2}$$

where, for $d \in \mathbb{N}$ and for integers $m_1, \dots, m_d$, $m := (m_1, \dots, m_d)$ is a multi-index with $0 \le m_1 \le \dots \le m_d$, $|m| := m_1 + \dots + m_d$, and $a_{d,m}$ is a constant. We will refer to equation (1.1) with P defined by (1.2) as equation (1.1)-(1.2). We will also consider equation (1.1) when the nonlinearity P has order $p \le k$.

The type of relation expressed in (1.2), between the degree and the order of each monomial of P, is present in the nonlinearities of the collection of equations known as the KdV (Korteweg-de Vries) hierarchy. This set of equations was introduced by Lax [15] in the process to determine the functions $u = u(x,t)$ for which the eigenvalues of the operator $L := \frac{d^2}{dx^2} - u(\cdot, t)$ remain constant as t evolves. This property had been already discovered by Gardner et al. in [3] for the solutions of the Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0.$$

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Lax [15] showed that this property holds for the solutions of the equations

$$\partial_t u + [B_k(u), L] = 0, \tag{1.3}$$

where $[B, L] := BL - LB$ denotes the commutator of B and L, and $B_k(u)$ is the skew-adjoint operator defined by

$$B_k(u) = b_k \frac{d^{2k+1}}{dx^{2k+1}} + \sum_{j=0}^{k-1} b_{k,j}(u) \frac{d^{2j+1}}{dx^{2j+1}} + \frac{d^{2j+1}}{dx^{2j+1}} b_{k,j}(u),$$

with the coefficients $b_{k,j}(u)$ chosen in such a way that the operator $[B_k(u), L]$ has order zero. It was proved in [16] that the equations in the KdV hierarchy (1.3) can be written in the form $\partial_t u + \partial_x G_{k+1}(u) = 0$, were the functions $G_k(u)$ are the gradients of the functionals $F_k(u)$ which define the conservation laws of the KdV equation. The gradients G_k satisfy the following recursion formula due to Lenard (see [4] and [20]):

$$\partial_x G_{k+1} = cJG_k\,, \quad \text{where } J = \partial_x^3 + \frac{2}{3} u \partial_x + \frac{1}{3} \partial_x u\,.$$

This formula can be applied to obtain a derivation of the equations in the hierarchy. Starting with $G_0(u) = 3$, with $k = 0$ we get the transport equation, with $k = 1$ the KdV equation, and, with $k = 2$, $k = 3$, and $k = 4$, we respectively find the equations

$$\begin{gathered}
\partial_t u + \partial_x^5 u - 10u\partial_x^3 u - 20\partial_x u \partial_x^2 u + 30u^2 \partial_x u = 0,\\
\partial_t u + \partial_x^7 u + 14u\partial_x^5 u + 42\partial_x u \partial_x^4 u + 70\partial_x^2 u \partial_x^3 u + 70u^2\partial_x^3 u\\
+ 280u\partial_x u \partial_x^2 u + 70(\partial_x u)^3 + 140u^3\partial_x u = 0,
\end{gathered}$$

$$\begin{aligned}
&\partial_t u + \partial_x^9 u + \sum_{\substack{m_1+m_2=7\\ 0\le m_1\le m_2}} a_{2,m}\, \partial_x^{m_1} u \partial_x^{m_2} u \\
&+ \sum_{\substack{m_1+m_2+m_3=5\\ 0\le m_1\le m_2\le m_3}} a_{3,m}\, \partial_x^{m_1} u \partial_x^{m_2} u \partial_x^{m_3} u + \dots + a_{5,m} u^4 \partial_x u = 0,
\end{aligned} \tag{1.4}$$

for certain constants $a_{d,m}$, with $d = 2, \dots, 5$ and $|m| = 2(5-d)+1$.

In spite of computational difficulties, it is possible to obtain exact expressions for all the equations in the hierarchy (see [1]). However, following a simple procedure, and without obtaining the explicit values for the coefficients, it can be proved (see [5]) that the equations in the KdV hierarchy (1.3) have the form of (1.1)-(1.2). When k is even we have made the change of variable $x \mapsto -x$ and thus the linear term $\partial_x^n u$ has been transformed into $(-1)^{k+1}\partial_x^n u$ in (1.1).

The aspects of local and global well-posedness of the initial value problem (IVP) associated with the general equation (1.1) have been considered in [10] and [11], where Kenig, Ponce, and Vega proved that the (IVP) is locally well-posed in weighted spaces $H^s(\mathbb{R}) \cap L^2(|x|^m\,dx)$ if $s \ge s_0(k)$, for some $s_0(k)$ and some integer $m = m(k)$.

For the (IVP) associated to (1.1)-(1.2), in [21], Saut proved the existence of global solutions for initial data in Sobolev spaces $H^m(\mathbb{R})$ for $m \ge k$, integer. By using a variant of Bourgain spaces, in [5], Grünrok proved the local well-posedness for the (IVP) of equation (1.1)-(1.2) in the context of the spaces

$$\widehat{H}_s^r(\mathbb{R}) := \{f \mid \|f\|_{s,r} := \|(1+\xi^2)^{s/2}\widehat{f}(\xi)\|_{L_\xi^{r'}} < \infty\},$$

with

$$r \in (1, \frac{2k}{2k-1}], \text{ and } s > k - \frac{3}{2} - \frac{1}{2k} + \frac{2k-1}{2r'}.$$

Here $\hat{}$ denotes the Fourier transform and $1/r + 1/r' = 1$. We also refer to the articles [9, 14, 17, 18, 19], which, among others, consider the problem of well-posedness for high order equations of KdV-type and especially for the equations of order five ($k = 2$).

Our main goal is to prove continuation principles for (1.1)-(1.2) with $n \geq 5$, which include the KdV hierarchy, and for the equations (1.1) with $n \geq 5$ and $p \leq k$. Roughly speaking, we will prove that if the difference $w := u_1 - u_2$ of two sufficiently smooth solutions of equation (1.1)-(1.2) decays as $\exp(-x_+^{4/3^+})$ at two different times, then $w \equiv 0$. (Here $x_+ := \frac{1}{2}(x + |x|)$, and $4/3^+$ means $4/3 + \epsilon$ for arbitrarily small $\epsilon > 0$). For (1.1) with $p \leq k$ we have a similar result if w decays as $\exp(-ax_+^{n/(n-1)})$ for $a > 0$ sufficiently large at two different times. This last result is coherent with the decay $\exp(-cx_+^{n/(n-1)})$ of the fundamental solution of the linear problem associated with (1.1) (see [23]). When the nonleariy P has higher order as in (1.1)-(1.2), it is then necessary to impose a stronger decay on w.

The aspect of unique continuation has been studied for a variety of non-linear dispersive equations, and especially for the KdV and Schrödinger equations. Saut and Sheurer [22] considered a class of nonlinear dispersive equations, which includes the KdV equation, and proved that if a solution u of one of such equations vanishes in an open set Ω of the space-time space, then u vanishes in all horizontal components of Ω, that is, in the set $\{(x,t) : \exists\, y \text{ with } (y,t) \in \Omega\}$.

By using methods of complex analysis, Bourgain [2] proved that if a solution u of the KdV equation is supported in a compact set $\{(x,t) : -B \leq x \leq B,\ t_0 \leq t \leq t_1\}$, then u vanishes identically.

Kenig, Ponce and Vega [12], considered a solution of the KdV equation which vanishes only in two half lines $[B, +\infty) \times \{t_0\}$ and $[B, +\infty) \times \{t_1\}$, and proved that this solution must be identically zero. A similar result was proved in [13] for the difference $w = u_1 - u_2$ of two solutions of the KdV equation. Escauriaza et al. [8] refined this result by only imposing the condition that $w(\cdot, t_0)$ and $w(\cdot, t_1)$ decay as $\exp(-ax_+^{\gamma})$, for $\gamma = 3/2$ and $a > 0$ sufficiently large, together with a additional hypothesis of polynomial decay for u_1 and u_2. This result is obtained by applying two types of estimates for the function w: Carleman type estimates, which express a boundedness of the inverse of the linear operator $\partial_t + \partial_x^3$ in $L^p - L^q$-spaces with exponential weight; and a so-called lower estimate which bounds the L^2-norm of w in a small rectangle at the origin with the H^2-norm of w in a distant rectangle $[R, R+1] \times [0, 1]$.

For the fifth-order equation (1.1) ($k = 2$), Dawson [6] proved a result similar to that in [8] with $\gamma = 4/3^+$ for the general case $p \leq n - 1 = 4$, and with $\gamma = 5/4$ for the case $p \leq 2$.

In this article we consider equations (1.1) and (1.1)-(1.2) with arbitrary order n and prove the continuation principles stated in Theorem 1.1 and 1.2 below. For that, we follow the method traced in [8]. The greatest difficulty in this process is to manage the huge amount of terms arising in the computations of the operators involved in the lower estimate. We consider that the main contribution of our work is the presentation of a clear and organized procedure to obtain the lower estimate (see Lemma 4.1 and Theorem 4.3).

We now state our main results.

Theorem 1.1. *For odd $n \geq 5$, $k = (n-1)/2$, and $\alpha > \frac{n+1}{3}$, suppose that u_1, u_2 are in $C([0,1]; H^{n+1}(\mathbb{R}) \cap L^2((1+x_+)^{2\alpha}\, dx))$ are two solutions of the equation*

$$\partial_t u + (-1)^{k+1}\partial_x^n u + P(u, \partial_x u, \dots, \partial_x^{n-2} u) = 0 \tag{1.5}$$

with P as in (1.2), and let $w := u_1 - u_2$. If

$$w(0), w(1) \in L^2(e^{2x_+^{4/3+\epsilon}}\, dx) \tag{1.6}$$

for some $\epsilon > 0$, then $w \equiv 0$.

The proof of this theorem can be adapted to obtain a similar continuation principle for equation (1.1) when $p \leq k$. In this case we require a weaker decay for $w(0)$ and $w(1)$ and consider some minor modifications in the polynomial decay hypothesis for u_1 and u_2. For the sake of simplicity we state this result without making special emphasis in the optimal value of α.

Theorem 1.2. *For odd $n \geq 5$, $k = (n-1)/2$, and $\alpha_0 > 0$ sufficiently large, suppose that u_1, $u_2 \in C([0,1]; H^{n+1}(\mathbb{R}) \cap L^2((1+|x|)^{2\alpha_0}\, dx))$ are two solutions of the equation*

$$\partial_t u + (-1)^{k+1}\partial_x^n u + P(u, \partial_x u, \dots, \partial_x^{p} u) = 0 \tag{1.7}$$

with $p \leq k$. Define $w := u_1 - u_2$. Then, there is $a > 0$, which depends only on n, such that if

$$w(0), w(1) \in L^2(e^{2a x_+^{n/(n-1)}}\, dx), \tag{1.8}$$

then $w \equiv 0$.

The article is organized as follows: In section 2 we prove that the exponential decay for w in the semi-axis $x > 0$ is preserved in time. In section 3 we establish the Carleman type estimates and in section 4 we prove the lower estimates. Finally we give the proofs of Theorems 1.1 and 1.2 in section 5.

Throughout the paper the letters C and c will denote diverse positive constants which may change from line to line, and whose dependence on certain parameters is clearly established in all cases. Sometimes, for a parameter a, we will use the notations C_a, $C(a)$, and c_a to make emphasis in the fact that the constants depend upon a. We frequently write $f(\cdot_s)$ to denote a function $s \mapsto f(s)$. For a set A, χ_A will denote the characteristic function of A. The symbols $\widehat{\ }$ and $\check{\ }$ will denote the Fourier and the inverse Fourier transform, respectively. The notations $\widehat{\ }^{x}$ and $\check{\ }^{\xi}$ will emphasize the facts that the Fourier transform and its inverse are taken with respect to specific variables x and ξ, respectively. For $1 \leq p, q < \infty$, $A, B \subseteq \mathbb{R}$, $D = A \times B$, and $f = f(x,t)$ we will denote

$$\|f\|^p_{L^p_x L^q_t(D)} := \int_A \Big(\int_B |f(x,t)|^q\, dt\Big)^{p/q}\, dx\,.$$

We will use similar definitions when $p = \infty$ or $q = \infty$ and also for $\|f\|_{L^q_t L^p_x(D)}$.

2. Exponential decay

In this section we prove that if the difference w of two solutions of (1.5) decays exponentially at $t = 0$, then this decay is preserved at all positive times. This property will be crucial for the application of the Carleman estimates in the proofs of Theorem 1.1 and 1.2.

Theorem 2.1. *For odd $n \geq 5$, $k = (n-1)/2$, and $\alpha > (n+1)/4$, let u_1, u_2 are in $C([0,1]; H^{n+1}(\mathbb{R}) \cap L^2((1+x_+)^{2\alpha}\, dx)))$ be two solutions of (1.1)-(1.2), and define $w := u_1 - u_2$. Let $\beta > 0$ and suppose that $w(0) \in L^2(e^{\beta x}\, dx)$. Then*

$$\sup_{t\in[0,1]} \|w(t)\|_{L^2(e^{\beta x}\, dx)} < \infty. \tag{2.1}$$

Proof. Let us denote $z_i = (u_i, \partial_x u_i, \dots \partial_x^{n-2} u_i)$, $i = 1, 2$. Then w is a solution of the differential equation

$$\partial_t w + (-1)^{k+1}\partial_x^n w + P(z_1) - P(z_2) = 0. \tag{2.2}$$

We will first prove that the theorem is valid provided we can construct a sequence $\{\phi_N\}_{N\in\mathbb{N}}$ of nondecreasing functions in $C^\infty(\mathbb{R})$ satisfying for all $x \in \mathbb{R}$ the conditions

$$\varphi_N(x) \to e^{\beta x} \quad \text{as } N \to \infty \quad \text{and} \quad 0 \leq \varphi_N(x) \leq C\, e^{\beta x}, \tag{2.3}$$

$$\varphi_N(x) \leq C_N (1+x_+)^{(k+2)/4}\,, \tag{2.4}$$

$$|\varphi_N^{(j)}(x)| \leq C_j \varphi_N'(x) \quad \text{for } j = 2, 3 \dots, n = 2k+1 \text{ and } \varphi_N'(x) \leq C\varphi_N(x), \tag{2.5}$$

$$\varphi_N(x) \leq C(1+x_+)\varphi_N'(x)\,, \tag{2.6}$$

where the constants C and C_j are independent of N.

We multiply (2.2) by $\varphi_N u$ and, for t fixed, integrate in $\mathbb{R}$. Thus, by applying integration by parts we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int \varphi_N w^2 = &-\frac{2k+1}{2}\int \varphi_N'(\partial_x^k w)^2 + c_{k-1}\int \varphi_N^{(3)}(\partial_x^{k-1} w)^2 + \dots \\ &+ c_1 \int \varphi_N^{(2k-1)}(\partial_x w)^2 + \frac{1}{2}\int \varphi_N^{(2k+1)} w^2 \\ &- \int (P(z_1) - P(z_2))\, \varphi_N w\,. \end{aligned} \tag{2.7}$$

The integration by parts is justified as follows: since there is a constant $C > 0$ such that $\|(1+x_+)^\alpha u_i(t)\|_{L^2} \leq C$, and $\|u_i(t)\|_{H^{n+1}(\mathbb{R})} \leq C$ for all $t \in [0,1]$ and $i = 1, 2$, by using integration by parts and truncation functions, it can be proved that the following interpolation property holds:

$$\|(1+x_+)^{\alpha(1-\frac{j}{(n+1)})}\partial_x^j u_i(t)\|_{L^2(\mathbb{R}} \leq C\,, \quad \text{for all } t \in [0,1]; i = 1, 2; \quad j = 0, \dots, n+1. \tag{2.8}$$

Since $\alpha > (n+1)/4$, it follows that, for $0 \leq j \leq k$, $(1+x_+)^{(k+2)/4}\partial_x^j w(t) \in L^2(\mathbb{R}$, and thus, from (2.4) and (2.5) $\varphi^{(l)}\partial_x^j w(t) \in L^2(\mathbb{R}$ for all positive integers l. This implies that all the terms which appear in the procedure to obtain (2.7) are in a right setting for the application of the integration by parts.

Let us estimate the last term on the rand-hand side of (2.7). From (1.2) we have that

$$P(z) = \sum_{d=2}^{k+1} A_d(z) \quad \text{where} \quad A_d(z) = \sum_{\substack{|m|=n-2(d-1)\\ 0\leq m_1\leq\dots\leq m_d}} a_{d,m}\, \partial_x^{m_1} u \dots \partial_x^{m_d} u\,, \tag{2.9}$$

and thus

$$|\int \big(P(z_1)-P(z_2)\big)\varphi_N\, w\, dx| = |\sum_{d=2}^{k+1}\int \big(A_d(z_1)-A_d(z_2)\big)\varphi_N\, w\, dx| \equiv |\sum_{d=2}^{k+1} I_d|\,. \tag{2.10}$$

It is easily seen that

$$\begin{aligned} A_d(z_1)-A_d(z_2) = \sum_{\substack{|m|=n-2(d-1)\\ 0\le m_1\le\dots\le m_d}} a_{d,m}\Big(&\partial_x^{m_1}w\partial_x^{m_2}u_1\dots\partial_x^{m_d}u_1 \\ &+\partial_x^{m_1}u_2\partial_x^{m_2}w\dots\partial_x^{m_d}u_1+\partial_x^{m_1}u_2\partial_x^{m_2}u_2\dots\partial_x^{m_d}w\Big). \end{aligned}\tag{2.11}$$

We estimate I_2, which, having the derivatives of the highest order, is the most critical term in (2.10). From (2.11),

$$\begin{aligned} I_2 &= \int \big(A_2(z_1)-A_2(z_2)\big)\varphi_N\, w\, dx \\ &= \sum_{\substack{m_1+m_2=n-2\\ 0\le m_1\le m_2}} a_{2,m}\int (\partial_x^{m_1}w\partial_x^{m_2}u_1+\partial_x^{m_1}u_2\partial_x^{m_2}w)\varphi_N\, w\, dx. \end{aligned}\tag{2.12}$$

We estimate only the second term on the right-hand side of (2.12), the other term being similar. We apply integration by parts to reduce the order of $\partial_x^{m_2}w$ and obtain that

$$\int (\partial_x^{m_1}u_2)\,(\partial_x^{m_2}w)\,\varphi_N\, w = \sum_{\substack{r_1+r_2+2r_3=m_1+m_2=n-2\\ r_1\ge m_1}} c_{r_1,r_2}\int (\partial_x^{r_1}u_2)\,\varphi_N^{(r_2)}(\partial_x^{r_3}w)^2 \tag{2.13}$$

To analyze the terms in this sum we consider the cases $r_2=0$ and $r_2\ge 1$:

(i) If $r_2=0$ and $r_3=0$, then $r_1=n-2$ and we bound the corresponding integral in (2.13) by

$$C\|\partial_x^{n-2}u_2(t)\|_{L^\infty}\int \varphi_N\, w^2 \le C\int \varphi_N\, w^2,$$

where C is independent of $t\in[0,1]$ by the Sobolev embedding of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$.

If $r_2=0$ and $r_3\ge 1$, then the maximum value of r_1 in (2.13) is $n-4$. Therefore, using the fact that $\varphi_N\le C(1+x_+)\varphi_N'$ we bound the corresponding integral in (2.13) by

$$C\max_{0\le r_1\le n-4}\|(1+x_+)\partial_x^{r_1}u_2(t)\|_{L^\infty}\int \varphi_N'\,(\partial_x^{r_3}w)^2. \tag{2.14}$$

From (2.8) it can be seen that if $\Psi\in C^\infty(\mathbb{R})$ is a truncation function with $\Psi\equiv 0$ in $(-\infty,1]$, and $\Psi\equiv 1$ in $[2,+\infty)$, then $\Psi(\cdot)(1+x_+)^{\alpha(1-\frac{j+1}{n+1})}\partial_x^j u_i(t)\in H^1(\mathbb{R})$, $i=1,2$, $j=0,1,\dots,n$, and

$$\begin{aligned} &\|\Psi(\cdot)(1+x_+)^{\alpha(1-\frac{j+1}{n+1})}\partial_x^j u_i(t)\|_{H^1(\mathbb{R})} \\ &\le C(\|(1+x_+)^\alpha u_i(t)\|_{L^2(\mathbb{R}}, \|u_i(t)\|_{H^{n+1}(\mathbb{R})}) \le C \quad \text{for all } t\in[0,1]. \end{aligned}\tag{2.15}$$

In particular, for $j=0,\dots,n-4$, $\alpha(1-\frac{j+1}{n+1})>\frac{n+1}{4}(1-\frac{n-3}{n+1})=1$, and thus from the Sobolev embedding of H^1 in L^∞ we conclude that

$$\max_{0\le j\le n-4}\|(1+x_+)\partial_x^j u_i(t)\|_{L^\infty(\mathbb{R})}\le C, \tag{2.16}$$

with C independent of $t\in[0,1]$. Thus (2.14) is bounded by $C\int\varphi_N'(\partial_x^{r_3}w)^2$.

(ii) If $r_2 \geq 1$, then $r_1 \leq n-3$. From (2.5), $\varphi^{(r_2)} \leq C_{r_2}\,\varphi' \leq C\varphi'$ for $1 \leq r_2 \leq n-2$. Thus we bound the corresponding term in (2.13) by

$$C \max_{0\leq r_1 \leq n-3} \|\partial_x^{r_1} u_2(t)\|_{L^\infty} \int \varphi_N'\,(\partial_x^{r_3} w)^2 \leq C \int \varphi_N'(\partial_x^{r_3} w)^2. \tag{2.17}$$

Now, let us determine the maximum value of r_3 appearing in (2.13). For that, we observe that $r_1 + r_2 + 2r_3 = n-2$ is odd, and thus the maximum value of r_3 occurs when $(r_1, r_2) = (0,1)$ or $(1,0)$, which then gives $r_3 \leq (n-3)/2 = k-1$.

Gathering the estimates of the cases (i) and (ii) above, and taking into account that $r_3 \leq k-1$, we conclude that

$$|I_2| \leq C \sum_{j=1}^{k-1} \int \varphi_N'(\partial_x^j w)^2\,dx + C\int \varphi_N w^2\,. \tag{2.18}$$

Proceeding in a similar way, we obtain the same bound for $|I_3|, \dots |I_{k+1}|$, and thus for the left hand side of (2.10). Therefore, returning to (2.7) and using the fact that, from condition (2.5), $|\phi^{(j)}| \leq C\,\phi'$, $j = 1, \dots 2k+1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int \varphi_N w^2 \leq -\frac{2k+1}{2}\int \varphi_N'(\partial_x^k w)^2 + C\sum_{j=1}^{k-1}\int \varphi_N'(\partial_x^j w)^2\,dx + C\int \varphi_N w^2 \tag{2.19}$$

To handle the terms in (2.19) having derivatives $\partial_x^j w$ with $j = 1, \dots, k-1$, we will show that given $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for $j = 1, \dots, k-1$

$$\int \varphi_N'(\partial_x^j w)^2 \leq \varepsilon \int \varphi_N'(\partial_x^k w)^2 + C_\varepsilon \int \varphi_N w^2\,. \tag{2.20}$$

In fact, we first prove that

$$\int \varphi_N'(\partial_x^j w)^2 \leq \varepsilon \int \varphi_N'(\partial_x^{j+1} w)^2 + C_\varepsilon \int \varphi_N w^2\,. \tag{2.21}$$

This can be seen by induction: by applying integration by parts, Young's inequality $|xy| \leq \frac{1}{2\varepsilon}x^2 + \frac{\varepsilon}{2}y^2$, and the properties of φ_N we can see that (2.21) is valid for $j = 1$. If we assume that (2.21) is valid for $j-1$, then, again integrating by parts and using Young's inequality,

$$\begin{aligned}\int \varphi_N'(\partial_x^j w)^2 &= \frac{1}{2}\int \varphi_N^{(3)}(\partial_x^{j-1} w)^2 - \int \varphi_N' \partial_x^{j-1} w\ \partial_x^{j+1} w \\ &\leq C\int \varphi_N'(\partial_x^{j-1} w)^2 + \frac{1}{2\varepsilon}\int \varphi_N'(\partial_x^{j-1} w)^2 + \frac{\varepsilon}{2}\int \varphi_N'(\partial_x^{j+1} w)^2.\end{aligned}$$

If we apply the induction hypothesis at level $j-1$, with $\frac{1/2}{C+1/(2\varepsilon)}$ instead of ε, then we have

$$\int \varphi_N'(\partial_x^j w)^2 \leq \frac{1}{2}\int \varphi_N'(\partial_x^j w)^2 + C_\varepsilon \int \varphi_N w^2 + \frac{\varepsilon}{2}\int \varphi_N'(\partial_x^{j+1} w)^2\,, \tag{2.22}$$

which gives (2.21). From a repeated application of (2.21) we obtain (2.20).

Therefore, taking into account that the first term on the right-hand side of (2.19) is negative, we can apply (2.20) with ε sufficiently small, to absorb with this negative term the integrals containing $(\partial_x^j w)^2$ in (2.19). Thus we conclude that

$$\frac{d}{dt}\int \varphi_N w^2 \leq C_\varepsilon \int \varphi_N w^2, \tag{2.23}$$

which, from Gronwall's inequality and (2.3) implies that

$$\int \varphi_N w(t)^2\,dx \le C\int \varphi_N w(0)^2\,dx \le C\int e^{\beta x}w(0)^2 dx\,,\quad \text{for all } t\in[0,1],$$

where C is independent of $t\in[0,1]$ and N. Since $\varphi_N(x)\to e^{\beta x}$ as $N\to\infty$, the conclusion of the theorem will follow by applying Fatou's Lemma on the left-hand side of the former inequality.

In this way, the proof of theorem 2.1 will be complete if we construct a sequence of functions φ_N, satisfying the conditions (2.3) to (2.6). For that we proceed as follows:

Let $\tilde\phi\in C^\infty(\mathbb{R})$ be a nonincreasing function such that $\tilde\phi(x)=1$ for $x\in(-\infty,0]$, and $\tilde\phi(x)=0$ for $x\in[1,\infty)$. For each $N\in\mathbb{N}$ let $\phi_N(x)\equiv\phi(x):=\tilde\phi(x-N)$. Thus ϕ is supported in $(-\infty,N+1]$, and $(1-\phi)$ in $[N,+\infty)$. We define

$$\theta_N(x)\equiv\theta(x):=\phi\beta e^{\beta x}+(1-\phi)\beta e^{\beta N}, \tag{2.24}$$

$$\varphi_N(x)\equiv\varphi(x):=\int_{-\infty}^{x}\theta(x')\,dx'\,. \tag{2.25}$$

Let us show that φ_N satisfies the conditions (2.3)–(2.6).

Taking into account the support of $(1-\phi)$, we see that $0\le\theta(x)\le\phi\beta e^{\beta x}+(1-\phi)\beta e^{\beta x}=\beta e^{\beta x}$. Thus, by integrating φ' we have that $0\le\varphi(x)\le e^{\beta x}$. Besides, from the definition of φ it is clear that $\varphi_N(x)\to e^{\beta x}$ as $N\to\infty$. Thus φ satisfies (2.3).

To prove (2.4) it suffices to observe that for $x\le N$, $\varphi(x)\le e^{\beta N}\le C_N(1+x_+)^{(k+2)/4}$, while for $x>N$,

$$\varphi(x)\le\int_{-\infty}^{N+1}\beta e^{\beta x'}dx'+\int_N^x\beta e^{\beta N}dx'\le e^{\beta(N+1)}+x\beta e^{\beta N}\le C_N(1+x_+)^{(k+2)/4}, \tag{2.26}$$

since $k\ge2$. Thus we have (2.4).

We proceed now to prove (2.6). For $x\le N$, $\varphi(x)=e^{\beta x}=\frac{1}{\beta}\varphi'(x)\le C(1+x_+)\varphi'(x)$. If $x>N$, then, from (2.26), and using the fact that $x\ge1$, we see that

$$\varphi(x)\le e^{\beta(N+1)}+x\beta e^{\beta N}\le(\frac{1}{\beta}+1)e^{\beta}x\beta e^{\beta N}\,. \tag{2.27}$$

On the other hand, for $x>N$,

$$x\varphi'(x)=x\theta(x)\ge N\phi\beta e^{\beta N}+x(1-\phi)\beta e^{\beta N}. \tag{2.28}$$

Therefore, from (2.27) and (2.28), taking into account the supports of ϕ and $(1-\phi)$ we observe that for $x>N+1$, $x\varphi'(x)\ge C\varphi(x)$, while for $N<x<N+1$, we conclude that

$$x\varphi'(x)\ge N\phi\beta e^{\beta N}+N(1-\phi)\beta e^{\beta N}=N\beta e^{\beta N}\ge\frac12(N+1)\beta e^{\beta N}\ge\frac12 x\beta e^{\beta N}\ge C\varphi(x),$$

from which (2.6) follows.

Finally, we verify (2.5). We observe that that for $j=1,2\dots$, and fixed $\beta>0$,

$$\begin{aligned}
|\varphi^{(j+1)}|=|\theta^{(j)}|&=|\phi\beta^{1+j}e^{\beta x}+\sum_{l=1}^{j}c_{j,l}\phi^{(l)}\beta^{j-l}\beta e^{\beta x}-\phi^{(j)}\beta e^{\beta N}|\\
&\le\beta^j\phi\beta e^{\beta x}+C_j(1+\beta)^{j-1}(\beta e^{\beta(N+1)}+\beta e^{\beta N})\chi_{[N,N+1]}\\
&\le\beta^j\theta+C_j\beta e^{\beta N}\chi_{[N,N+1]}
\end{aligned}$$

$$\begin{aligned}&= \beta^j\theta + C_j(\beta\phi e^{\beta N} + (1-\phi)\beta e^{\beta N})\chi_{[N,N+1]}\\ &\le \beta^j\theta + C_j\theta = C_j\varphi',\end{aligned}$$

where C_j depends on β and j but is independent of N. Thus, the first inequality in (2.5) is proved. For the inequality $\phi' \le C\phi$ in (2.5) we proceed by integrating the inequality $\phi'' \le C\phi'$ already established. This completes the proof of Theorem 2.1. $\square$

Remark 2.2. For the case of equation (1.1), with $p \le k$, we can establish a result similar to Theorem 2.1, by making minor modifications and some simplifications in the former proof. In the simple case $p \le 1$, for example for the equation

$$\partial_t u + (-1)^{k+1}\partial_x^n u = -u\partial_x u,$$

it is possible to follow the procedure of the proof of Theorem 2.1 to establish, without the hypothesis of polynomial decay, that the exponential decay at $t = 0$ is preserved for $t \in [0,1]$. This can be done by taking $\varphi_N(x) := \int_{-\infty}^x \theta_N(x')\,dx'$ as in (2.25), with $\theta_N(x) \equiv \theta(x) := \phi\beta e^{\beta x} + (1-\phi)\beta e^{-\beta(x-2N)}$, instead of the functions θ_N defined in (2.24). This functions φ_N are bounded and satisfy (2.3) and (2.5) which is enough for this case.

3. Estimates of Carleman type

In this section we obtain boundedness properties of the linear operator $(\partial_t + (-1)^{k+1}\partial_x^n)^{-1}$, and its spatial derivatives up to order $n-1$, in spaces of the type $L^p - L^q$ with exponential weight $e^{\lambda x}$. We keep our exposition simple since we only use values of p and q in the set $\{1, 2, +\infty\}$.

Let $D := \mathbb{R} \times [0,1]$ and, for $R \in \mathbb{R}$, let $D_R := \{(x,t) \mid x \ge R\,,\ t \in [0,1]\}$. We will denote

$$\begin{gathered}\|\cdot\|_{L^p_x L^q_T} := \|\cdot\|_{L^p_x L^q_t(D)}, \quad \|\cdot\|_{L^p_{x\ge R} L^q_T} := \|\cdot\|_{L^p_x L^q_t(D_R)},\\ \|\cdot\|_{L^q_T L^p_x} := \|\cdot\|_{L^q_t L^p_x(D)}, \quad \|\cdot\|_{L^q_T L^p_{x\ge R}} := \|\cdot\|_{L^q_t L^p_x(D_R)}.\end{gathered}$$

Theorem 3.1. *For $n = 2k+1$ with $k \in \mathbb{N}$, let v be a function in $C([0,1]; H^n(\mathbb{R})) \cap C^1([0,1]; L^2(\mathbb{R})$ such that $\operatorname{supp} v(t) \subset [-M, M]$ for all $t \in [0,1]$, for some $M > 0$. Then, for $\lambda > 2$ we have*

$$\|e^{\lambda x} v\|_{L^\infty_T L^2_x} \le C\|e^{\lambda x}(|v(0)| + |v(1)|)\|^2_L(\mathbb{R} + C\|e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)v\|_{L^1_T L^2_x}. \tag{3.1}$$

$$\begin{aligned}\sum_{j=1}^{n-1} \|e^{\lambda x}\partial_x^j v\|_{L^\infty_x L^2_T} \le C\lambda^{n-1}\|(|J^n(e^{\lambda x}v(1))| + |J^n(e^{\lambda x}v(0))|\|^2_L(\mathbb{R}\\ + \|e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)v\|_{L^1_x L^2_T},\end{aligned} \tag{3.2}$$

where C is independent of $\lambda > 2$ and M, and $(Jf)\widehat{\ } := (1+|\xi|^2)^{1/2}\widehat{f}$.

Reasoning formally, suppose that $e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)g = h$, and denote $f = e^{\lambda x}g$ and $T_0 = [e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)e^{-\lambda x}]^{-1}$. Then, $f = T_0 h$. Since $e^{\lambda x}\partial_x e^{-\lambda x} f = (\partial_x - \lambda)f$, we have that $e^{\lambda x}\partial_x^n e^{-\lambda x} f = (\partial_x - \lambda)^n f$, and thus the multipier operator representing T_0 via the Fourier transform is given by

$$(T_0 h)\widehat{\ }(\xi,\tau) = \frac{\widehat{h}}{i\tau + (-1)^{k+1}(i\xi - \lambda)^n} \equiv m_0\widehat{h}. \tag{3.3}$$

We will write m_0 as

$$m_0 = \frac{-i}{\tau - (\xi + i\lambda)^n}. \tag{3.4}$$

Since for a positive integer j, $e^{\lambda x}\partial_x^j g = (\partial_x - \lambda)^j f$, we have that

$$\begin{aligned} e^{\lambda x}\partial_x^j g &= (\partial_x - \lambda)^j T_0 h \equiv T_j h = [(i\xi - \lambda)^j m_0 \widehat{h}]\check{} \\ &= \Big[\frac{-i^{j+1}(\xi + i\lambda)^j}{\tau - (\xi + i\lambda)^n}\widehat{h}\Big]\check{} \equiv [m_j \widehat{h}]\check{}. \end{aligned} \tag{3.5}$$

Lemma 3.2. *Let $h \in L^1(\mathbb{R}^2)$. Then there is $C > 0$ independent of h and $\lambda > 0$ such that*

$$\|T_0 h\|_{L_t^\infty L_x^2} \le C\|h\|_{L_t^1 L_x^2}. \tag{3.6}$$

Proof. Let $a(\xi)$ and $b(\xi)$ be the real and imaginary parts of $-(\xi + i\lambda)^n$, respectively. Then

$$m_0 = \frac{-i}{\tau + a(\xi) + ib(\xi)}.$$

We recall that for $a \in \mathbb{R}$ and $b \ne 0$

$$\begin{aligned} &\Big(\frac{1}{\tau + a + ib}\Big)\check{}^{\tau}(t) \\ &= ce^{-iat}[e^{-bt}\chi_{[0,\infty]}(t)\chi_{(0,\infty)}(b) + e^{bt}\chi_{[-\infty,0)}(t)\chi_{(-\infty,0)}(b)] =: G_{a,b}(t). \end{aligned} \tag{3.7}$$

Since $|G_{a,b}| \le c$, by taking inverse Fourier transform in the variable τ and using convolutions, it follows that for $t \in \mathbb{R}$

$$|[T_0 h(t)]\widehat{}(\xi)| = |\int_{-\infty}^{\infty} G_{a(\xi),b(\xi)}(t - s)\widehat{h(s)}(\xi)\,ds| \le C\int_{-\infty}^{\infty} |\widehat{h(s)}(\xi)|\,ds,$$

for those values of ξ such that $b(\xi) \ne 0$ (a finite set). In this way, applying Plancherel's identity and Minkowski's integral inequality we obtain (3.6). □

Lemma 3.3. *Let $h \in L^1(\mathbb{R}^2)$. Then there is $C > 0$, independent of h and $\lambda > 2$, such that*

$$\|T_j h\|_{L_x^\infty L_t^2} \le C\|h\|_{L_x^1 L_t^2} \quad \textit{for } j = 1, \dots, n-1. \tag{3.8}$$

Proof. From (3.4) and (3.5), it follows that

$$(T_j h)\widehat{} = (i\xi - \lambda)^j m_0 \widehat{h} = \frac{-i^{j+1}(\xi + i\lambda)^j \widehat{h}}{\tau - (\xi + i\lambda)^n} = m_j \widehat{h}. \tag{3.9}$$

Let us denote $\theta := (\xi + i\lambda)/\tau^{1/n}$. Then

$$m_j = \frac{C}{\tau^{1-j/n}}\frac{\theta^j}{1 - \theta^n} = \frac{C}{\tau^{1-j/n}}\sum_{l=1}^{n}\frac{c_l}{\theta - r_l}, \tag{3.10}$$

where $r_l := a_l + ib_l$, $l = 1, \dots, n$, are the n^{th}-roots of 1, and the c_l can be computed by L'Hopital's rule to obtain that

$$c_l = \lim_{\theta \to r_l}\frac{(\theta - r_l)\theta^j}{1 - \theta^n} = -\frac{1}{nr_l^{n-j-1}}.$$

Therefore,

$$m_j = \frac{C}{\tau^{1-(j+1)/n}}\sum_{l=1}^{n}\frac{c_l}{\xi + i\lambda - \tau^{1/n}r_l} = \frac{C}{\tau^{1-(j+1)/n}}\sum_{l=1}^{n}\frac{c_l}{\xi - \tau^{1/n}a_l + i(\lambda - \tau^{1/n}b_l)}.$$

Taking the inverse Fourier transform with respect to the variable ξ, observing that for fixed λ, $\lambda - \tau^{1/m} b_l \neq 0$ for all l, except for a finite number of values of τ, and applying (3.7) (with ξ and x instead of τ and t, respectively) we obtain a collection of bounded functions $G_1, \dots, G_l$ of x and τ such that

$$[m_j(\cdot_\xi, \tau)]^{\check{}\,\xi}\,(x) = \frac{C}{\tau^{1-(j+1)/n}} \sum_{l=1}^{n} c_l\, G_l(x,\tau).$$

If $|\tau| > 1$, then it is clear that

$$|[m_j(\cdot, \tau)]^{\check{}\,\xi}\,(x)| \leq C, \tag{3.11}$$

with C independent of λ, x and $|\tau| > 1$. We can use (3.9) to prove that this function is bounded also for $|\tau| \leq 1$. To do this we will consider only the case $j = n-1 = 2k$, the other cases being similar. Let us observe from (3.5) that

$$\begin{aligned} \big|m_{2k} - \frac{i^n}{\xi + i\lambda}\big| &= \big|\frac{(\xi+i\lambda)^{n-1}}{\tau - (\xi+i\lambda)^n} + \frac{1}{\xi+i\lambda}\big| \\ &= \big|\frac{\tau}{(\tau-(\xi+i\lambda)^n)(\xi+i\lambda)}\big| \leq \frac{2}{|\xi+i\lambda|^{n+1}} \end{aligned}$$

which belongs to L^1_ξ, since $\lambda > 2$ and $|\tau| \leq 1$. From the Fourier inversion formula it can be seen that $|[|\xi+i\lambda|^{-n-1}]^{\check{}}(x)| \leq C$, with C independent of $\lambda > 2$. Thus, by taking inverse Fourier transform with respect to the variable ξ and taking into account that from (3.7) $[(\xi+i\lambda)^{-1}]^{\check{}\,\xi}\,(x)$ is a bounded function of x, with bound independent of λ, we see, together with the estimate already obtained for $|\tau| \leq 1$, that (3.11) is valid for all x and all but a finite number of values of τ.

Hence, we can apply basic properties of convolution and Plancherel's identity to conclude that

$$\|T_{2k}h\|_{L^\infty_x L^2_t} \leq C\|h\|_{L^1_x L^2_t},$$

which concludes the proof of Lemma 3.3. $\square$

Proof of Theorem 3.1. We extend v to all $t \in \mathbb{R}$ with value zero in $\mathbb{R} - [0,1]$, and call this extension again v. For $\varepsilon > 0$, we consider a function $\eta := \eta_\varepsilon \in C^\infty(\mathbb{R}_t)$ such that $\eta_\varepsilon = 0$ in $\mathbb{R} - [0,1]$, $\eta_\varepsilon = 1$ in $[\varepsilon, 1-\varepsilon]$, $\eta' \geq 0$ in $[0,\varepsilon]$, $\eta' \leq 0$ in $[1-\varepsilon, 1]$, and $\eta_{\epsilon'} \leq \eta_\varepsilon$ if $\varepsilon' < \varepsilon$. Define $g := \eta_\varepsilon(\cdot_t)v$. Then

$$e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)g = e^{\lambda x}\eta'_\varepsilon v + \eta_\varepsilon e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)v \equiv h_{1,\varepsilon} + h_{2,\varepsilon} \equiv h_1 + h_2.$$

Then, from (3.5),

$$e^{\lambda x}\partial_x^j g = T_j h_1 + T_j h_2 \quad \text{for } j = 0, \dots, n-1. \tag{3.12}$$

From Lemma 3.3,

$$\|T_j h_2\|_{L^\infty_x L^2_t} \leq C\|h_2\|_{L^1_x L^2_t} = C\|\eta_\varepsilon e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)v\|_{L^1_x L^2_t}. \tag{3.13}$$

For $T_j h_1$, we see from (3.5) that $(T_j h_1)\widehat{\ } = C(\xi+i\lambda)^j (T_0 h_1)\widehat{\ }$, and apply the Sobolev embedding from $H^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$, Plancherel's identity, and Lemma 3.2 to conclude that

$$\begin{aligned}
\|T_jh_1\|_{L^\infty_xL^2_T} &\leq C\|JT_jh_1\|_{L^2_TL^2_x} \\
&\leq C\|(1+|\xi|)(|\xi|^j+\lambda^j)(T_0h_1)\widehat{\ }^x\|_{L^2_TL^2_\xi} \\
&\leq C\|(1+\lambda)^j(1+|\xi|)^{j+1}(T_0h_1)\widehat{\ }^x\|_{L^2_TL^2_\xi} \\
&\leq C(1+\lambda)^j\|T_0J^{j+1}h_1\|_{L^\infty_tL^2_x} \\
&\leq C\lambda^j\|J^{j+1}h_1\|_{L^1_tL^2_x} \\
&= C\lambda^j\|\eta'_\varepsilon J^{j+1}(e^{\lambda x}v)\|_{L^1_tL^2_x} \\
&\leq C\lambda^{n-1}\|\eta'_\varepsilon J^n(e^{\lambda x}v)\|_{L^1_tL^2_x}.
\end{aligned} \tag{3.14}$$

Hence, from (3.12), (3.13), and (3.14) we have that

$$\|\eta_\varepsilon e^{\lambda x}\partial_x^jv\|_{L^\infty_xL^2_T} \leq C\lambda^{n-1}\|\eta'_\varepsilon J^n(e^{\lambda x}v)\|_{L^1_tL^2_x} + C\|\eta_\varepsilon e^{\lambda x}(\partial_t+(-1)^{k+1}\partial_x^n)v\|_{L^1_xL^2_t}.$$

We now make $\varepsilon\to 0$ in this inequality and apply Fatou's Lemma on the left-hand side and the monotone convergence theorem in the second term of the right-hand side. For the first term on the right-hand side we use the fact that $|\eta'_\varepsilon|$ acts as an approximation of the identity on each one of the time intervals $(0,\varepsilon)$ and $(1-\varepsilon,1)$. Thus, we obtain (3.2) after adding up in j.

The proof of (3.1) is similar but we use Lemma 3.2 instead of Lemma 3.3. This completes the proof. □

4. Lower estimates

In this section we prove that the L^2-norm of w in a small rectangle $Q=[0,1]\times[r,1-r]$, with $r\in(0,1)$ can be bounded by a multiple of the H^{n-1} norm of w in a distant rectangle $[R,R+1]\times[0,1]$.

Lemma 4.1. *Let $\phi\in C^\infty([0,1])$ be a function with $\phi(0)=\phi(1)=0$, and for $R>1$ and $\alpha>0$ define $\psi_\alpha(x,t):=\psi(x,t):=\alpha(\frac{x}{R}+\phi(t))^2$, $x\in\mathbb{R}$, $t\in[0,1]$. For $n=2k+1$, suppose that $g\in C([0,1];H^n(\mathbb{R}))\cap C^1([0,1];L^2(\mathbb{R}))$ is such that $g(0)=g(1)=0$ and the support of g is contained in the set*

$$A_{1,5}:=\{(x,t):t\in[0,1],\ 1\leq\frac{x}{R}+\phi(t)\leq 5\}.$$

Then, there are constants $C=C(n)>0$ and $\overline{C}=\overline{C}(n,\|\phi'\|_{L^\infty},\|\phi''\|_{L^\infty})>1$ such that

$$\sum_{j=0}^{n-1}\frac{\alpha^{n-j-\frac12}}{R^{n-j}}\|e^{\psi_\alpha}\partial_x^jg\| \leq C\|e^{\psi_\alpha}(\partial_t+(-1)^{k+1}\partial_x^n)g\| \quad \text{for all } \alpha>\overline{C}R^{n/(n-1)}, \tag{4.1}$$

where $\|\cdot\|:=\|\cdot\|_{L^2(D)}$.

Proof. Let $f:=e^\psi g$ and observe that $e^\psi\partial_x^jg=e^\psi\partial_x^je^{-\psi}f=(\partial_x-\psi_x)^jf$, and $e^\psi\partial_tg=(\partial_t-\psi_t)f$. Therefore, if we denote

$$T:=e^\psi(\partial_t+(-1)^{k+1}\partial_x^n)e^{-\psi}=(\partial_t-\psi_t)+(-1)^{k+1}(\partial_x-\psi_x)^n,$$

then, to prove the inequality in (4.1) we must prove that

$$\sum_{j=0}^{n-1}\frac{\alpha^{n-j-\frac12}}{R^{n-j}}\|(\partial_x-\psi_x)^jf\|\leq C\|Tf\|. \tag{4.2}$$

Let $B := -\psi_x = -\frac{2\alpha}{R}\varphi$, where $\varphi(x,t) := \frac{x}{R} + \phi(t)$. We will study the operator $(\partial_x - \psi_x)^n = (\partial_x + B)^n$ in the following manner.

Each one of the terms in the expansion of this operator is of the form $T_1 \dots T_n$, where each T_i is either ∂_x or B. For m and l with $m+l=n$ we will denote by $[m,l]$ the sum of the terms $T_1 \dots T_n$ in this expansion with $T_i = \partial_x$ for m values of i, and $T_i = B$ for l values of i. The number of terms of $[m,l]$ in the binomial expansion of $(\partial_x + B)^n$ is then given by $\binom{n}{l} = \binom{n}{m} := n!/m!\,l!$. Applying integration by parts in D, for the class of functions satisfying the hypotheses given for g, we observe that $[m,l]$ is a symmetric operator if m is even and is an antisymmetric operator if m is odd.

In this way, we write

$$(-1)^{k+1}T = (-1)^{k+1}(\partial_t - \psi_t) + \sum_{\substack{0\le m\le n\\ l+m=n}} [m,l] := S + A, \tag{4.3}$$

where

$$\begin{aligned} S &= [n-1,1] + [n-3,3] + \dots + [2,n-2] + [0,n] + (-1)^k\psi_t \equiv S_1 + (-1)^k\psi_t, \\ A &= [n,0] + [n-2,2] + \dots + [3,n-3] + [1,n-1] - (-1)^k\partial_t \equiv A_1 - (-1)^k\partial_t \,. \end{aligned} \tag{4.4}$$

Let us denote by $\langle\cdot,\cdot\rangle$ the inner product in the (real) space $L^2(D)$. Then

$$\|Tf\|^2 = \langle (S+A)f, (S+A)f\rangle = \|Sf\|^2 + \|Af\|^2 + 2\langle Sf, Af\rangle \ge 2\langle Sf, Af\rangle. \tag{4.5}$$

Now,

$$\langle Sf, Af\rangle = \langle S_1 f, A_1 f\rangle - (-1)^k\langle S_1 f, \partial_t f\rangle + (-1)^k\langle \psi_t f, A_1 f\rangle - \langle \psi_t f, \partial_t f\rangle. \tag{4.6}$$

We will now estimate each one of the four terms on the right hand side of (4.6).

To estimate $\langle S_1 f, A_1 f\rangle$ we observe that this product is a sum of terms of the form $\langle [m,n-m]f, [r,n-r]f\rangle$, with m even and r odd, say $m = 2k_1$, $r = 2k_2+1$, $k_1, k_2 \in \{0,\dots,k\}$. Using the fact that $B_{xx} = \psi_{xxx} = 0$, we can apply integration by parts to obtain

$$\langle [m,n-m]f, [r,n-r]f\rangle = \sum_{j=0}^{k_1+k_2} \int_D P_{k_1,k_2,j}(B, B_x)(\partial_x^j f)^2 \tag{4.7}$$

where $P_{k_1,k_2,j}(B,B_x) = c_{k_1,k_2,j} B^{(n-m)+(n-r)-\nu} B_x^\nu$, with $\nu + 2j = m + r$. Since $B = -\psi_x = -\frac{2\alpha}{R}\varphi$, $B_x = -\psi_{xx} = -\frac{2\alpha}{R^2}$, we have that

$$\begin{aligned} P_{k_1,k_2,j}(B,B_x) &= -c_{k_1,k_2,j}(2\alpha)^{2n-m-r}\frac{\varphi^{2n-m-r-\nu}}{R^{2n-m-r+\nu}} \\ &= -(2\alpha)^{2n-2k_1-2k_2-1} c_{k_1,k_2,j}\frac{\varphi^{2n-4k_1-4k_2+2j-2}}{R^{2n-2j}} \\ &\equiv \alpha^{2n-2k_1-2k_2-1} Q_{k_1,k_2,j} \end{aligned} \tag{4.8}$$

Therefore,

$$\begin{aligned} \langle S_1 f, A_1 f\rangle &= \sum_{k_1,k_2=0}^{k} \langle [2k_1, n-2k_1]f, [2k_2+1, n-2k_2-1]f\rangle \\ &= \sum_{k_1,k_2=0}^{k} \alpha^{2n-2k_1-2k_2-1} \sum_{j=0}^{k_1+k_2} \int Q_{k_1,k_2,j}(\partial_x^j f)^2 \end{aligned}$$

$$= \sum_{j=0}^{2k} \sum_{\substack{k_1+k_2\geq j\\ 0\leq k_1,k_2\leq k}} \alpha^{2n-2k_1-2k_2-1} \int Q_{k_1,k_2,j}(\partial_x^j f)^2. \tag{4.9}$$

For each j in the former expression we will separate the term having the highest power of α. Thus we write

$$\begin{aligned} \langle S_1 f, A_1 f\rangle =& \sum_{j=0}^{2k} \alpha^{2n-2j-1} \sum_{\substack{k_1+k_2=j\\ 0\leq k_1,k_2\leq k}} \int Q_{k_1,k_2,j}(\partial_x^j f)^2 \\ &+ \sum_{j=0}^{2k} \sum_{\substack{k_1+k_2>j\\ 0\leq k_1,k_2\leq k}} \alpha^{2n-2k_1-2k_2-1} \int Q_{k_1,k_2,j}(\partial_x^j f)^2 \\ \equiv& \sum_{j=0}^{2k} I_j + \sum_{j=0}^{2k} II_j . \end{aligned} \tag{4.10}$$

We will concentrate upon the terms I_j and will refer to the terms II_j as lower order terms (l.o.t.).

For each j we will now compute the term I_j. If $m \leq n$ and $l = n-m$, then $[m,l]$ is a sum of operators of the form $T = T_1T_2\dots T_n$ where $T_i = \partial_x$ for m indices i, and $T_i = B$ for the remaining l indices i. We apply the product rule for derivatives to expand T and consider the terms in its expansion containing the derivatives of highest order: ∂_x^m and ∂_x^{m-1}. This leads to (see the illustration below)

$$\begin{aligned} T = T_1\dots T_n = B^l\partial_x^m &+ [B^{r_1}(\partial_x B^{l-r_1}) + B^{r_2}(\partial_x B^{l-r_2}) + \dots \\ &+ B^{r_m}(\partial_x B^{l-r_m})]\partial_x^{m-1} + \text{l.d.t.}, \end{aligned} \tag{4.11}$$

where $r_1, r_2, \dots r_n$ depend upon the position of the m operators ∂_x in the expression of T, and the notation l.d.t. stands for "lower derivative terms".

To illustrate this, let us take for example the case with $n = 9$, $m = 3$, $l = 6$, and consider the operator $T = B\partial_x B\partial_x BB\partial_x BB$. Then,

$$T = B^6\partial_x^3 + B^4(\partial_x B^2)\partial_x^2 + B^2(\partial_x B^4)\partial_x^2 + B(\partial_x B^5)\partial_x^2 + \text{l.d.t.}$$

But, the operator $T^{(*)} := T_n\dots T_2T_1$, is also present in the expansion of $[m,l]$, and

$$\begin{aligned} T^{(*)} &= T_n\dots T_1 \\ &= B^l\partial_x^m + [B^{l-r_1}(\partial_x B^{r_1}) + B^{l-r_2}(\partial_x B^{r_2}) + \dots \\ &\quad + B^{l-r_m}(\partial_x B^{r_m})]\partial_x^{m-1} + \text{l.d.t.} \end{aligned} \tag{4.12}$$

Therefore, if $T \neq T^*$ we have from (4.11) and (4.12) that

$$T + T^{(*)} = 2B^l\partial_x^m + m(\partial_x B^l)\partial_x^{m-1} + \text{l.d.t.} \tag{4.13}$$

It can be seen that if $T = T^{(*)}$, then the same expression is valid. Since there are $\binom{n}{m}$ terms in the expansion of $[m,l]$ we conclude from (4.13) that

$$[m,l] = \binom{n}{m}[B^l\partial_x^m + \frac{ml}{2}B^{l-1}B_x\partial_x^{m-1}] + \text{l.d.t.} \tag{4.14}$$

In this way, for $m = 2k_1$, $r = 2k_2+1$, $j = k_1+k_2 = \frac{1}{2}(m+r-1)$, $l = n-m$, $s = n-r$, we apply integration by parts to observe that

$$\langle [m,l]f, [r,s]f\rangle$$

$$
\begin{aligned}
&= \binom{n}{m}\binom{n}{r}\Big(\int B^{l+s}\partial_x^m f\,\partial_x^r f + \frac{sr}{2}\int B^{l+s-1}B_x\,\partial_x^m f\,\partial_x^{r-1} f + \frac{ml}{2} \\
&\quad \times \int B^{l+s-1}B_x\,\partial_x^{m-1} f\partial_x^r f\Big) + \text{l.o.t.} \\
&= \frac{1}{2}\binom{n}{m}\binom{n}{r}(-1)^{\frac{1}{2}(m+1-r)}\Big((m-r)(l+s) + rs - ml\Big) \\
&\quad \times \int B^{l+s-1}B_x(\partial_x^{\frac{1}{2}(m+r-1)} f)^2 + \text{l.o.t.} \\
&= \frac{1}{2}\binom{n}{m}\binom{n}{r}(-1)^{k_1-k_2} n(m-r)\int B^{l+s-1}B_x(\partial_x^j f)^2 + \text{l.o.t.} \\
&= (2\alpha)^{2n-2j-1}\frac{1}{2}\binom{n}{m}\binom{n}{r}(-1)^j n(m-r)\int (\frac{\varphi}{R})^{2n-2j-2}\,\frac{-1}{R^2}(\partial_x^j f)^2 + \text{l.o.t.}
\end{aligned}
$$

According to the definition of I_j in (4.10), and from the first equality in (4.9), to obtain I_j we add the high order terms terms in the former expression with $k_1 + k_2 = j$ and $0 \le k_1, k_2 \le k$. In this way, we obtain that

$$
I_j = A_j^n\,\frac{n}{2}\,\frac{(2\alpha)^{2n-2j-1}}{R^{2n-2j}}\int \varphi^{2n-2j-2}(\partial_x^j f)^2, \tag{4.15}
$$

Where

$$
A_j^n := (-1)^j \sum_{\substack{k_1+k_2=j\\ 0\le k_1,k_2\le k}} (2k_2 + 1 - 2k_1)\binom{n}{2k_1}\binom{n}{2k_2+1}. \tag{4.16}
$$

We will prove in Lemma 4.2 below that

$$
A_j^n = n\binom{n-1}{j} \ge n, \quad \text{for all integers } n \ge 3 \text{ odd and all } j \le n-1, \tag{4.17}
$$

and in particular all the coefficients A_j^n are positive.

Regarding the lower terms II_j, we see from (4.10) and (4.8) that

$$
II_j = -\sum_{\substack{k_1+k_2>j\\ 0\le k_1,k_2\le k}} c_{k_1,k_2,j}\,\frac{(2\alpha)^{2n-2(k_1+k_2)-1}}{R^{2n-2j}}\int \varphi^{2n-4k_1-4k_2+2j-2}(\partial_x^j f)^2, \tag{4.18}
$$

Thus, from (4.10), using (4.15) and (4.18) we have

$$
\begin{aligned}
&\langle S_1 f, A_1 f\rangle \\
&= \sum_{j=0}^{2k} A_j^n\,\frac{n}{2}\,\frac{(2\alpha)^{2n-2j-1}}{R^{2n-2j}}\int \varphi^{2n-2j-2}(\partial_x^j f)^2 \\
&\quad - \sum_{j=0}^{2k}\sum_{\substack{k_1+k_2>j\\ 0\le k_1,k_2\le k}} c_{k_1,k_2,j}\,\frac{(2\alpha)^{2n-2(k_1+k_2)-1}}{R^{2n-2j}}\int \varphi^{2n-4k_1-4k_2+2j-2}(\partial_x^j f)^2.
\end{aligned} \tag{4.19}
$$

We now turn our attention to the product $\langle \psi_t f, A_1 f\rangle$ in (4.6). For $r = 2k_1 + 1$, $k_1 = 0, \dots, k$ and $s = n - r$, taking into account that $\psi_{txx=0}$, and using integration by parts we see that

$$
\langle [r,s]f, \psi_t f]\rangle = \sum_{j=0}^{k_1}\int P_{k_1,j}(B, B_x, \psi_t, \psi_{tx})(\partial_x^j f)^2, \tag{4.20}
$$

where

$$P_{k_1,j}(B,B_x,\psi_t,\psi_{tx}) = c'_{k_1,j}B^{s-\nu}B_x^{\nu}\psi_t + c''_{k_1,j}B^{s-(\nu-1)}B_x^{\nu-1}\psi_{tx} \tag{4.21}$$

with $\nu+2j=r$. Since $\psi_t = 2\alpha(\frac{x}{R}+\phi)\phi' = 2\alpha\varphi\phi'$ and $\psi_{tx} = 2\alpha\frac{\phi'}{R}$, we have that

$$\begin{aligned}
P_{k_1,j}(B,B_x,\psi_t,B_t) &= (2\alpha)^{s+1}\Big(c'_{k_1,j}\frac{\varphi^{s-\nu}}{R^{s-\nu}}\frac{1}{R^{2\nu}}\,\varphi\phi' + c''_{k_1,j}\frac{\varphi^{s-\nu+1}}{R^{s-\nu+1}}\frac{1}{R^{2\nu-2}}\frac{\phi'}{R}\Big)\\
&\equiv c_{k_1,j}\,\alpha^{s+1}\frac{\varphi^{s-\nu+1}}{R^{s+\nu}}\phi' = \alpha^{n-2k_1}\,c_{k_1,j}\frac{\varphi^{n-4k_1+2j-1}}{R^{n-2j}}\phi'\\
&\equiv \alpha^{n-2k_1}Q_{k_1,j}.
\end{aligned}$$

In this way, from (4.20) and proceeding as we did to obtain (4.9),

$$\begin{aligned}
&\langle\psi_t f, A_1 f\rangle\\
&= \sum_{k_1=0}^{k}\langle[2k_1+1, n-(2k_1+1)]f, \psi_t f\rangle\\
&= \sum_{k_1=0}^{k}\alpha^{n-2k_1}\sum_{j=0}^{k_1}\int Q_{k_1,j}(\partial_x^j f)^2 = \sum_{j=0}^{k}\sum_{k_1=j}^{k}\alpha^{n-2k_1}\int Q_{k_1,j}(\partial_x^j f)^2\\
&= \sum_{j=0}^{k}\alpha^{n-2j}\int Q_{j,j}(\partial_x^j f)^2 + \sum_{j=0}^{k}\sum_{k_1=j+1}^{k}\alpha^{n-2k_1}\int Q_{k_1,j}(\partial_x^j f)^2\\
&= \sum_{j=0}^{k}c_{j,j}\frac{\alpha^{n-2j}}{R^{n-2j}}\int\varphi^{n-2j-1}\phi'(\partial_x^j f)^2\\
&\quad + \sum_{j=0}^{k}\sum_{k_1=j+1}^{k}c_{k_1,j}\frac{\alpha^{n-2k_1}}{R^{n-2j}}\int\varphi^{n-4k_1+2j-1}\phi'(\partial_x^j f)^2.
\end{aligned} \tag{4.22}$$

To estimate the term $\langle S_1 f, \partial_t f\rangle$ in (4.6) we notice that, since S_1 is the sum of operators of the form $T_1,\dots T_n$ as described above, from the product rule for differentiation we see that $\partial_t[(S_1 f)] = S_{1t}f + S_1\partial_t f$, where S_{1t} is a sum of compositions of the form $Q_1Q_2\dots Q_n$ where one of the $Q_i's$ is B_t and the others are either B or ∂_x. In this way, by applying integration by parts we conclude that

$$\langle S_1 f, \partial_t f\rangle = -\langle S_{1t}f + S_1\partial_t f, f\rangle = -\langle S_{1t}f, f\rangle - \langle\partial_t f, S_1 f\rangle.$$

Therefore, $\langle S_1 f, \partial_t f\rangle = -\frac{1}{2}\langle S_{1t}f, f\rangle$. Proceeding as we did to obtain (4.20) to (4.22) we see that $\langle S_{1t}f, f\rangle$ has the same form as the right hand side of (4.22).

To conclude the computation of (4.6) we see that

$$\langle\psi_t f, \partial_t f\rangle = -\frac{1}{2}\int\psi_{tt}f^2 = -\alpha\int((\phi')^2+\varphi\phi'')f^2. \tag{4.23}$$

We return to (4.6) and compare the terms of the form α^N/R^M (for diverse integer powers N and M) in (4.19), (4.22), and (4.23), especially, we compare the highest order terms

$$\frac{\alpha^{2n-2j-1}}{R^{2n-2j}}\quad\text{(in (4.19))}\quad\text{and}\quad\frac{\alpha^{n-2j}}{R^{n-2j}}\quad\text{(in (4.22))}.$$

Since by the hypotheses in this lemma, $1\le\varphi\le 5$ in the support of f, there is a constant $\overline{C} = \overline{C}(n, \|\phi'\|_{L^\infty}+\|\phi''\|_{L^\infty}) > 1$ such that if we take $\alpha > \overline{C}R^{n/(n-1)}$, then the leading terms in (4.19) with coefficient $\frac{1}{2}nA_j^n(2\alpha)^{2n-2j-1}/R^{2n-2j}$ absorb the other terms appearing in (4.19), as well as all terms in (4.22), and (4.23). Therefore, from (4.6) and (4.5) we have

$$\|Tf\|^2 \geq 2\langle Sf, Af\rangle \geq C\sum_{j=0}^{n-1} A_j^n \frac{\alpha^{2n-2j-1}}{R^{2n-2j}} \int (\partial_x^j f)^2, \tag{4.24}$$

where C depends only upon n.

To prove (4.2) we must obtain an expression similar to (4.24) with the integrals $\int(\partial_x^j f)^2$ replaced by $\int((\partial_x + B)^j f)^2$. To do that, using integration by parts, we observe that for $m = 1, \dots, n-1$,

$$\begin{aligned}
&\int ((\partial_x + B)^m f)^2 \\
&= \int (\partial_x^m f)^2 + \sum_{j=0}^{m-1} \sum_{\substack{r,s\geq 0\\ r+2s+2j=2m}} c_{m,s,j} \int B^r B_x^s (\partial_x^j f)^2 \\
&= \int (\partial_x^m f)^2 + \sum_{j=0}^{m-1} \sum_{\substack{r,s\geq 0\\ r+2s+2j=2m}} c_{m,s,j} \frac{(2\alpha)^{r+s}}{R^{r+2s}} \int \varphi^r (\partial_x^j f)^2 \\
&= \int (\partial_x^m f)^2 + \sum_{j=0}^{m-1} \sum_{s=0}^{m-j} c_{m,s,j} \frac{(2\alpha)^{2m-2j-s}}{R^{2m-2j}} \int \varphi^{2m-2s-2j} (\partial_x^j f)^2 \,.
\end{aligned} \tag{4.25}$$

Since $\alpha > \overline{C} R^{n/(n-1)} > 1$, and in the support of f, $1 \leq \varphi \leq 5$, from (4.25) it follows that

$$\int \big((\partial_x + B)^m f\big)^2 \leq \int (\partial_x^m f)^2 + C' \sum_{j=0}^{m-1} \frac{\alpha^{2m-2j}}{R^{2m-2j}} \int (\partial_x^j f)^2, \tag{4.26}$$

where $C' > 0$ depends only on n. Now, from (4.24) we see that, if K_{n-1} is a constant with $0 < K_{n-1} \leq A_{n-1}^n = A_0^n = n$, then

$$\|Tf\|^2 \geq CK_{n-1} \frac{\alpha}{R^2} \int (\partial_x^{n-1} f)^2 + C \sum_{j=0}^{n-2} A_j^n \frac{\alpha^{2n-2j-1}}{R^{2n-2j}} \int (\partial_x^j f)^2,$$

Thus, applying (4.26) with $m = n-1$, we conclude that

$$\begin{aligned}
&\|Tf\|^2 \\
&\geq CK_{n-1} \frac{\alpha}{R^2} \Big[\int ((\partial_x + B)^{n-1} f)^2 - C' \sum_{j=0}^{n-2} \frac{\alpha^{2n-2-2j}}{R^{2n-2-2j}} \int (\partial_x^j f)^2 \Big] \\
&\quad + C \sum_{j=0}^{n-2} A_j^n \frac{\alpha^{2n-2j-1}}{R^{2n-2j}} \int (\partial_x^j f)^2 \\
&= CK_{n-1} \frac{\alpha}{R^2} \int ((\partial_x + B)^{n-1} f)^2 + C \sum_{j=0}^{n-2} (A_j^n - K_{n-1} C') \frac{\alpha^{2n-2j-1}}{R^{2n-2j}} \int (\partial_x^j f)^2.
\end{aligned}$$

Therefore, by choosing $K_{n-1} = \min\{A^n_{n-1}, \frac{1}{2}A^n_{n-2}/C', \dots, \frac{1}{2}A^n_0/C'\}$, we obtain that

$$\|Tf\|^2 \geq CK_{n-1}\frac{\alpha}{R^2}\int((\partial_x+B)^{n-1}f)^2 + C\sum_{j=0}^{n-2}\frac{1}{2}A^n_j\frac{\alpha^{2n-2j-1}}{R^{2n-2j}}\int(\partial_x^j f)^2, \quad (4.27)$$

and in this way we obtain an expression similar to (4.24) with the first integral term $\int(\partial_x^{n-1}f)^2$ replaced by $\int((\partial_x+B)^{n-1})f)^2$. Notice that from (4.17), $K_{n-1} > 0$. Proceeding in a similar manner, using (4.27), successively applying (4.26) with $m = n-2, n-3, \dots, 1$, and taking adequate values of $K_{n-2}, \dots, K_1 > 0$, we can perform the replacement of the remaining integrals. Since the $\min\{K_1, \dots, K_{n-1}\} > 0$, (4.2) follows and the proof is complete. $\square$

We now prove that all the coefficients A^n_j defined in (4.16) are positive.

Lemma 4.2. *For integer $k \geq 1$, let $n = 2k+1$. Then, for $j = 0, \dots, n-1$,*

$$A^n_j := (-1)^j \sum_{\substack{k_1+k_2=j\\ 0\leq k_1,k_2\leq k}}(2k_2+1-2k_1)\binom{n}{2k_1}\binom{n}{2k_2+1} = n\binom{n-1}{j}. \quad (4.28)$$

Proof. Using the formula $(1+x)^n = \sum_{r=0}^n \binom{n}{r}x^r$ we see that for $x, \beta \in \mathbb{R}$, $x \neq 0$,

$$\begin{aligned}
h_\beta(x) &:= \frac{1}{4}\Big((1+\beta x)^n + (1-\beta x)^n\Big)\Big((1+\beta/x)^n - (1-\beta/x)^n\Big)\\
&= \sum_{k_1,k_2=0}^{k}\binom{n}{2k_1}\binom{n}{2k_2+1}\beta^{2k_1+2k_2+1}x^{2k_1-2k_2-1}\\
&= \sum_{j=0}^{2k}\sum_{\substack{k_1+k_2=j\\ 0\leq k_1,k_2\leq k}}\binom{n}{2k_1}\binom{n}{2k_2+1}\beta^{2j+1}x^{2k_1-2k_2-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
h'_\beta(1) &= \sum_{j=0}^{2k}\beta^{2j+1}\sum_{\substack{k_1+k_2=j\\ 0\leq k_1,k_2\leq k}}(2k_1-2k_2-1)\binom{n}{2k_1}\binom{n}{2k_2+1}\\
&= \sum_{j=0}^{2k}(-1)^{j+1}A^n_j\beta^{2j+1}.
\end{aligned} \quad (4.29)$$

But

$$\begin{aligned}
\frac{4}{n\beta}h'_\beta(x) &= \Big((1+\beta x)^{n-1} - (1-\beta x)^{n-1}\Big)\Big((1+\beta/x)^n - (1-\beta/x)^n\Big)\\
&\quad + \Big((1+\beta x)^n + (1-\beta x)^n\Big)\Big((1+\beta/x)^{n-1} + (1-\beta/x)^{n-1}\Big)\Big(-\frac{1}{x^2}\Big).
\end{aligned}$$

Therefore, with $x = 1$,

$$\begin{aligned}
\frac{4}{n\beta}h'_\beta(1) &= \Big((1+\beta)^{n-1} - (1-\beta)^{n-1}\Big)\Big((1+\beta)^n - (1-\beta)^n\Big)\\
&\quad - \Big((1+\beta)^n + (1-\beta)^n\Big)\Big((1+\beta)^{n-1} + (1-\beta)^{n-1}\Big)\\
&= -2(1+\beta)^{n-1}(1-\beta)^n - 2(1+\beta)^n(1-\beta)^{n-1} = -4(1-\beta^2)^{n-1}.
\end{aligned}$$

In this way,

$$h'_\beta(1)=\sum_{j=0}^{n-1}(-1)^{j+1}n\binom{n-1}{j}\beta^{2j+1},$$

which, with together with (4.29), gives (4.28). $\square$

In the following theorem we apply Lemma 4.1 to the difference w of two solutions of equation (1.1) to obtain a bound of the L^2-norm of w in a small rectangle with the H^{n-1} norm of w in a distant rectangle $[R-1,R]\times[0,1]$.

Theorem 4.3. *Let $\varepsilon>0$. For $r\in(0,1)$, $R>2$ and $u_1,u_2\in C([0,1];H^n(\mathbb{R}))$ solutions of equation* (1.1) *define $w=u_1-u_2$, $Q=[0,1]\times[r,1-r]$, and*

$$A_R(w)=\int_0^1\int_R^{R+1}\sum_{j=0}^{n-1}(\partial_x^j w)^2\,dx\,dt\,. \tag{4.30}$$

Suppose that $\|w\|_{L^2(Q)}\geq\delta>0$. Then there exist constants $C>0$, $C_*=C_*(n,r)>0$, and $R_0>1$ such that

$$\|w\|_{L^2(Q)}\leq Ce^{C_*R^\gamma}A_R(w)\quad \text{for all } R>R_0\,, \tag{4.31}$$

where

$$\gamma\equiv\gamma_{n,p}=\begin{cases}\frac{n}{n-1} & \text{if } p=0,\dots,k=\frac{n-1}{2},\\ \frac{2(n-p)}{2(n-p)-1}+\epsilon & \text{if } p=k+1,\dots,n-1.\end{cases} \tag{4.32}$$

Notice that for the KdV Hierarchy, $p=n-2$, and thus we have an exponential $e^{C_*R^{4/3+}}$ in (4.31).

Proof. From (1.1), by a process similar to that used to obtain (2.11), it can be seen that w is a solution of the equation

$$\partial_t w+(-1)^{k+1}\partial_x^n w=-[P(z_1)-P(z_2)]=-\sum_{j=0}^{p}F_j\partial_x^j w\,, \tag{4.33}$$

where each F_j is a polynomial in $\partial_x^{j_1}u_1$ and $\partial_x^{j_2}u_2$ with $j_1,j_2\leq p\leq n-1$. From the embedding of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$, the functions $F_j=F_j(x,t)$ are bounded in $D=\mathbb{R}\times[0,1]$. Let $\phi\in C^\infty([0,1])$ be a function such that $\phi\equiv0$ in $[0,\frac{r}{2}]\cup[1-\frac{r}{2},1]$, $\phi\equiv4$ in $[r,1-r]$, ϕ increasing in $[\frac{r}{2},r]$, and decreasing in $[1-r,1-\frac{r}{2}]$. Let us choose functions $\widetilde{\mu}$, $\widetilde{\theta}\in C^\infty(\mathbb{R})$, such that $\widetilde{\mu}\equiv0$ in $(-\infty,1]$, $\widetilde{\mu}\equiv1$ in $[2,\infty)$, $\widetilde{\theta}\equiv1$ in $(-\infty,0]$, $\widetilde{\theta}\equiv0$ in $[1,\infty)$, and define $\mu_R(x,t)\equiv\mu(x,t):=\widetilde{\mu}(\frac{x}{R}+\phi(t))$ and $\theta_R(x)\equiv\theta(x):=\widetilde{\theta}(x-R)$. Let $g:=\mu\theta w$. Then, it can be seen that in the support of $\mu\theta$, $1\leq\frac{x}{R}+\phi(t)\leq5$. Besides $g(0)=g(1)=0$. Thus g satisfies the hypotheses of Lemma 4.1. With $\psi=\alpha(\frac{x}{R}+\phi(t))^2$, $\alpha>0$, as in the statement of Lemma 4.1, we compute

$$\begin{aligned}&e^\psi(\partial_t g+(-1)^{k+1}\partial_x^n g)\\&=e^\psi\Big(\mu_t\theta w+\mu\theta\partial_t w+\mu\theta(-1)^{k+1}\partial_x^n w+\sum_{r=1}^{n}c_{n,r}\partial_x^r(\mu\theta)\partial_x^{n-r}w\Big)\\&=e^\psi\Big(-\sum_{j=0}^{p}F_j\mu\theta\partial_x^j w+\mu_t\theta w+\sum_{r=1}^{n}c_{n,r}\mu\partial_x^r\theta\partial_x^{n-r}w\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{n}\sum_{s=1}^{r}c_{n,r}c_{r,s}\partial_x^s\mu\partial_x^{r-s}\theta\partial_x^{n-r}w\Big)\\
&=e^{\psi}\Big(-\sum_{j=0}^{p}F_j\partial_x^j(\mu\theta w)+\sum_{j=1}^{p}F_j\sum_{r=1}^{j}c_{j,r}\partial_x^r(\mu\theta)\partial_x^{j-r}w\\
&+\mu_t\theta w+\mu\sum_{r=1}^{n}c_{n,r}\partial_x^r\theta\partial_x^{n-r}w+\sum_{r=1}^{n}\sum_{s=1}^{r}c_{n,r}c_{r,s}\partial_x^s\mu\partial_x^{r-s}\theta\partial_x^{n-r}w\Big)\\
&=e^{\psi}\Big(-\sum_{j=0}^{p}F_j\partial_x^j g+\mu\sum_{j=1}^{p}F_j\sum_{r=1}^{j}c_{j,r}\partial_x^r\theta\partial_x^{j-r}w\\
&+\sum_{j=1}^{p}\sum_{r=1}^{j}\sum_{s=1}^{r}F_jc_{j,r}c_{r,s}\partial_x^s\mu\partial_x^{r-s}\theta\partial_x^{j-r}w\\
&+\mu_t\theta w+\mu\sum_{r=1}^{n}c_{n,r}\partial_x^r\theta\partial_x^{n-r}w+\sum_{r=1}^{n}\sum_{s=1}^{r}c_{n,r}c_{r,s}\partial_x^s\mu\partial_x^{r-s}\theta\partial_x^{n-r}w\Big).
\end{aligned}
$$

To obtain a bound for the L^2-norm of the right-hand side of the former expression we take into account the following facts: The derivatives $\partial_x^s\mu$ $(s\geq 1)$ are supported in $\subseteq\{(x,t)\mid 1<\frac{x}{R}+\phi(t)<2\}$ and thus $e^{\psi}\leq e^{4\alpha}$ in this set whose area is of order R. Also, $\partial_x^r\theta$ $(r\geq 1)$ is supported in $[R,R+1]\times[0,1]$, and $e^{\psi}\leq e^{25\alpha}$ in this rectangle. Besides, w, the functions F_j, and all the derivatives of μ, θ, and w, are bounded by a constant independent of R. From this considerations, and applying Lemma 4.1 we obtain

$$
\begin{aligned}
&\sum_{j=0}^{n-1}\frac{\alpha^{n-j-1/2}}{R^{n-j}}\|e^{\psi}\partial_x^j g\|\\
&\leq C\|e^{\psi}(\partial_t+(-1)^{k+1}\partial_x^n)g\|\\
&\leq C\sum_{j=0}^{p}\|e^{\psi}\partial_x^j g\|+Ce^{25\alpha}\sum_{j=0}^{n-1}\|\partial_x^j w\|_{L^2([R,R+1]\times[0,1])}+CR^{1/2}e^{4\alpha}.
\end{aligned}
$$

Let $\overline{C}$ be the constant in Lemma 4.1. Then $\overline{C}=\overline{C}(n,r)$. If we take $\alpha=(1+\overline{C})R^{1+s}$, with $s\geq\frac{1}{n-1}$, then $\alpha>\overline{C}R^{\frac{n}{n-1}}$, and for $j=1,\dots,p$

$$
\frac{\alpha^{n-j-1/2}}{R^{n-j}}=(1+\overline{C})^{n-j-\frac12}R^{-\frac12+s(n-j-\frac12)}\geq R^{s(n-p-\frac12)-\frac12}
$$

and therefore, after discarding the terms with $j>p$ on the left-hand side of (4.34), and bearing in mind the definition of $A_R(w)$ given in (4.30), we have that

$$
\sum_{j=0}^{p}R^{s(n-p-\frac12)-\frac12}\|e^{\psi}\partial_x^j g\|\leq C\sum_{j=0}^{p}\|e^{\psi}\partial_x^j g\|+Ce^{25\alpha}A_R(w)+CR^{1/2}e^{4\alpha}. \tag{4.34}
$$

We will choose $s\geq\frac{1}{n-1}$ in such a way that $s(n-p-1/2)-1/2>0$; that is, $s>1/(2n-2p-1)$. Then, by making R sufficiently large we can make the left-hand side of (4.34) more than twice the first term on the right-hand side, allowing the absorsion of this last term to obtain that

$$\sum_{j=0}^{p} R^{s(n-p-\frac{1}{2})-\frac{1}{2}}\|e^{\psi}\partial_x^j g\| \le Ce^{25\alpha}A_R(w) + CR^{1/2}e^{4\alpha}. \tag{4.35}$$

To choose the appropriate value of s we see that if $p \le (n-1)/2 = k$, then

$$\frac{1}{(2n-2p-1)} \le \frac{1}{n} < \frac{1}{n-1},$$

and we choose $s = 1/(n-1)$. Then with $\alpha = (1+\overline{C})R^{1+s} = (1+\overline{C})R^{(n-1)/n}$, we have (4.35) for large R.

If $(n-1)/2 \le p \le n-1$, that is if $(n+1)/2 \le p \le n-1$, then, with $\epsilon > 0$ and $s = 1/(2n-2p-1)+\epsilon$; that is, with $\alpha = (1+\overline{C})R^{\frac{2(n-p)}{2(n-p)-1}+\epsilon}$, we obtain (4.35) for large R.

Since $\frac{x}{R} + \phi(t) \ge 4$ in $Q := [0,1]\times[r,1-r]$ and $\mu \equiv 1$, and $\theta \equiv 1$ in Q, we can replace the left-hand side of (4.35) by a smaller amount to conclude that for R sufficiently large

$$e^{16\alpha}\|w\|_{L^2(Q)} \le Ce^{25\alpha}A_R(w) + CR^{1/2}e^{4\alpha}, \tag{4.36}$$

Hence, with γ as in (4.32), and $\alpha = (1+\overline{C})R^{\gamma}$,

$$e^{16(1+\overline{C})R^{\gamma}}\|w\|_{L^2(Q)} \le Ce^{25(1+\overline{C})R^{\gamma}}A_R(w) + CR^{1/2}e^{4(1+\overline{C})R^{\gamma}},$$

Since $\|w\|_{L^2(Q)} \ge \delta > 0$, by making R sufficiently large we can absorb the last term on the right-hand side of the former inequality with the left-hand side to obtain (4.31) with $C_* = 9(1+\overline{C})$, which completes the proof of Theorem 4.3. $\square$

5. Proofs of Theorems 1.1 and 1.2

For Theorem 1.1 we present a proof which can be adapted with minor changes to prove Theorem 1.2.

Proof of Theorem 1.1. From hypothesis (1.6), $e^{x_+^{4/3+\epsilon}}w(0) \in L^2(\mathbb{R})$, then it follows that $\|e^{ax_+^{4/3+\epsilon/2}}w(0)\|_{L^2(\mathbb{R}} \le C_a < \infty$ for all $a > 0$. The same property holds for $w(1)$. Also, by an interpolation argument similar to that in (2.8),

$$\|e^{ax_+^{4/3+\epsilon/2}}\partial_x^j w(i)\|_{L^2(\mathbb{R}} \le C_a < \infty \quad \text{for all } a>0,\ j=1,\dots,n,\ i=0,1. \tag{5.1}$$

Suppose that w does not vanish identically in $D := \mathbb{R}\times[0,1]$. Then, there is a rectangle $Q := [x_0, x_0+1]\times[r,1-r]$, for some $x_0 \in \mathbb{R}$ and $r \in (0,1)$, such that $\|w\|_{L^2(Q)} > 0$. If we consider translations $\tilde{u}_i$ of u_i, defined by $\tilde{u}_i(x,t) := u_i(x+x_0,t)$, $i = 1,2$, then, it can be seen that $\tilde{u}_1$, $\tilde{u}_2$, and $\tilde{w} := \tilde{u}_1 - \tilde{u}_2$, satisfy the hypotheses of Theorem 1.1. In this way, making a translation if necessary, we can suppose without loss of generality that $Q = [0,1]\times[r,1-r]$.

Let $\eta \in C^\infty(\mathbb{R})$ be a function supported in $(0,1)$ and such that $\int \eta = 1$. For $R > 1$ and $N > 4R+1$, define $\phi_{R,N}(x) \equiv \phi(x) := \int_{-\infty}^{x} \eta(x'-R) - \eta(x'-N))\,dx'$. Then $\phi_{R,N} = 1$ in $[R+1,N]$, $\operatorname{supp}\phi_{R,N} \subseteq [R,N+1]$ and $|\phi_{R,N}^{(j)}| \le c_j$ with c_j independent of R and N. We will apply Theorem 3.1 to the function $v_{R,N} \equiv v :=$

$\phi_{R,N}\, w$. From (4.33) with $p = n-2$, v satisfies

$$
\begin{aligned}
&\partial_t v + (-1)^{k+1}\partial_x^n v\\
&= \phi(\partial_t w + (-1)^{k+1}\partial_x^n w) + \sum_{j=1}^{n} c_{n,j}\phi^{(j)}\partial_x^{n-j} w\\
&= -\sum_{j=0}^{p} \phi F_j \partial_x^j w + \sum_{j=1}^{n} c_{n,j}\phi^{(j)}\partial_x^{n-j} w\\
&= -\sum_{j=0}^{p} F_j\partial_x^j(\phi w) + \sum_{j=1}^{p} F_j \sum_{r=1}^{j} c_{j,r}\phi^{(r)}\partial_x^{j-r} w + \sum_{j=1}^{n} c_{n,j}\phi^{(j)}\partial_x^{n-j} w.
\end{aligned}
\tag{5.2}
$$

For $\lambda > 2$ we now apply together (3.1) and (3.2) in Theorem 3.1 to v. We use (5.2), Hölder's inequality, and the fact that $\|\cdot\|_{L^2_T L^2_x} \le \|\cdot\|_{L^\infty_T L^2_x}$ and $\|\cdot\|_{L^1_T L^2_x} \le \|\cdot\|_{L^2_T L^2_x}$, and take into account that ϕ is supported in $[R, N+1]$ and its derivatives $\phi^{(j)}$ are supported in $[R, R+1] \cup [N, N+1]$, to obtain

$$
\begin{aligned}
&\|e^{\lambda x}\phi w\|_{L^2_T L^2_x} + \sum_{j=1}^{n-1} \|e^{\lambda x}\partial_x^j(\phi w)\|_{L^\infty_x L^2_T}\\
&\le C\lambda^{n-1}\||J^n(e^{\lambda x}\phi w(1))| + |J^n(e^{\lambda x}\phi w(0))|\|^2_L(\mathbb{R}\\
&\quad + C\|e^{\lambda x}(\partial_t + (-1)^{k+1}\partial_x^n)v\|_{L^1_T L^2_x \cap L^1_x L^2_T}\\
&\le C\lambda^{2n-1}\sum_{j=0}^{1} \|e^{\lambda x}(|w(j)| + |\partial_x^n w(j)|)\|_{L^2([R,\infty))}\\
&\quad + C\|F_0\|_{L^2_T L^\infty_{x\ge R}}\|e^{\lambda x}\phi w\|_{L^2_T L^2_x} + C\|F_0\|_{L^2_{x\ge R} L^\infty_T}\|e^{\lambda x}\phi w\|_{L^2_x L^2_T}\\
&\quad + C\sum_{j=1}^{p} \|F_j\|_{L^2_{x\ge R} L^\infty_T}\|e^{\lambda x}\partial_x^j(\phi w)\|_{L^\infty_x L^2_T}\\
&\quad + C\sum_{j=1}^{p} \|F_j\|_{L^1_{x\ge R} L^\infty_T}\|e^{\lambda x}\partial_x^j(\phi w)\|_{L^\infty_x L^2_T}\\
&\quad + Ce^{\lambda(R+1)}\sum_{j=0}^{n-1} \|\partial_x^j w\|_{L^\infty_T L^2_x} + C\sum_{j=0}^{n-1} e^{\lambda(N+1)}\|\partial_x^j w\|_{L^\infty_T L^2_{x\ge N}}\\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}
\tag{5.3}
$$

where C does not depend on λ, N, and R.

Taking into account the specific form of the polynomial P in (1.2), and since $\sum_{j=0}^{n-1} F_j\partial_x^j w = P(z_1) - P(z_2)$, we see from (2.9) and (2.11) that each F_j is a polynomial in $\partial_x^{j_1} u_1$ and $\partial_x^{j_2} u_2$ with $j_1, j_2 \le n-3$, except for F_0 which has a term $\partial_x^{n-2} u_1$ coming from the quadratic term $w\partial_x^{n-2} u_1$ in $A_2(z_1) - A_2(z_2)$ (see (2.12)). From (2.15) it follows that

$$
\|(1 + x_+)^{\alpha(1-\frac{l+1}{n+1})}\partial_x^l u_i(t)\|_{L^\infty(\mathbb{R})} \le C\,, \quad \text{for all } t \in [0,1],\ l = 0,\dots,n. \tag{5.4}
$$

For $l \le n-3$, $\alpha(1-\frac{j+1}{n+1}) > \frac{n+1}{3}(1-\frac{n-2}{n+1}) = 1$. Therefore, $\|(1+x_+)^{1^+} F_j\|_{L^\infty_T L^\infty_x} < \infty$ for $j = 1,\dots,p$. In a similar way, with $l = n-2$ in (5.4), we see that $\|(1 + x_+)^{2/3} F_0\|_{L^\infty_T L^\infty_x} < \infty$. From this decay of the functions F_j we conclude that, by taking R sufficiently large, the norms involving these functions in II and III of (5.3) can be made small in such a way that II and III can be absorbed by the left-hand side of (5.3).

After we perform this absorbtion, and taking into account that $\phi \equiv 1$ in $[4R, 4R+1]$, we replace the left-hand side of (5.3) by a smaller amount to obtain

$$
\begin{aligned}
&\|e^{\lambda x} w\|_{L^2([4R,4R+1]\times[0,1])} + \sum_{j=1}^{n-1} \|e^{\lambda x}\partial_x^j w\|_{L^2([4R,4R+1]\times[0,1])} \\
&\le C\lambda^{2n-1} \sum_{j=0}^{1} \|e^{\lambda x}(|w(j)| + |\partial_x^n w(j))|\|_{L^2([R,\infty))} \\
&\quad + Ce^{\lambda(R+1)} \sum_{j=0}^{n-1} \|\partial_x^j w\|_{L_T^\infty L_x^2} + Ce^{\lambda(N+1)} \|\partial_x^j w\|_{L_T^\infty L^2_{x\ge N}} \\
&=: \mathrm{I} + \mathrm{IV}.
\end{aligned}
\tag{5.5}
$$

From the decay hypothesis (1.6) of w and the exponential decay preservation proved in Theorem 2.1, it follows that $\|e^{\lambda x} w(t)\|_L^2(\mathbb{R} \le C_\lambda < \infty$, for all $\lambda > 0$ and all $t \in [0,1]$. From an interpolation argument similar to that in (2.8), we also have that $\|e^{\lambda x}\partial_x^j w(t)\|_L^2(\mathbb{R} \le C_\lambda$, for $j = 1, \dots, n$. Therefore,

$$
\begin{aligned}
\mathrm{IV} &\le Ce^{\lambda(R+1)} \sum_{j=0}^{n-1} \|\partial_x^j w\|_{L_T^\infty L_x^2} + Ce^{\lambda(N+1)} e^{-2\lambda N} \|e^{2\lambda x}\partial_x^j w\|_{L_T^\infty L_x^2} \\
&\le Ce^{\lambda(R+1)} + C_\lambda e^{-\lambda(N-1)}\,,
\end{aligned}
$$

Then, as $N \to \infty$, from (5.5) we conclude that

$$
\begin{aligned}
&e^{4R\lambda} \sum_{j=0}^{n-1} \|\partial_x^j w\|_{L^2([4R,4R+1]\times[0,1])} \\
&\le C\lambda^{2n-1} \sum_{j=0}^{1} \|e^{\lambda x}(|w(j)| + |\partial_x^n w(j))|\|_{L^2([R,\infty))} + Ce^{\lambda(R+1)}.
\end{aligned}
\tag{5.6}
$$

For $a > 0$ to be determined later, we take $\lambda = aR^{1/3+\epsilon/2}$. Since $\lambda x = aR^{1/3+\epsilon/2} x \le ax^{4/3+\epsilon/2}$ for $x \ge R$, from (5.6) we have

$$
\begin{aligned}
&e^{4aR^{4/3+\epsilon/2}} \sum_{j=0}^{n-1} \|\partial_x^j w\|_{L^2([4R,4R+1]\times[0,1])} \\
&\le Ca^{2n-1} R^{(2n-1)(\frac{1}{3}+\epsilon/2)} \sum_{j=0}^{1} \|e^{ax^{4/3+\epsilon/2}}(|w(j)| + |\partial_x^n w(j))|\|_{L^2(\mathbb{R})} \\
&\quad + Ce^{aR^{1/3+\epsilon/2}(R+1)},
\end{aligned}
$$

and thus, from the definition of $A_R(w)$ given in (4.30) and from (5.1),

$$
e^{4aR^{4/3+\epsilon/2}} A_R(w) \le C_a R^{(2n-1)(1/3+\epsilon/2)} + Ce^{2aR^{4/3+\epsilon/2}} \le C_a e^{2aR^{4/3+\epsilon/2}}. \tag{5.7}
$$

We now apply Theorem 4.3 with $p = n-2$ to obtain

$$
\|w\|_{L^2(Q)} \le Ce^{C_* R^{4/3+\epsilon/2}} A_R(w)
$$

$$\le C e^{C_* R^{4/3+\epsilon/2}} C_a e^{-2aR^{4/3+\epsilon/2}} = C_a e^{(C_*-2a)R^{4/3+\epsilon/2}},$$

Where $C^* = C^*(r)$. If we fix $a > C_*/2$, then by taking $R \to \infty$ we conclude that $\|w\|_{L^2(Q)} = 0$, which contradicts the original assumption $\|w\|_{L^2(Q)} \neq 0$. Then we conclude that $w \equiv 0$, and Theorem 1.1 is proved. □

Proof of Theorem 1.2. From Remark 2.2, w satisfies (2.23). With $\beta = 1$, after applying Gronwall's inequality and taking $N \to \infty$, we can conclude that for $t_0 \in [0, 1]$,

$$\int e^x w(t)^2 \le C \int e^x w(t_0)^2 \quad \text{for all } t \in [t_0, 1]. \tag{5.8}$$

By making the change of variables $x \mapsto -x$ and $t \mapsto 1-t$, and taking into account that $w \in C([0,1]; H^{n+1}(\mathbb{R}) \cap L^2((1+x_-)^{2\alpha_0}\,dx)$ we can also see that

$$\int e^{-x} w(t)^2 \le C \int e^{-x} w(t_0)^2 \quad \text{for all } t \in [0, t_0]. \tag{5.9}$$

Thus we can conclude that if $w(t_0) = 0$, then $w \equiv 0$.

We will find a constant $a > 0$ such that if $w(0), w(1) \in L^2(e^{ax^{n/(n-1)}})$, then $w \equiv 0$. We reason by contradiction. Suppose that w does not vanish identically in $D := \mathbb{R} \times [0,1]$. Then, by the uniqueness argument just given, w does not vanish identically in $D_0 := \mathbb{R} \times [1/3, 2/3]$. Therefore, there is a rectangle $Q := [x_0, x_0+1] \times [1/3, 2/3]$ such that $\|w\|_{L^2(Q)} > 0$. By making a translation if necessary, we can suppose without loss of generality that $Q = [0,1] \times [1/3, 2/3]$. We now continue applying the same arguments used to prove Theorem 1.1, using $\lambda = aR^{1/(n-1)}$ instead of $aR^{1/3+\epsilon/2}$. In this case we apply Theorem 4.3 with $p \le k$, and $C_* = C_*(1/3)$ and choose $a = \frac{C_*}{2} + 1 > \frac{C_*}{2}$, which gives a value of a which depends only on n. □

References

[1] Avramidi, I.; Schiming, R.; *A new explicit expression for the Korteweg-De Vries hierarchy*, Math. Nachr. **219** (2000), 45-64.

[2] Bourgain, J.; *On the compactness of the support of solutions of dispersive Equations,* IMRN, International Mathematics Research Notices **9** (1997), 437-444.

[3] Gardner, C.; Greene, J.; Kruskal, M.; Miura R.; *Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett.* **19** (1967), 1095-1097.

[4] Gardner, C.; Greene, J.; Kruskal, M.; Miura R.; *Korteweg-de Vries equation and generalization. VI. Methods for exact solution*, Comm. Pure Appl. Math. **27** (1974), 97-133.

[5] Grünrock, A.; *On the hierarchies of higher order mKdV and KdV equations*, Central European Journal of Mathematics **8** (2010), 500-536.

[6] Dawson, L.; *Uniqueness properties of higher order dispersive equations*, J. Differential Equations **236** (2007), 199-236.

[7] Escauriaza, L.; Kenig, C.; Ponce, G.; Vega, L.; *On unique continuation of solutions of Schrödinger equations*, Comm. Partial Differential Equations **31** (2006), 1811-1823.

[8] Escauriaza, L.; Kenig, C.; Ponce, G.; Vega, L.; *On uniqueness properties of solutions of the k-generalized KdV equations,* J. Funct. Anal. **244** (2007), 504-535.

[9] Kenig, C.; Pilod, D.; *Well-posedness for the fifth-order KdV equation in the energy space*, to appear in Trans. Amer. Math. Soc.

[10] Kenig, C.; Ponce G.; Vega L.; *On the hierarchy of the generalized KdV equations*, in: Proc. Singular Limits of Dispersive Waves, Lyon, 1991. NATO Adv. Sci. Inst. Ser. B Phys., **320** (1994), 347-356.

[11] Kenig, C.; Ponce G.; Vega L.; *Higher-order nonlinear dispersive equations*, Proc. Amer. Math. Soc., **122** (1994), 157-166.

[12] Kenig, C.; Ponce, G.; Vega, L.; *On the support of solutions to the g-KdV equation,* Ann. Inst. H. Poincaré Anal. Non linéaire **19** (2002), 191-208.

[13] Kenig, C.; Ponce, G.; Vega, L.; *On the unique continuation of solutions to the generalized KdV equation*, Mat. Res. Letters **10** (2003), 833-846.
[14] Kwon, S.; *On the fifth-order KdV equation: local well-posedness and lack of uniform continuity of the solution map*, J. Differential Equations **245** (2008) 2627-2659.
[15] Lax, P. D.; *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968) 467-490.
[16] Lax, P. D.; *Almost periodic solutions of the KdV equation, SIAM Rev.* **8** (3) (1976), 351-375.
[17] Linares, F.; *A higher order modified Korteweg-de Vries equation*, Mat. Apl. Comput., **14** (1995), 253-267.
[18] Pilod, D.; *On the Cauchy problem for higher-order nonlinear dispersive equations*, J. Differential Equations **245** (2008), 2055-2077.
[19] Ponce, G.; *Lax pairs and higher order models for water waves*, J. Differential Equations **102** (1993), 360-381.
[20] Praught, J.; Smirnov, R.; *Andrew Lenard: A Mystery Unraveled*, Symmetry, Integrability and Geometry: Methods and Applications **1** (2005), Paper 005, 7 pages.
[21] Saut, J. C.; *Quelques generalisations de l'equation de Korteweg - de Vries, II*, J. Differential Equations, **33** (1979), 320-335.
[22] Saut, J. C.; Scheurer, B.; *Unique continuation for some evolution equations,* J. Differential Equations **66** (1987), 118-139.
[23] Sidi, A.; Sulem, C.; Sulem, P.-L.; *On the long time behaviour of a generalized KdV equation*, Acta Appl. Math. **7** (1986), 35-47.

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