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ELECTROMAGNETIC TRANSMISSION PROBLEMS WITH A LARGE PARAMETER IN WEIGHTED SOBOLEV SPACES

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Abstract. We present an a priori estimate for an electromagnetic transmission problem in unbounded exterior domains in $\mathbb{R}^3$. We consider Maxwell's equations in two sub-domains, the bounded interior representing a conducting material (metal) and the unbounded exterior representing an insulating material (air). The behavior of the solution at infinity is described by means of families of weighted Sobolev spaces, so-called Beppo-Levi spaces [11]. We prove the existence and uniqueness of the solution.

1. The electromagnetic transmission problem

Let $\Omega^{\rm cd}$ be a bounded region in $\mathbb{R}^3$ representing a metallic conductor and $\Omega^{\rm is} := \mathbb{R}^3 \backslash \overline{\Omega^{\rm cd}}$. Let latter represents the air. The parameters ε_0, μ_0, σ denote permittivity, permeability, and conductivity in $\Omega^{\rm cd}$. We assume $\sigma = 0$ in $\Omega^{\rm is}$. All fields are time-harmonic with frequency ω. As in [13] we neglect conduction currents in the air and displacement currents in the metal. Thus we consider

$$\begin{aligned} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0, &\quad \text{in } \Omega^{\rm cd} \cup \Omega^{\rm is} \quad \text{(Faraday's law)}, \\ \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \sigma)\mathbf{E} = \mathbf{J}, &\quad \text{in } \Omega^{\rm cd} \cup \Omega^{\rm is} \quad \text{(Ampere's law)}, \end{aligned} \tag{1.1}$$

where $\mathbf{E}$ denotes the electric field, $\mathbf{H}$ the magnetic field and $\mathbf{J}$ the electric current. Across the interface Σ the tangential components of both $\mathbf{E}$ and $\mathbf{H}$ must be continuous; i.e. $\mathbf{E}_T^{\rm is} = \mathbf{E}_T^{\rm cd}$, $\mathbf{H}_T^{\rm is} = \mathbf{H}_T^{\rm cd}$. Furthermore the Silver-Müller radiation condition is assumed to hold at infinity (see (1.2) below).

Following Peron [17] we introduce a large parameter $\rho = \sqrt{\frac{\sigma}{\omega\varepsilon_0}} > 0$ and set $\mu = \sqrt{\mu_0/\varepsilon_0}$, $\varepsilon(\rho) = \frac{1}{\mu}(1_{\Omega^{\rm is}} + (1+i\rho^2)1_{\Omega^{\rm cd}})$, and $\mathbf{F} = i\kappa\mathbf{J}$. Then, defining $\widehat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$,

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equation (1.1) and the Silver-Müller radiation condition become

$$\begin{gathered}
\operatorname{curl}\mathbf{E}_\rho - i\kappa\mu\mathbf{H}_\rho = 0, \quad \text{in } \Omega^{\rm cd}\cup\Omega^{\rm is} \quad \text{(Faraday's law)},\\
\operatorname{curl}\mathbf{H}_\rho + i\kappa\varepsilon(\rho)\mathbf{E}_\rho = \frac{1}{i\kappa}\mathbf{F}, \quad \text{in } \Omega^{\rm cd}\cup\Omega^{\rm is} \quad \text{(Amper's law)},\\
|\mathbf{H}_\rho\times\widehat{\mathbf{x}} - \mathbf{E}_\rho| = o\big(\frac{1}{|\mathbf{x}|}\big), \quad \text{as } |\mathbf{x}|\to\infty \quad \text{(Silver-Müller radiation condition)}.
\end{gathered} \tag{1.2}$$

The first two equations in (1.2) reduce to

$$\frac{1}{\mu}\operatorname{curl}\operatorname{curl}\mathbf{E}_\rho - \kappa^2\varepsilon(\rho)\mathbf{E}_\rho = \mathbf{F}, \quad \text{in } \Omega^{\rm cd}\cup\Omega^{\rm is},$$

setting $\mathbf{H}_\rho = \frac{1}{i\kappa\mu}\operatorname{curl}\mathbf{E}_\rho$. The Silver-Müller radiation condition at infinity becomes

$$|\operatorname{curl}\mathbf{E}_\rho\times\widehat{\mathbf{x}} - \mathbf{E}_\rho| = o\big(\frac{1}{|\mathbf{x}|}\big), \quad \text{as } |\mathbf{x}|\to\infty. \tag{1.3}$$

Peron [17] considers problem (1.2) in a bounded domain Ω which is split into the conductor $\Omega^{\rm cd}$ and the insulator $\Omega^{\rm is}$, with either Dirichlet or Neumann condition. In our case Ω is unbounded and the boundary conditions are replaced by the Silver-Müller decay condition at infinity. Problem (1.2) may be analyzed using $H^s_{\rm loc}(\mathbb{R}^3)$ spaces or Beppo-Levi spaces. The reader is referred to Costabel and Stephan [5] and Giroire [8] for applications of these spaces to boundary value problems involving the Laplacian operator. Nedelec [14] uses of $H^s_{\rm loc}$ spaces for the study of problems in electromagnetic theory. Let

$$\mathbf{L}^2(\Omega^{\rm cd}) = (L^2(\Omega^{\rm cd}))^3 := \big\{\mathbf{u}:\Omega^{\rm cd}\to\mathbb{R}^3 : \int_{\Omega^{\rm cd}}|\mathbf{u}|^2dx<\infty\big\},$$

with norm

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega^{\rm cd})} = \Big(\int_{\Omega^{\rm cd}}|\mathbf{u}|^2dx\Big)^{1/2}.$$

Let also

$$\begin{gathered}
\mathbf{H}(\operatorname{curl},\Omega^{\rm cd}) = \{\mathbf{u}\in\mathbf{L}^2(\Omega^{\rm cd}) : \operatorname{curl}\mathbf{u}\in\mathbf{L}^2(\Omega^{\rm cd})\},\\
\mathbf{H}(\operatorname{div},\Omega^{\rm cd}) = \{\mathbf{u}\in\mathbf{L}^2(\Omega^{\rm cd}) : \operatorname{div}\mathbf{u}\in L^2(\Omega^{\rm cd})\},
\end{gathered}$$

with norms

$$\begin{gathered}
\|\mathbf{u}\|^2_{\mathbf{H}(\operatorname{curl},\Omega^{\rm cd})} = \|\operatorname{curl}\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})},\\
\|\mathbf{u}\|^2_{\mathbf{H}(\operatorname{div},\Omega^{\rm cd})} = \|\operatorname{div}\mathbf{u}\|^2_{L^2(\Omega^{\rm cd})} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})},
\end{gathered}$$

respectively. As in [17] we define

$$\mathbf{X}(\Omega^{\rm cd}) = \mathbf{H}(\operatorname{curl},\Omega^{\rm cd})\cap\mathbf{H}(\operatorname{div},\Omega^{\rm cd}),$$

with norm

$$\|\mathbf{u}\|^2_{\mathbf{X}(\Omega^{\rm cd})} = \|\operatorname{curl}\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})} + \|\operatorname{div}\mathbf{u}\|^2_{L^2(\Omega^{\rm cd})} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})},$$

and

$$\begin{gathered}
\mathbf{X}_T(\Omega^{\rm cd}) = \{\mathbf{u}\in\mathbf{X}(\Omega^{\rm cd}) : [\mathbf{n}\cdot\mathbf{u}] = 0, \text{ on } \Sigma\},\\
\mathbf{X}_N(\Omega^{\rm cd}) = \{\mathbf{u}\in\mathbf{X}(\Omega^{\rm cd}) : [\mathbf{n}\times\mathbf{u}] = 0, \text{ on } \Sigma\},\\
\mathbf{X}_T(\Omega^{\rm cd},\rho) = \{\mathbf{u}\in\mathbf{H}(\operatorname{curl},\Omega^{\rm cd}) : \varepsilon(\rho)\mathbf{u}\in\mathbf{H}(\operatorname{div},\Omega^{\rm cd}),\ [\mathbf{n}\cdot\mathbf{u}] = 0, \text{ on } \Sigma\},
\end{gathered}$$

$$\mathbf{X}_N(\Omega^{\rm cd},\rho)=\{\mathbf{u}\in\mathbf{H}(\mathrm{curl},\Omega^{\rm cd}):\varepsilon(\rho)\mathbf{u}\in\mathbf{H}(\mathrm{div},\Omega^{\rm cd}),\ [\mathbf{n}\times\mathbf{u}]=0,\ \text{on }\Sigma\}.$$

with norm

$$\|\mathbf{u}\|^2_{\mathbf{X}(\Omega^{\rm cd},\rho)}=\|\operatorname{curl}\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})}+\|\operatorname{div}(\varepsilon(\rho)\mathbf{u})\|^2_{L^2(\Omega^{\rm cd})}+\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})},$$

Note that $\mathbf{X}_T(\Omega^{\rm cd})$, $\mathbf{X}_N(\Omega^{\rm cd})$, $\mathbf{X}_T(\Omega^{\rm cd},\rho)$ and $\mathbf{X}_N(\Omega^{\rm cd},\rho)$ are Hilbert spaces. Also let $\mathfrak{D}$ denote the space of all C^∞-functions defined in $\mathbb{R}^3$ with compact support and $\mathfrak{D}'$ its topological dual space (space of distributions, see [18]).

In what follows we define several spaces of distributions which turn out to be Hilbert spaces. For detailed proof of the corresponding facts the reader is referred to [2, 6, 11] and [14, Section 2.5.4].

For $\mathbf{x}\in\mathbb{R}^3$, let $\ell(\|\mathbf{x}\|)=\sqrt{1+x_1^2+x_2^2+x_3^2}$, and

$$\mathbf{W}(\mathrm{curl},\mathbb{R}^3)=\{\mathbf{u}\in\mathfrak{D}'(\mathbb{R}^3):\ell(\|\mathbf{x}\|)^{-1}\mathbf{u}\in\mathbf{L}^2(\mathbb{R}^3),\ \operatorname{curl}\mathbf{u}\in\mathbf{L}^2(\mathbb{R}^3)\},$$
$$\mathbf{W}(\mathrm{div},\mathbb{R}^3)=\{\mathbf{u}\in\mathfrak{D}'(\mathbb{R}^3):\ell(\|\mathbf{x}\|)^{-1}\mathbf{u}\in\mathbf{L}^2(\mathbb{R}^3),\ \operatorname{div}\mathbf{u}\in L^2(\mathbb{R}^3)\}.$$

Note that $\mathbf{W}(\mathrm{curl},\mathbb{R}^3)$ and $\mathbf{W}(\mathrm{div},\mathbb{R}^3)$ are Hilbert spaces equipped with the norms

$$\|\mathbf{u}\|^2_{\mathbf{W}(\mathrm{curl},\mathbb{R}^3)}=\|\operatorname{curl}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}+\|\ell(\|\mathbf{x}\|)^{-1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)},$$

and

$$\|\mathbf{u}\|^2_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)}=\|\operatorname{div}\mathbf{u}\|^2_{L^2(\mathbb{R}^3)}+\|\ell(\|\mathbf{x}\|)^{-1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}.$$

Furthermore we will use the space

$$\mathbb{X}(\mathbb{R}^3)=\mathbf{W}(\mathrm{curl},\mathbb{R}^3)\cap\mathbf{W}(\mathrm{div},\mathbb{R}^3),$$

subject to the norm

$$\|\mathbf{u}\|^2_{\mathbb{X}(\mathbb{R}^3)}=\|\operatorname{curl}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}+\|\operatorname{div}\mathbf{u}\|^2_{L^2(\mathbb{R}^3)}+\|\ell(\|\mathbf{x}\|)^{-1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)},$$

and

$$\mathbb{X}_T(\mathbb{R}^3)=\{\mathbf{u}\in\mathbb{X}(\mathbb{R}^3):[\mathbf{n}\cdot\mathbf{u}]=0,\ \text{on }\Sigma\},$$
$$\mathbb{X}_N(\mathbb{R}^3)=\{\mathbf{u}\in\mathbb{X}(\mathbb{R}^3):[\mathbf{n}\times\mathbf{u}]=0,\ \text{on }\Sigma\},$$
$$\mathbb{X}_T(\mathbb{R}^3,\rho)=\{\mathbf{u}\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3):\varepsilon(\rho)\mathbf{u}\in\mathbf{W}(\mathrm{div},\mathbb{R}^3),[\mathbf{n}\cdot\mathbf{u}]=0,\ \text{on }\Sigma\},$$
$$\mathbb{X}_N(\mathbb{R}^3,\rho)=\{\mathbf{u}\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3):\varepsilon(\rho)\mathbf{u}\in\mathbf{W}(\mathrm{div},\mathbb{R}^3),[\mathbf{n}\times\mathbf{u}]=0,\ \text{on }\Sigma\},$$
$$\mathbb{X}_{TN}(\mathbb{R}^3,\rho)=\{\mathbf{u}\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3):\varepsilon(\rho)\mathbf{u}\in\mathbf{W}(\mathrm{div},\mathbb{R}^3),[\mathbf{n}\times\mathbf{u}]=[\mathbf{n}\cdot\mathbf{u}]=0,\ \text{on }\Sigma\},$$

with norm

$$\|\mathbf{u}\|^2_{\mathbb{X}_{TN}(\mathbb{R}^3,\rho)}=\|\operatorname{curl}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}+\|\operatorname{div}(\varepsilon(\rho)\mathbf{u})\|^2_{L^2(\mathbb{R}^3)}+\|\ell(\|\mathbf{x}\|)^{-1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}.$$

Note that $\mathbb{X}_T(\mathbb{R}^3)$, $\mathbb{X}_N(\mathbb{R}^3)$, $\mathbb{X}_T(\mathbb{R}^3,\rho)$, and $\mathbb{X}_N(\mathbb{R}^3,\rho)$ are Hilbert spaces. For m in $\mathbb{N}\cup\{0\}$ and k in $\mathbb{Z}$, we define

$$\mathbf{L}^2_{m,k}(\mathbb{R}^3):=\big\{\mathbf{u}:\mathbb{R}^3\to\mathbb{R}^3:\forall\alpha\in\mathbb{N}^3,0\le|\alpha|\le m,\ell(\|\mathbf{x}\|)^{|\alpha|-m+k}\mathbf{u}\in\mathbf{L}^2(\mathbb{R}^3)\big\},$$

with norm

$$\|\mathbf{u}\|_{\mathbf{L}^2_{m,k}(\mathbb{R}^3)}=\|\ell(\|\mathbf{x}\|)^{|\alpha|-m+k}\mathbf{u}\|_{\mathbf{L}^2(\mathbb{R}^3)},$$

where $\mathbf{L}^2_{m,k}(\mathbb{R}^3)=\big(L^2_{m,k}(\mathbb{R}^3)\big)^3$.

Next we extend [12, Theorems 1.2.16 and 1.2.17] for unbounded domains.

Lemma 1.1. *Let* $\mathbf{F} \in \mathbf{L}^2(\mathbb{R}^3)$*. Let* $\mathbf{E}_\rho$ *and* $\mathbf{H}_\rho$ *in* $\mathbf{L}^2(\mathbb{R}^3)$ *be a solution to* (1.2)*. Then,* $\mathbf{E}_\rho, \mathbf{H}_\rho \in \mathbf{W}(\operatorname{curl}, \mathbb{R}^3)$ *if and only if* $\mathbf{E}_\rho^{\rm cd}, \mathbf{H}_\rho^{\rm cd} \in \mathbf{H}(\operatorname{curl}, \Omega^{\rm cd})$ *and* $\mathbf{E}_\rho^{\rm is}, \mathbf{H}_\rho^{\rm is} \in \mathbf{W}(\operatorname{curl}, \Omega^{\rm is})$ *and* $[\mathbf{n}\times\mathbf{E}_\rho]_\Sigma = 0$*,* $[\mathbf{n}\times\mathbf{H}_\rho]_\Sigma = 0$*, where* $[\mathbf{u}]_\Sigma = \mathbf{u}^{\rm is} - \mathbf{u}^{\rm cd}$ *denotes the jump across* Σ*.*

Proof. "$\Rightarrow$" If $\mathbf{E}_\rho, \mathbf{H}_\rho \in \mathbf{W}(\operatorname{curl}, \mathbb{R}^3)$, then by definition $\mathbf{E}_\rho^{\rm cd}, \mathbf{H}_\rho^{\rm cd} \in \mathbf{H}(\operatorname{curl}, \Omega^{\rm cd})$ and $\mathbf{E}_\rho^{\rm is}, \mathbf{H}_\rho^{\rm is} \in \mathbf{W}(\operatorname{curl}, \Omega^{\rm is})$. Thus for $\mathbf{u}_\rho = \mathbf{E}_\rho$ or $\mathbf{u}_\rho = \mathbf{H}_\rho$, we have

$$\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho dx = \int_{\Omega^{\rm cd}} \mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho^{\rm cd} dx + \int_{\Omega^{\rm is}} \mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho^{\rm is} dx,$$

and

$$\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \mathbf{u}_\rho\cdot\operatorname{curl}\mathbf{v} dx = \int_{\Omega^{\rm cd}} \mathbf{u}_\rho^{\rm cd}\cdot\operatorname{curl}\mathbf{v} dx + \int_{\Omega^{\rm is}} \mathbf{u}_\rho^{\rm is}\cdot\operatorname{curl}\mathbf{v} dx,$$

for all $\mathbf{v} \in \mathbf{W}(\operatorname{curl}, \mathbb{R}^3)$.

In $\Omega^{\rm cd}$, integrating by parts (see [12, Theorem 1.2.17]) gives

$$\int_{\Omega^{\rm cd}} [\mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho^{\rm cd} dx - \mathbf{u}_\rho^{\rm cd}\cdot\operatorname{curl}\mathbf{v}] dx = \int_\Sigma [\mathbf{n}\times(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd})]\cdot(\mathbf{n}\times\mathbf{v}) ds.$$

where

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd}) = \mathbf{n}(\mathbf{n}\cdot\mathbf{u}_\rho^{\rm cd}) - \mathbf{u}_\rho^{\rm cd}(\mathbf{n}\cdot\mathbf{n}) = \mathbf{n}(\mathbf{n}\cdot\mathbf{u}_\rho^{\rm cd}) - \mathbf{u}_\rho^{\rm cd}.$$

Thus

$$[\mathbf{n}\times(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd})]\cdot(\mathbf{n}\times\mathbf{v}) = [\mathbf{n}(\mathbf{n}\cdot\mathbf{u}_\rho^{\rm cd}) - \mathbf{u}_\rho^{\rm cd}]\cdot(\mathbf{n}\times\mathbf{v}) = -\mathbf{u}_\rho^{\rm cd}\cdot(\mathbf{n}\times\mathbf{v}),$$

and

$$-\mathbf{u}_\rho^{\rm cd}\cdot(\mathbf{n}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd}).$$

Hence

$$\int_{\Omega^{\rm cd}} \mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho^{\rm cd} dx = \int_{\Omega^{\rm cd}} \mathbf{u}_\rho^{\rm cd}\cdot\operatorname{curl}\mathbf{v} dx + \int_\Sigma \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd}) ds.$$

Let B_R be a ball with radius $R > 0$ containing $\Omega^{\rm cd}$. Let $\Omega_R = B_R \cap \Omega^{\rm is}$. Hence $\partial\Omega_R = \partial B_R \cup \Sigma$.

In the domain Ω_R, we have, integrating by parts, (see [12, Theorem 1.2.17]),

$$\int_{\Omega_R} \mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho^{\rm is} dx = \int_{\Omega_R} \mathbf{u}_\rho^{\rm is}\cdot\operatorname{curl}\mathbf{v} dx - \int_\Sigma \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm is}) ds + \int_{\partial B_R} \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm is}) ds. \quad (1.4)$$

Due to the Silver-Müller radiation conditions (see (1.3))

$$\begin{aligned} \Big|\int_{\partial B_R} \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm is}) ds\Big| &\le \int_{\partial B_R} |\mathbf{v}||\mathbf{n}\times\mathbf{u}_\rho^{\rm is}| ds \\ &\le \int_{\partial B_R} |\mathbf{v}||\mathbf{n}||\mathbf{u}_\rho^{\rm is}||\sin\theta| ds \\ &\le \int_{\partial B_R} \frac{C_1}{R^2}\frac{C_2}{R^2} ds = \frac{C}{R^2} \to 0, \quad \text{as } R \to \infty. \end{aligned}$$

Hence, in (1.4) taking the limit as $R \to \infty$, we have

$$\int_{\Omega^{\rm is}} \mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho^{\rm is} dx = \int_{\Omega^{\rm is}} \mathbf{u}_\rho^{\rm is}\cdot\operatorname{curl}\mathbf{v} dx - \int_\Sigma \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm is}) ds.$$

Thus

$$\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \mathbf{v}\cdot\operatorname{curl}\mathbf{u}_\rho dx = \int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \mathbf{u}_\rho\cdot\operatorname{curl}\mathbf{v} dx + \int_\Sigma \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd} - \mathbf{n}\times\mathbf{u}_\rho^{\rm is}) ds,$$

yielding

$$\int_\Sigma \mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd}-\mathbf{n}\times\mathbf{u}_\rho^{\rm is})ds=0,$$

and therefore $[\mathbf{n}\times\mathbf{u}_\rho]_\Sigma=0$.

"$\Leftarrow$" If $\mathbf{E}_\rho^{\rm cd},\mathbf{H}_\rho^{\rm cd}\in\mathbf{H}(\mathrm{curl},\Omega^{\rm cd})$, $\mathbf{E}_\rho^{\rm is},\mathbf{H}_\rho^{\rm is}\in\mathbf{W}(\mathrm{curl},\Omega^{\rm is})$, $[\mathbf{n}\times\mathbf{E}_\rho]_\Sigma=0$, and $[\mathbf{n}\times\mathbf{H}_\rho]_\Sigma=0$. Then for $\mathbf{v}\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3)$ and $\mathbf{u}_\rho=\mathbf{E}_\rho$ or $\mathbf{u}_\rho=\mathbf{H}_\rho$, integrating by parts,

$$\begin{aligned}&\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}}\mathbf{u}_\rho\cdot\mathrm{curl}\,\mathbf{v}dx\\&=\int_{\Omega^{\rm cd}}\mathbf{u}_\rho^{\rm cd}\cdot\mathrm{curl}\,\mathbf{v}dx+\int_{\Omega^{\rm is}}\mathbf{u}_\rho^{\rm is}\cdot\mathrm{curl}\,\mathbf{v}dx,\\&=\int_{\Omega^{\rm cd}}\mathbf{v}\cdot\mathrm{curl}\,\mathbf{u}_\rho^{\rm cd}dx+\int_{\Omega^{\rm is}}\mathbf{v}\cdot\mathrm{curl}\,\mathbf{u}_\rho^{\rm is}dx+\int_\Sigma\mathbf{v}\cdot(\mathbf{n}\times\mathbf{u}_\rho^{\rm cd}-\mathbf{n}\times\mathbf{u}_\rho^{\rm is})ds.\end{aligned}$$

Hence $\mathbf{n}\times\mathbf{u}_\rho^{\rm cd}-\mathbf{n}\times\mathbf{u}_\rho^{\rm is}=0$ implies

$$\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}}\mathbf{u}_\rho\cdot\mathrm{curl}\,\mathbf{v}dx=\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}}\mathbf{v}\cdot\mathrm{curl}\,\mathbf{u}_\rho dx.$$

□

Our next Lemma adapts [17, Lemmas 2.7 and 2.8] for exterior domains in weighted Sobolev spaces.

Lemma 1.2. *Let* $\mathbf{F}\in\mathbf{W}(\mathrm{div},\mathbb{R}^3)$. *Let* $\mathbf{E}_\rho$ *and* $\mathbf{H}_\rho$ *in* $\mathbf{L}^2(\mathbb{R}^3)$ *solutions of* (1.2). *If* $\mathbf{E}_\rho,\mathbf{H}_\rho\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3)$, *then* $\varepsilon(\rho)\mathbf{E}_\rho,\mathbf{H}_\rho\in\mathbf{W}(\mathrm{div},\mathbb{R}^3)$, $[\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)]_\Sigma=0$, *and* $[\mathbf{n}\cdot\mathbf{H}_\rho]_\Sigma=0$. *Furthermore,*

$$\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)=-\frac{1}{\kappa^2}\,\mathrm{div}\,\mathbf{F},\quad \mathrm{div}\,\mathbf{H}_\rho=0\quad \textit{in }\mathbf{L}^2(\mathbb{R}^3).$$

Proof. If $\mathbf{F}\in\mathbf{W}(\mathrm{div},\mathbb{R}^3)$ and $\mathbf{E}_\rho,\mathbf{H}_\rho\in\mathbf{L}^2(\mathbb{R}^3)$ are solutions of (1.2), then applying divergence operator in (1.2), we have

$$\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)=-\frac{1}{\kappa^2}\,\mathrm{div}\,\mathbf{F},\quad \mathrm{div}\,\mathbf{H}_\rho=0\quad \mathrm{in}\mathbf{L}^2(\mathbb{R}^3),$$

and $\varepsilon(\rho)\mathbf{E}_\rho\in\mathbf{W}(\mathrm{div},\mathbb{R}^3)$, $\mathbf{H}_\rho\in\mathbf{W}(\mathrm{div},\mathbb{R}^3)$. Now, for $\mathbf{u}_\rho=\varepsilon(\rho)\mathbf{E}_\rho$ or $\mathbf{H}_\rho$, we have

$$\begin{aligned}\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}}\phi\,\mathrm{div}\,\mathbf{u}_\rho dx&=\int_{\Omega^{\rm cd}}\phi\,\mathrm{div}\,\mathbf{u}_\rho^{\rm cd}dx+\int_{\Omega^{\rm is}}\phi\,\mathrm{div}\,\mathbf{u}_\rho^{\rm is}dx,\\\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}}\mathbf{u}_\rho\cdot\nabla\phi dx&=\int_{\Omega^{\rm cd}}\mathbf{u}_\rho^{\rm cd}\cdot\nabla\phi dx+\int_{\Omega^{\rm is}}\mathbf{u}_\rho^{\rm is}\cdot\nabla\phi dx,\end{aligned}$$

for all $\phi\in\mathbf{V}=H_0^1(\Omega^{\rm cd})\cup\mathbb{W}_0^1(\Omega^{\rm is})$. In $\Omega^{\rm cd}$, integrating by parts (see [12, Theorem 1.2.16]),

$$\int_{\Omega^{\rm cd}}\phi\,\mathrm{div}\,\mathbf{u}_\rho^{\rm cd}dx=\int_{\Omega^{\rm cd}}\mathbf{u}_\rho^{\rm cd}\cdot\nabla\phi dx-\int_\Sigma(\mathbf{n}\cdot\mathbf{u}_\rho^{\rm cd})\phi ds.$$

Let be a ball B_R with radius $R>0$ containing $\Omega^{\rm cd}$. Let $\Omega_R=B_R\cap\Omega^{\rm is}$. Hence $\partial\Omega_R=\partial B_R\cup\Sigma$. In the domain Ω_R, we have, integrating by parts, (see [12, Theorem 1.2.16]),

$$\int_{\Omega_R}\phi\,\mathrm{div}\,\mathbf{u}_\rho^{\rm is}dx$$

$$
\begin{aligned}
&= \int_{\Omega_R} \mathbf{u}_\rho^{\rm is} \cdot \nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot \mathbf{u}_\rho^{\rm is})\phi ds - \int_{\partial B_R} (\mathbf{n}\cdot \mathbf{u}_\rho^{\rm is})\phi ds \\
&= \int_{\Omega_R} \mathbf{u}_\rho^{\rm is} \cdot \nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot \mathbf{u}_\rho^{\rm is})\phi ds - \int_{\partial B_R} (\mathbf{n}\cdot [\mathbf{u}_\rho^{\rm is} + \mathbf{U}_\rho^{\rm is}\times \mathbf{n}])\phi ds,
\end{aligned}
$$

where, if $\mathbf{u}_\rho^{\rm is} = \varepsilon(\rho)\mathbf{E}_\rho$, $\mathbf{U}_\rho^{\rm is} = -\varepsilon(\rho)\mathbf{H}_\rho$ or if $\mathbf{u}_\rho^{\rm is} = \mathbf{H}_\rho$, $\mathbf{U}_\rho^{\rm is} = \mathbf{E}_\rho$ and $\mathbf{n}\cdot[\mathbf{U}_\rho^{\rm is}\times\mathbf{n}] = 0$. Due to the Silver-Müller radiation conditions (see (1.3))

$$
\int_{\partial B_R} \mathbf{n}\cdot[\mathbf{u}_\rho^{\rm is} + \mathbf{U}_\rho^{\rm is}\times\mathbf{n}]\cdot\phi ds \to 0 \quad \text{as} R\to\infty.
$$

Hence

$$
\int_{\Omega^{\rm is}} \phi \operatorname{div}\mathbf{u}_\rho^{\rm is} dx = \int_{\Omega^{\rm is}} \mathbf{u}_\rho^{\rm is}\cdot\nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot\mathbf{u}_\rho^{\rm is})\phi ds.
$$

Altogether we have

$$
\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \phi \operatorname{div}\mathbf{u}_\rho dx = \int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \mathbf{u}_\rho\cdot\nabla\phi dx + \int_\Sigma (\mathbf{n}\cdot\mathbf{u}_\rho^{\rm is} - \mathbf{n}\cdot\mathbf{u}_\rho^{\rm cd})\cdot\phi ds
$$

then

$$
\int_\Sigma (\mathbf{n}\cdot\mathbf{u}_\rho^{\rm is} - \mathbf{n}\cdot\mathbf{u}_\rho^{\rm cd})\cdot\phi ds = 0,
$$

for all ϕ implies $[\mathbf{n}\cdot\mathbf{u}_\rho]_\Sigma = 0$. □

For $\mathbf{E}_\rho, \mathbf{E}' \in \widetilde{\mathbf{W}}(\mathrm{curl},\mathbb{R}^3) := \{\mathbf{E}\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3) : \mathbf{E}\in\mathbf{L}^2(\mathbb{R}^3)\}$, set

$$
b_\rho(\mathbf{E}_\rho,\mathbf{E}') := \int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \Big(\frac{1}{\mu}\operatorname{curl}\mathbf{E}_\rho\cdot\operatorname{curl}\overline{\mathbf{E}'} - \kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\mathbf{E}'}\Big)dx. \tag{1.5}
$$

Proposition 1.3. *If $\mathbf{F}\in\mathbf{L}^2(\mathbb{R}^3)$, $\mathbf{E}_\rho,\mathbf{H}_\rho\in\mathbf{L}^2(\mathbb{R}^3)$ satisfy* (1.2)*, then, $\mathbf{E}_\rho\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3)$ and for all $\mathbf{E}'\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3)$,*

$$
b_\rho(\mathbf{E}_\rho,\mathbf{E}') = \int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \mathbf{F}\cdot\overline{\mathbf{E}'}dx. \tag{1.6}
$$

The proof of Proposition 1.3 is a minor modification of the proof [17, Proposition 3], and is left for the reader.

Proposition 1.4. *If $\mathbf{E}_\rho\in\mathbf{W}(\mathrm{curl},\mathbb{R}^3)$ satisfies* (1.6)*, then $\mathbf{E}_\rho$ satisfies (in the sense of distributions):*

$$
\begin{gathered}
\operatorname{curl}\operatorname{curl}\mathbf{E}_\rho - \kappa^2\mathbf{E}_\rho = \mu\mathbf{F}^{\rm is}, \quad \text{in } \Omega^{\rm is}, \\
\operatorname{curl}\operatorname{curl}\mathbf{E}_\rho - \kappa^2(1+i\rho^2)\mathbf{E}_\rho = \mu\mathbf{F}^{\rm cd}, \quad \text{in } \Omega^{\rm cd}, \\
[\mathbf{n}\times\mathbf{E}_\rho]_\Sigma = 0, \quad [\mathbf{n}\times\operatorname{curl}\mathbf{E}_\rho]_\Sigma = 0, \quad \text{on } \Sigma,
\end{gathered} \tag{1.7}
$$

with Silver-Müller condition

$$
|\operatorname{curl}\mathbf{E}_\rho\times\widehat{\mathbf{x}} - \mathbf{E}_\rho| = o\big(\frac{1}{|\mathbf{x}|}\big), \quad \text{as } |\mathbf{x}|\to\infty,
$$

On the other hand, if $\mathbf{E}_\rho$ solves (1.7)*, then*

$$
\frac{1}{\mu}\operatorname{curl}\operatorname{curl}\mathbf{E}_\rho - \kappa^2\varepsilon(\rho)\mathbf{E}_\rho = \mathbf{F}, \quad \text{in } \Omega^{\rm cd}\cup\Omega^{\rm is}, \tag{1.8}
$$

$$
\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho) = -\frac{1}{\kappa^2}\operatorname{div}\mathbf{F}, \quad \text{in } \Omega^{\rm cd}\cup\Omega^{\rm is}. \tag{1.9}
$$

Proof. We follow the proof of [17, Proposition 2.15] to show that $\mathbf{E}_\rho$ satisfies the first two equations in (1.7).

Taking $\mathbf{E}' \in \mathfrak{D}'(\mathbb{R}^3)$ with support in $\Omega^{\rm is}$ as test function in (1.6) and using

$$\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \operatorname{curl}\mathbf{E}_\rho \cdot \operatorname{curl}\overline{\mathbf{E}'}dx = \langle \operatorname{curl}\operatorname{curl}\mathbf{E}_\rho, \mathbf{E}'\rangle_{\Omega^{\rm is}},$$

we see that the first equation in (1.7) is satisfied. Taking $\mathbf{E}' \in \mathfrak{D}'(\mathbb{R}^3)$ with support in $\Omega^{\rm cd}$ as test function in (1.6) and using

$$\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \operatorname{curl}\mathbf{E}_\rho \cdot \operatorname{curl}\overline{\mathbf{E}'}dx = \langle \operatorname{curl}\operatorname{curl}\mathbf{E}_\rho, \mathbf{E}'\rangle_{\Omega^{\rm cd}},$$

we see that the second equation in (1.7) is satisfied. By Lemma 1.1, the third relation (1.7) holds.

Let B_R be a ball with radius $R > 0$ containing $\Omega^{\rm cd}$. Let $\Omega = \Omega^{\rm cd} \cup (\Omega^{\rm is} \cap B_R)$. Hence $\partial(\Omega^{\rm is} \cap B_R) = \Sigma \cup \partial B_R = \partial\Omega$, for all $\mathbf{E}, \mathbf{H} \in \mathbf{H}(\operatorname{curl}, \Omega)$,

$$\int_\Omega \operatorname{curl}\mathbf{E} \cdot \overline{\mathbf{H}} - \mathbf{E} \cdot \operatorname{curl}\overline{\mathbf{H}})dx = \langle \mathbf{n} \times \mathbf{E}, \mathbf{H}_\tau\rangle_{\partial\Omega}, \tag{1.10}$$

where $\mathbf{H}_\tau = (\mathbf{n} \times \mathbf{H}) \times \mathbf{n}$. From the first equation in (1.7) we have $\operatorname{curl}\mathbf{E}^{\rm is} \in \mathbf{H}(\operatorname{curl}, \Omega^{\rm is} \cap B_R)$. Applying formula (1.10) in $\Omega^{\rm is} \cap B_R$ to $\mathbf{E} = \operatorname{curl}\mathbf{E}^{\rm is}_\rho$ and $\mathbf{H} = \mathbf{E}' \in \mathbf{H}(\operatorname{curl}, \Omega)$, we have

$$\begin{aligned} &\int_{\Omega^{\rm is}\cap B_R} \operatorname{curl}\mathbf{E}^{\rm is}_\rho \cdot \operatorname{curl}\overline{(\mathbf{E}')^{\rm is}}dx \\ &= \int_{\Omega^{\rm is}\cap B_R} \operatorname{curl}\operatorname{curl}\mathbf{E}^{\rm is}_\rho \cdot \overline{(\mathbf{E}')^{\rm is}}dx + \langle \operatorname{curl}\mathbf{E}^{\rm is}_\rho \times \mathbf{n}, (\mathbf{E}')^{\rm is}_\tau\rangle_{\partial(\Omega^{\rm is}\cap B_R)}. \end{aligned} \tag{1.11}$$

Applying formula (1.10) in $\Omega^{\rm cd}$ to $\mathbf{E} = \operatorname{curl}\mathbf{E}^{\rm cd}_\rho$ and $\mathbf{H} = \mathbf{E}' \in \mathbf{H}(\operatorname{curl}, \Omega)$, we have

$$\begin{aligned} &\int_{\Omega^{\rm cd}} \operatorname{curl}\mathbf{E}^{\rm cd}_\rho \cdot \operatorname{curl}\overline{(\mathbf{E}')^{\rm cd}}dx \\ &= \int_{\Omega^{\rm cd}} \operatorname{curl}\operatorname{curl}\mathbf{E}^{\rm cd}_\rho \cdot \overline{(\mathbf{E}')^{\rm cd}}dx + \langle \operatorname{curl}\mathbf{E}^{\rm cd}_\rho \times \mathbf{n}, (\mathbf{E}')^{\rm cd}_\tau\rangle_{\Sigma}. \end{aligned} \tag{1.12}$$

In (1.11),

$$\langle \operatorname{curl}\mathbf{E}^{\rm is}_\rho \times \mathbf{n}, (\mathbf{E}')^{\rm is}_\tau\rangle_{\partial(\Omega^{\rm is}\cap B_R)} = \langle \operatorname{curl}\mathbf{E}^{\rm is}_\rho \times \mathbf{n}, (\mathbf{E}')^{\rm is}_\tau\rangle_{\Sigma} + \langle \operatorname{curl}\mathbf{E}^{\rm is}_\rho \times \mathbf{n}, (\mathbf{E}')^{\rm is}_\tau\rangle_{\partial B_R}.$$

Applying Silver-Müller radiation condition yields

$$\begin{aligned} |\langle \operatorname{curl}\mathbf{E}^{\rm is}_\rho \times \mathbf{n}, (\mathbf{E}')^{\rm is}_\tau\rangle_{\partial B_R}| &= \Big| \int_{\partial B_R} \operatorname{curl}\mathbf{E}^{\rm is}_\rho \times \mathbf{n} \cdot (\mathbf{E}')^{\rm is}_\tau ds\Big| \\ &\le \int_{\partial B_R} |\operatorname{curl}\mathbf{E}^{\rm is}_\rho \times \mathbf{n}||(\mathbf{n} \times \mathbf{E}') \times \mathbf{n}|ds \\ &\le \int_{\partial B_R} |\operatorname{curl}\mathbf{E}^{\rm is}_\rho||\mathbf{n}||\sin\theta_1||\mathbf{n}||\mathbf{n} \times \mathbf{E}'||\sin\theta_2|ds \\ &\le \int_{\partial B_R} C_1|\operatorname{curl}\mathbf{E}^{\rm is}_\rho||\mathbf{n}||\mathbf{E}'||\sin\theta_3|ds \\ &\le \int_{\partial B_R} C_1|\operatorname{curl}\mathbf{E}^{\rm is}_\rho||\mathbf{E}'|ds \\ &= C_1\frac{C_2}{R^4}R^2 = \frac{C}{R^2} \to 0, \quad \text{as } R \to \infty. \end{aligned}$$

Then, by the dominated convergence Theorem,

$$\begin{aligned} &\int_{\Omega^{\rm is}} \operatorname{curl} \mathbf{E}_\rho^{\rm is} \cdot \operatorname{curl} \overline{(\mathbf{E}')^{\rm is}} dx \\ &= \int_{\Omega^{\rm is}} \operatorname{curl} \operatorname{curl} \mathbf{E}_\rho^{\rm is} \cdot \overline{(\mathbf{E}')^{\rm is}} dx + \langle \operatorname{curl} \mathbf{E}_\rho^{\rm is} \times \mathbf{n}, (\mathbf{E}')_\tau^{\rm is} \rangle_\Sigma. \end{aligned} \tag{1.13}$$

From (1.12) and (1.13),

$$\begin{aligned} &\int_{\Omega^{\rm cd} \cup \Omega^{\rm is}} \operatorname{curl} \mathbf{E}_\rho \cdot \operatorname{curl} \overline{\mathbf{E}'} dx \\ &= \int_{\Omega^{\rm cd} \cup \Omega^{\rm is}} \operatorname{curl} \operatorname{curl} \mathbf{E}_\rho \cdot \overline{\mathbf{E}'} dx + \langle [\operatorname{curl} \mathbf{E}_\rho \times \mathbf{n}]_\Sigma, \mathbf{E}'_\tau \rangle_\Sigma, \end{aligned}$$

for all $\mathbf{E}' \in \mathbf{W}(\operatorname{curl}, \mathbb{R}^3)$. From (1.6) and the first two equations in (1.7),

$$\langle [\operatorname{curl} \mathbf{E}_\rho \times \mathbf{n}]_\Sigma, \mathbf{E}'_\tau \rangle_\Sigma = 0,$$

which proofs the fourth equation in (1.7). $\square$

Next we consider a *regularized version* of problem (1.6). Namely: We consider finding $\mathbf{E}_\rho \in \mathbb{X}_T(\mathbb{R}^3, \rho)$, such that, for all $\mathbf{E}'_\rho \in \mathbb{X}_T(\mathbb{R}^3, \rho)$,

$$\begin{aligned} &\int_{\Omega^{\rm cd} \cup \Omega^{\rm is}} \Big(\frac{1}{\mu} \operatorname{curl} \mathbf{E}_\rho \cdot \operatorname{curl} \overline{\mathbf{E}'_\rho} + \alpha \operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho) \cdot \operatorname{div}(\overline{\varepsilon(\rho)\mathbf{E}'_\rho}) - \kappa^2 \varepsilon(\rho) \mathbf{E}_\rho \cdot \overline{\mathbf{E}'_\rho} \Big) dx \\ &= \langle f, \mathbf{E}'_\rho \rangle, \end{aligned} \tag{1.14}$$

where

$$\langle f, \mathbf{E}'_\rho \rangle = \int_{\Omega^{\rm cd} \cup \Omega^{\rm is}} \Big(\mathbf{F} \cdot \overline{\mathbf{E}'_\rho} - \frac{\alpha}{\kappa^2} \operatorname{div} \mathbf{F} \cdot \operatorname{div}(\overline{\varepsilon(\rho)\mathbf{E}'_\rho}) \Big) dx, \tag{1.15}$$

and where $\alpha > 0$.

The next theorem, extends [17, Theorem 2.21] (see also Costabel et al. [4]), which corresponds to Peron's theorem [17, Theorem 2.21] and is its modification for an unbounded exterior domain and weighted spaces.

Theorem 1.5. *There exists $\alpha > 0$, independent of ρ, such that if $\mathbf{E}_\rho \in \mathbb{X}_T(\mathbb{R}^3, \rho)$ is a solution of* (1.14)–(1.15) *for $\mathbf{F} \in \mathbf{W}_0(\operatorname{div}, \mathbb{R}^3)$, then*

$$\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho) + \frac{1}{\kappa^2} \operatorname{div} \mathbf{F} = 0, \quad \text{in } \Omega^{\rm cd} \cup \Omega^{\rm is}. \tag{1.16}$$

Furthermore $\mathbf{E}_\rho$ and $\mathbf{H}_\rho = \frac{1}{i\omega\varepsilon_0} \operatorname{curl} \mathbf{E}_\rho$ satisfy Maxwell's equations (1.2).

Proof. Let us define the operator $\Delta^N_{\varepsilon(\rho)}$ from $\mathbb{W}^1_0(\mathbb{R}^3)$ to $\mathbb{W}^1_0(\mathbb{R}^3)'$ mapping φ to $\operatorname{div}(\varepsilon(\rho)\nabla\varphi)$, where $\operatorname{div}(\varepsilon(\rho)\nabla\varphi) \in \mathbb{W}^1_0(\mathbb{R}^3)'$ defined for any $\psi \in \mathbb{W}^1_0(\mathbb{R}^3)$ by

$$\int_{\Omega^{\rm cd} \cup \Omega^{\rm is}} \varepsilon(\rho) \nabla\varphi \cdot \overline{\nabla\psi} dx.$$

see [17, Theorem 2.21]. Defining the domain of $\Delta^N_{\varepsilon(\rho)}$ by

$$\mathbf{D}(\Delta^N_{\varepsilon(\rho)}) = \{\varphi \in \mathbb{W}^1_0(\mathbb{R}^3) | \quad \operatorname{div}(\varepsilon(\rho)\nabla\varphi) \in L^2(\mathbb{R}^3)\},$$

$\nabla\varphi \in \mathbb{X}_T(\mathbb{R}^3, \rho)$ for $\varphi \in \mathbf{D}(\Delta^N_{\varepsilon(\rho)})$.

Let $\mathbf{E}_\rho$ satisfy (1.14). Choosing $\mathbf{E}' = \nabla\varphi$ with $\varphi \in \mathbf{D}(\Delta^N_{\varepsilon(\rho)})$, (1.14) gives

$$\begin{aligned}\int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} &(\alpha \operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot \operatorname{div}(\overline{\varepsilon(\rho)\nabla\varphi}) - \kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx \\ &= \int_{\Omega^{\rm cd}\cup\Omega^{\rm is}} \Big(\mathbf{F}\cdot\overline{\nabla\varphi} - \frac{\alpha}{\kappa^2}\operatorname{div}\mathbf{F}\cdot\operatorname{div}(\overline{\varepsilon(\rho)\nabla\varphi})\Big)dx.\end{aligned}\tag{1.17}$$

Since $\varepsilon(\rho)\mathbf{E}_\rho, \mathbf{F} \in \mathbf{W}_0(\operatorname{div},\mathbb{R}^3)$ and $\varphi \in \mathbb{W}^1_0(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\Omega^{\rm cd}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx + \int_{\Omega^{\rm is}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx.$$

Green's formula in $\Omega^{\rm cd}$, yields

$$\int_{\Omega^{\rm cd}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\Omega^{\rm cd}} \kappa^2\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi})dx + \int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho\overline{\nabla\varphi}))dS.$$

Let B_R be a ball with radius $R>0$ containing $\Omega^{\rm cd}$, with $\partial\Omega_R = \partial B_R\cup\Sigma$. Let Ω_R be as above. Integrating by parts,

$$\int_{\Omega_R} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx = \int_{\Omega_R} \kappa^2\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi})dx + \int_{\partial\Omega_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds,\tag{1.18}$$

and

$$\int_{\partial\Omega_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds = \int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds + \int_{\partial B_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi})ds.$$

As in the proof of Lemma 1.2, applying the Silver-Müller condition (see (1.7)),

$$\int_{\partial B_R} \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi}ds \to 0, \quad \text{as } R\to\infty,$$

Hence, by (1.18),

$$\begin{aligned}&\int_{\Omega^{\rm is}} -\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi})dx \\ &= \int_{\Omega^{\rm is}} \kappa^2\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi}dx - \int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho\overline{\nabla\varphi}))ds - \int_\Sigma \kappa^2\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)\overline{\varphi}ds = 0,\end{aligned}$$

for all $\varphi\in\mathbf{D}(\Delta^N_{\varepsilon(\rho)})$ yielding $[\mathbf{n}\cdot(\varepsilon(\rho)\mathbf{E}_\rho)] = 0$ on Σ. Now

$$\int_{\mathbb{R}^3}\mathbf{F}\cdot\overline{\nabla\varphi}dx = \int_{\Omega^{\rm cd}}\mathbf{F}\cdot\overline{\nabla\varphi}dx + \int_{\Omega^{\rm is}}\mathbf{F}\cdot\overline{\nabla\varphi}dx,$$

Green's formula in $\Omega^{\rm cd}$, yields

$$\int_{\Omega^{\rm cd}}\mathbf{F}\cdot\overline{\nabla\varphi}dx = -\int_{\Omega^{\rm cd}}\operatorname{div}\mathbf{F}\cdot\overline{\varphi}dx - \int_\Sigma(\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds.$$

Again, we choose $R>0$ such that B_R contains $\Omega^{\rm cd}$. Using again that $\Omega^{\rm is} = \bigcup_{R>0}\Omega_R$ and that $\partial\Omega_R = \partial B_R\cup\Sigma$.

Applying the divergence theorem to $\mathbf{F} = i\kappa\operatorname{curl}\mathbf{H} - \kappa^2\varepsilon(\rho)\mathbf{E}$ in Ω_R, and the Silver-Müller condition, we have

$$\int_{\partial B_R}(\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds\to 0, \quad \text{as } R\to\infty.$$

Hence

$$\int_{\Omega^{\rm is}}\mathbf{F}\cdot\overline{\nabla\varphi}dx = -\int_{\Omega^{\rm is}}\operatorname{div}\mathbf{F}\cdot\overline{\varphi}dx + \int_\Sigma(\mathbf{n}\cdot\mathbf{F})\overline{\varphi}ds.$$

Then, we have

$$\begin{aligned}\int_{\mathbb{R}^3} -\kappa^2 \varepsilon(\rho)\mathbf{E}_\rho \cdot \overline{\nabla\varphi})dx &= \int_{\mathbb{R}^3} \kappa^2 \operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot \overline{\varphi}dx + \int_\Sigma \kappa^2 \mathbf{n}\cdot[\varepsilon(\rho)\mathbf{E}_\rho]\overline{\varphi}ds \\ &= \int_{\mathbb{R}^3} \kappa^2 \operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot \overline{\varphi}dx\end{aligned}$$

and

$$\int_{\mathbb{R}^3} \mathbf{F}\cdot \overline{\nabla\varphi}dx = -\int_{\mathbb{R}^3} \operatorname{div}\mathbf{F}\cdot\overline{\varphi}dx.$$

Similarly, according to (1.17) there holds

$$\begin{aligned}&\int_{\mathbb{R}^3} \Big(\alpha \operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot \operatorname{div}(\overline{\varepsilon(\rho)\nabla\varphi}) + \frac{\alpha}{\kappa^2}\operatorname{div}\mathbf{F}\cdot\operatorname{div}(\overline{\varepsilon(\rho)\nabla\varphi}) \\ &- \kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\varphi} - \mathbf{F}\cdot\overline{\nabla\varphi}\Big)dx = 0.\end{aligned}$$

Then

$$\begin{aligned}&\int_{\mathbb{R}^3} \Big(\alpha \operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot \operatorname{div}(\overline{\varepsilon(\rho)\nabla\varphi}) + \frac{\alpha}{\kappa^2}\operatorname{div}\mathbf{F}\cdot\operatorname{div}(\overline{\varepsilon(\rho)\nabla\varphi}) \\ &+ \kappa^2\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\varphi} + \operatorname{div}\mathbf{F}\cdot\overline{\varphi}\Big)dx = 0.\end{aligned}$$

Therefore, for all $\varphi \in \mathbf{D}(\Delta^N_{\varepsilon(\rho)})$,

$$\int_{\mathbb{R}^3} \big(\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho) + \frac{1}{\kappa^2}\operatorname{div}\mathbf{F}\big)\cdot(\alpha\operatorname{div}(\overline{\varepsilon(\rho)\nabla\varphi}) + \kappa^2\overline{\varphi})dx = 0. \tag{1.19}$$

The sesquilinear form associated with the operator $-\Delta^N_{\varepsilon(\rho)}$ is uniformly coercive on $\mathbb{W}^1_0(\mathbb{R}^3)$, because (see Giroire [8])

$$\operatorname{Re}\Big(\int_{\mathbb{R}^3}\varepsilon(\rho)\nabla\varphi\cdot\overline{\nabla\varphi}dx\Big) = \frac{1}{\mu}|\varphi|^2_{\mathbb{W}^1_0(\mathbb{R}^3)} \geq C\|\varphi\|^2_{\mathbb{W}^1_0(\mathbb{R}^3)}. \tag{1.20}$$

Next, we follow again Peron [17] and examine the real non-zero eigenvalues λ of $-\Delta^N_{\varepsilon(\rho)}$; i.e,

$$-\Delta^N_{\varepsilon(\rho)}\varphi = \lambda\varphi \quad \text{in } \mathbb{R}^3, \tag{1.21}$$

which, after integration by parts, gives

$$\int_{\mathbb{R}^3}\varepsilon(\rho)\nabla\varphi\cdot\overline{\nabla\varphi}dx = \lambda\int_{\mathbb{R}^3}\varphi\cdot\overline{\varphi}dx.$$

Now (1.20) gives $\lambda \geq C$ and we take $\alpha > 0$ large enough such that $\frac{\kappa^2}{\alpha} < C$. Then $\frac{\kappa^2}{\alpha}$ is not an eigenvalue of $-\Delta^N_{\varepsilon(\rho)}$. Consequently (1.19) implies

$$\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho) + \frac{1}{\kappa^2}\operatorname{div}\mathbf{F} = 0, \quad \text{in } \mathbb{R}^3.$$

This way, from (1.14) and (1.15),

$$\int_{\mathbb{R}^3}\Big(\frac{1}{\mu}\operatorname{curl}\mathbf{E}_\rho\cdot\operatorname{curl}\overline{\mathbf{E}'_\rho} - \kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\mathbf{E}'_\rho}\Big)dx = \int_{\mathbb{R}^3}\mathbf{F}\cdot\overline{\mathbf{E}'_\rho}dx.$$

for all $\mathbf{E}'_\rho \in \mathbb{X}_T(\mathbb{R}^3,\rho)$.

We define $\mathbf{H}_\rho$ from Faraday's law by $\mathbf{H}_\rho = \frac{1}{i\omega\varepsilon_0}\operatorname{curl}\mathbf{E}_\rho$ in $\mathbb{R}^3$. Then from Proposition 2, it follows that

$$\int_{\mathbb{R}^3}\Big(\frac{i\omega\varepsilon_0}{\mu}\operatorname{curl}\mathbf{H}_\rho\cdot\overline{\mathbf{E}'_\rho}-\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\mathbf{E}'_\rho}\Big)dx=\int_{\mathbb{R}^3}\mathbf{F}\cdot\overline{\mathbf{E}'_\rho}dx,$$

for all $\mathbf{E}'_\rho\in\mathbb{X}_T(\mathbb{R}^3,\rho)$, implying

$$\operatorname{curl}\mathbf{H}_\rho+i\kappa\varepsilon(\rho)\mathbf{E}_\rho=\frac{1}{i\kappa}\mathbf{F},\quad \text{in } \mathbb{R}^3.$$

□

Remark 1.6. An easy modification of the proof of the Theorem 1 shows that there exists $\beta>0$, independent of ρ, such that if $\mathbf{E}_\rho\in\mathbb{X}_N(\mathbb{R}^3,\rho)$ is a solution to

$$\int_{\mathbb{R}^3}\Big(\frac{1}{\mu}\operatorname{curl}\mathbf{E}_\rho\cdot\operatorname{curl}\overline{\mathbf{E}'_\rho}+\beta\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\operatorname{div}(\overline{\varepsilon(\rho)\mathbf{E}'_\rho})-\kappa^2\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\mathbf{E}'_\rho}\Big)dx=\langle f,\mathbf{E}'_\rho\rangle, \tag{1.22}$$

and (1.15) for some $\mathbf{F}\in\mathbf{W}(\operatorname{div},\mathbb{R}^3)$, then

$$\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)+\frac{1}{\kappa^2}\operatorname{div}\mathbf{F}=0,\quad \text{in } \mathbb{R}^3. \tag{1.23}$$

Furthermore, $\mathbf{E}_\rho$ and $\mathbf{H}_\rho=\frac{1}{i\omega\varepsilon_0}\operatorname{curl}\mathbf{E}_\rho$ solve (1.2). This results corresponds directly to [17, Theorem 2.22].

In this part, we give a variational formulation for the term $\varphi_\rho\in\mathcal{V}$, with $\mathcal{V}=H^1_0(\Omega_-)\cup\mathbb{W}^1_0(\Omega_+)$, (see [16, Chapter 2] and [15]), which appears in the decomposition of the electrical field, to see Theorem 2.1. Again, we extend the ideas of Peron [17] to prove Lemmas 1.7 and 1.8 for the unbounded exterior domain. Our Lemma 1.7 corresponds to [17, Lemma 2.33] and gives the appropriate setting for an unbounded exterior domain.

Lemma 1.7. *Let $\mathbf{E}_\rho\in\mathbb{X}_T(\mathbb{R}^3,\rho)$ satisfy (1.14)-(1.15) for $\mathbf{F}\in\mathbf{W}_0(\operatorname{div},\mathbb{R}^3)$, and let $(\mathbf{w}_\rho,\varphi_\rho)\in\mathbf{W}^1_0(\mathbb{R}^3)\times\mathcal{V}$ with $\operatorname{div}\mathbf{w}_\rho=0$ given by Theorem 2.1. Then, φ_ρ solves the variational problem: Find $\varphi_\rho\in\mathcal{V}$, such that for all $\psi\in\mathcal{V}$,*

$$\int_{\mathbb{R}^3}\varepsilon(\rho)\nabla\varphi_\rho\cdot\overline{\nabla\psi}dx=\frac{1}{\kappa^2}\int_{\mathbb{R}^3}\operatorname{div}\mathbf{F}\cdot\overline{\psi}dx+\frac{1}{\mu}i\rho^2\int_\Sigma\mathbf{w}_\rho\cdot\mathbf{n}|_\Sigma\overline{\psi}ds. \tag{1.24}$$

Proof. Due to Theorem 2.1 there exists an unique couple $(\mathbf{w}_\rho,\varphi_\rho)\in\mathbb{W}^1_0(\mathbb{R}^3)\times\mathcal{V}$ such that $\mathbf{E}_\rho=\mathbf{w}_\rho+\nabla\varphi_\rho$. Thus we have

$$\int_{\mathbb{R}^3}\varepsilon(\rho)\nabla\varphi_\rho\cdot\overline{\nabla\psi}dx=\int_{\mathbb{R}^3}\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\psi}dx-\int_{\mathbb{R}^3}\varepsilon(\rho)\mathbf{w}_\rho\cdot\overline{\nabla\psi}dx,\quad\forall\psi\in\mathcal{V}.$$

Then, since $\varepsilon(\rho)\mathbf{E}_\rho\in\mathbf{W}_0(\operatorname{div},\mathbb{R}^3)$, there holds

$$\int_{\mathbb{R}^3}\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\psi}dx=-\int_{\mathbb{R}^3}\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\overline{\psi}dx,$$

so, due to Theorem 1.5,

$$\int_{\mathbb{R}^3}\varepsilon(\rho)\mathbf{E}_\rho\cdot\overline{\nabla\psi}dx=\frac{1}{\kappa^2}\int_{\mathbb{R}^3}\operatorname{div}\mathbf{F}\cdot\overline{\psi}dx.$$

Next, we have

$$\int_{\mathbb{R}^3}\varepsilon(\rho)\mathbf{w}_\rho\cdot\overline{\nabla\psi}dx=\int_{\Omega^{\rm is}}\varepsilon(\rho)^{\rm is}\mathbf{w}_\rho\cdot\overline{\nabla\psi}dx+\int_{\Omega^{\rm cd}}\varepsilon(\rho)^{\rm cd}\mathbf{w}_\rho\cdot\overline{\nabla\psi}dx,$$

and, by integration by parts,

$$\int_{\Omega^{\rm cd}} \varepsilon(\rho)^{\rm cd}\mathbf{w}_\rho \cdot \overline{\nabla\psi}dx = \int_{\Omega^{\rm cd}} \operatorname{div}(\varepsilon(\rho)^{\rm cd}\mathbf{w}_\rho)\overline{\psi}dx - \int_\Sigma (\varepsilon(\rho)^{\rm cd}\mathbf{w}_\rho\cdot\mathbf{n})\overline{\psi}ds.$$

Let B_R be a ball with radius $R>0$ containing $\Omega^{\rm cd}$. We have

$$\begin{aligned}\int_{\Omega_R} \varepsilon(\rho)^{\rm is}\mathbf{w}_\rho \cdot \overline{\nabla\psi}dx &= \int_{\Omega_R} \operatorname{div}(\varepsilon(\rho)^{\rm is}\mathbf{w}_\rho)\overline{\psi}dx + \int_\Sigma (\varepsilon(\rho)^{\rm is}\mathbf{w}_\rho\cdot\mathbf{n})\overline{\psi}ds \\ &\quad + \int_{\partial B_R} (\varepsilon(\rho)^{\rm is}\mathbf{w}_\rho\cdot\mathbf{n})\overline{\psi}ds.\end{aligned}$$

Applying the Silver-Müller condition,

$$\int_{\partial B_R} (\varepsilon(\rho)^{\rm is}\mathbf{w}_\rho\cdot\mathbf{n})\overline{\psi}ds = \int_{\partial B_R} (\varepsilon(\rho)^{\rm is}[\mathbf{w}_\rho - \mathbf{w}_\rho\times\mathbf{n}]\cdot\mathbf{n})\overline{\psi}ds \to 0 \quad \text{as } R\to\infty,$$

Hence

$$\int_{\Omega^{\rm is}} \varepsilon(\rho)^{\rm is}\mathbf{w}_\rho \cdot \overline{\nabla\psi}dx = \int_{\Omega^{\rm is}} \operatorname{div}(\varepsilon(\rho)^{\rm is}\mathbf{w}_\rho)\overline{\psi}dx + \int_\Sigma (\varepsilon(\rho)^{\rm is}\mathbf{w}_\rho\cdot\mathbf{n})\overline{\psi}ds.$$

Thus

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\mathbf{w}_\rho \cdot \overline{\nabla\psi}dx = \int_{\mathbb{R}^3} \operatorname{div}(\varepsilon(\rho)\mathbf{w}_\rho)\overline{\psi}dx + \int_\Sigma (\varepsilon(\rho)^{\rm is} - \varepsilon(\rho)^{\rm cd})\mathbf{w}_\rho\cdot\mathbf{n}|_\Sigma\overline{\psi}ds.$$

Since $\operatorname{div}\mathbf{w}_\rho = 0$ in $\mathbb{R}^3$, we obtain (1.24) because (see (1.2))

$$\varepsilon(\rho)^{\rm is} - \varepsilon(\rho)^{\rm cd} = -\frac{1}{\mu}i\rho^2.$$

□

Analogously we obtain the following counterpart of [17, Lemma 2.34].

Lemma 1.8. *Let $\mathbf{E}_\rho \in \mathbb{X}_N(\mathbb{R}^3,\rho)$ solution of (1.22)-(1.15) associated with $\mathbf{F}\in\mathbf{W}(\operatorname{div},\mathbb{R}^3)$, and let $(\mathbf{w}_\rho,\varphi_\rho)\in\mathbb{X}_N(\mathbb{R}^3)\times\mathcal{V}$ given by Theorem 2.1. Then, φ_ρ is solution of following variational problem: Find $\varphi\in\mathcal{V}$, such that for all $\psi\in\mathcal{V}$,*

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\nabla\varphi\cdot\overline{\nabla\psi}dx = \frac{1}{\kappa^2}\int_{\mathbb{R}^3}\operatorname{div}\mathbf{F}\cdot\overline{\psi}dx + \frac{1}{\mu}i\rho^2\int_\Sigma \mathbf{w}_\rho\cdot\mathbf{n}|_\Sigma\overline{\psi}ds. \tag{1.25}$$

2. Decomposition of vector fields and compact embedding in weighted spaces

In this section we collect the tools needed in the proof of our a priori estimate (Theorem 3.1), namely a vector Helmholtz decomposition in $\mathbb{R}^3$ and a compactness results (Lemma 2.3) for the embedding in weighted spaces.

First we consider the vector potential of divergence-free vector fields and present results for a Helmholtz decomposition by Girault [6]. The weighted Sobolev spaces used here were introduced and studied by Hanouzet in [9]. For any multi-index α in $\mathbb{N}^3$, we denote by ∂^α the differential operator of order α:

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}, \quad \text{with } |\alpha| = \alpha_1+\alpha_2+\alpha_3.$$

Then, for all m in $\mathbb{N}$ and all k in $\mathbb{Z}$, we define the weighted Sobolev space

$$\mathbb{W}_k^m(\Omega^{\rm is}) := \big\{v\in\mathfrak{D}'(\Omega^{\rm is}) : \forall\alpha\in\mathbb{N}^3,\ 0\le|\alpha|\le m,\ \ell(r)^{|\alpha|-m+k}\partial^\alpha v\in L^2(\Omega^{\rm is})\big\}, \tag{2.1}$$

which is a Hilbert space with the norm:

$$\|v\|_{\mathbb{W}_k^m(\Omega^{\rm is})} = \Big\{ \sum_{|\alpha|=0}^{m} \|\ell(r)^{|\alpha|-m+k}\partial^\alpha v\|^2_{L^2(\Omega^{\rm is})}\Big\}^{1/2}.$$

Hence

$$\mathbb{W}_0^0(\Omega^{\rm is}) = L^2(\Omega^{\rm is}), \quad \mathbb{W}_{-1}^0(\mathbb{R}^3) = L^2_{0,-1}(\mathbb{R}^3).$$

For all $n \in \mathbb{Z}$, $\mathbf{P}_n$ denotes the space of all polynomials (in three variables) of degree at most n, with the convention that the space is reduced to zero when n is negative.

We denote by $\mathcal{P}_n$ is the subspace of all harmonic polynomials of $\mathbf{P}_n$, again with the convention that the space is reduced to zero when n is negative. For all integers $k \geq 0$, we define the following subspace of $(\mathcal{P}_k)^3$,

$$\mathcal{G}_k := \{\nabla q : q \in \mathcal{P}_{k+1}\}.$$

Note that $\mathcal{G}_0 = \mathbb{R}^3$. The following result is based on the paper by Girault [6]. In the case of a bounded domain, there are two classical orthogonal decompositions of vector fields: a decomposition in $\mathbf{L}^2$ and a decomposition in H^1_0 (cf. for example [7]). The following theorem establishes the analogue of the decomposition in $\mathbf{L}^2$ for vector fields in $\mathbb{R}^3$. Let

$$\mathbf{V}_k^m(\mathbb{R}^3) := \{\mathbf{v} \in \mathbb{W}_k^m(\mathbb{R}^3)^3 : \operatorname{div}\mathbf{v} = 0\},$$
$$\mathcal{C}_k := \{\operatorname{curl}\mathbf{q} : \mathbf{q} \in (\mathcal{P}_{k+1})^3\},$$

with the usual convention that $\mathcal{C}_k = \{0\}$, when $k < 0$, observe that $\mathcal{C}_0 = \mathbb{R}^3 = \mathcal{G}_0$. In addition, for all $k \geq 1$, $\mathcal{G}_k \subset \mathcal{C}_k$, but the inverse inclusion is false.

Theorem 2.1 (Girault [6, Theorem 5.1]). *Let the integers m and k belong to $\mathbb{Z}$ and let $\mathbf{u}$ be a vector field in $\mathbb{W}^m_{m+k}(\mathbb{R}^3)^3$.*

(1) If $k \leq 1$, $\mathbf{u}$ may be decomposed as

$$\mathbf{u} = \nabla p + \operatorname{curl}\Phi, \tag{2.2}$$

where Φ is unique in $\mathbf{V}^{m+1}_{m+k}(\mathbb{R}^3)/\mathcal{C}_{-k-1}$ and p is uniquely determined by $\mathbf{u}$ and Φ in $\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)/\mathbb{R}$, or $\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)$ if $k = 0$ or 1. They satisfy the bounds:

$$\|\Phi\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)^3/\mathcal{C}_{-k-1}} + \|p\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)/\mathbb{R}} \leq C\|\mathbf{u}\|_{\mathbb{W}^m_{m+k}(\mathbb{R}^3)^3}, \tag{2.3}$$

with the convention that the quotient norm of p is replaced by $\|p\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)}$ when $k = 0$ or 1.

(2) If $k \geq 2$ has the decomposition (2.2) with a unique p in $\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)$ and a unique Φ in $\mathbf{V}^{m+1}_{m+k}(\mathbb{R}^3)$ if and only if $\mathbf{u}$ is orthogonal to $\mathcal{C}_{k-2}$ (for the duality paring). The analogue of (2.3) holds:

$$\|\Phi\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)^3} + \|p\|_{\mathbb{W}^{m+1}_{m+k}(\mathbb{R}^3)} \leq C\|\mathbf{u}\|_{\mathbb{W}^m_{m+k}(\mathbb{R}^3)^3}, \tag{2.4}$$

(3) When both m and k belong to $\mathbb{N}$, the decomposition is orthogonal for the scalar product of $\mathbf{L}^2(\mathbb{R}^3)$.

Now, this part is concerned with compact embedding in weighted Sobolev spaces for unbounded domains, and is based on Avantaggiati and Troisi [1]. Let Ω be an unbounded domain of $\mathbb{R}^n$, satisfying the cone property, and $\delta \in C^0(\overline{\Omega})$, a positive continuous function divergent for $|\mathbf{x}| \to \infty$, satisfying also:

(1) There exist two open and separated subsets Ω_1 and Ω_2 of $\mathbb{R}^n$, such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ and
$$\delta(\mathbf{x}) \le 1, \quad \forall \mathbf{x} \in \Omega_1,\ \delta(\mathbf{x}) \ge 1,\ \forall \mathbf{x} \in \Omega_2.$$
We will put also, $\Omega_0 = \Omega$.

(2) For each $\mathbf{x}_0 \in \Omega_i$, $i = 0, 1, 2$, let
$$A_i(\mathbf{x}_0) = \Omega_i \cap \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \delta(\mathbf{x}_0)\}.$$
We assume that there c_1 and c_2 are two positive constants independent of $\mathbf{x}_0$ and $\mathbf{x}$, and
$$c_1\delta(\mathbf{x}_0) \le \delta(\mathbf{x}) \le c_2\delta(\mathbf{x}_0), \quad \forall \mathbf{x} \in A_i(\mathbf{x}_0),$$

(3) If $\varphi_i(\mathbf{x}, \mathbf{x}_0)$ is the characteristic function of the set $A_i(\mathbf{x}_0)$, then the inequalities
$$c_3\delta^n(\mathbf{x}) \le \int_{\Omega_i} \varphi_i(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}_0 \le c_4\delta^n(\mathbf{x}), \quad \forall \mathbf{x} \in A_i(\mathbf{x}_0),$$
hold, where c_3 and c_4 are two positive constants independent of $\mathbf{x}$.

If $s, \lambda \in \mathbb{R}$ and $0 < p \le \infty$, we will denote by $\widetilde{L}^p_{s,\lambda}(\Omega)$ the space of the functions $u(\mathbf{x})$, such that $\delta^s\big(\frac{\delta}{1+\delta^2}\big)^\lambda u \in L^p(\Omega)$, with norm
$$\|u\|_{\widetilde{L}^p_{s,\lambda}(\Omega)} := \|\delta^s\big(\frac{\delta}{1+\delta^2}\big)^\lambda u\|_{L^p(\Omega)}. \tag{2.5}$$
If $s, \lambda \in \mathbb{R}$, $r \in \mathbb{N}_0$ and $p \in (1, \infty)$, we will denote by $W^{r,p}_{s,\lambda}(\Omega)$ the space of the distributions u on Ω, such that $\partial^\alpha u \in \widetilde{L}^p_{s+|\alpha|-r,\lambda}(\Omega)$ for $|\alpha| \le r$, with norm
$$\|u\|_{W^{r,p}_{s,\lambda}(\Omega)} := \Big[\sum_{k=0}^{r} \|\partial^k u\|^p_{\widetilde{L}^p_{s+|\alpha|-r,\lambda}(\Omega)}\Big]^{1/p}. \tag{2.6}$$
We observe that $W^{r,p}_{s,\lambda}(\Omega)$ is continuously embedded in
$$W^{k,p}_{s+k-r+t,\lambda+\tau}(\Omega), \quad \text{for } k \le r, \tau \ge 0,\ t \in [-\tau, \tau]. \tag{2.7}$$
Therefore,
$$W^{0,p}_{s,\lambda}(\Omega) = \widetilde{L}^p_{s,\lambda}(\Omega).$$
We have also $L^2_{0,-1}(\Omega) = \widetilde{L}^2_{-1,1}(\Omega)$.

The next theorem is due to Avantaggiati and Troisi [1, Theorem 6.1].

Theorem 2.2. *There are real numbers s, λ, r, p, where $r \in \mathbb{Z}_+$ and $p > 1$, such that for each non-negative integer $k < r$, for each real number $\tau > 0$, and for each $t \in (-\tau, \tau)$, the injection*
$$W^{r,p}_{s,\lambda}(\Omega) \hookrightarrow W^{k,p}_{s+k-r+t,\lambda+\tau}(\Omega) \tag{2.8}$$
is compact.

As a consequence of the above results, we have the following Lemma.

Lemma 2.3. *The emmbeding of $\mathbf{PH}^1(\mathbb{R}^3)$ into $\mathbf{L}^2_{0,-1}(\mathbb{R}^3)$ is compact.*

Proof. First, we observe that by definition $\widetilde{L}^2_{-1,1}(\Omega) = L^2_{0,-1}(\Omega) = W^{0,2}_{-1,1}(\Omega)$. On the other hand choosing $t = s = \lambda = k = 0$, $\tau = r = 1$, $p = 2$ in (2.8) gives the compact embedding $W^{1,2}_{0,0}(\Omega) \subset\subset W^{0,2}_{-1,1}(\Omega)$. Hence $W^{1,2}_{0,0}(\Omega) \subset\subset L^2_{0,-1}(\Omega)$ where we can set $\Omega = \mathbb{R}^3$.

Furthermore $\varphi \in PH^1(\mathbb{R}^3) := \{\varphi = (\varphi^{\rm is}, \varphi^{\rm cd}) : \varphi^{\rm is} \in \mathbb{W}^1_0(\Omega^{\rm is}), \varphi^{\rm cd} \in H^1(\Omega^{\rm cd})\}$, due to the definition of $\mathbb{W}^1_0$, gives that $\nabla\varphi \in L^2$ and hence $\nabla\varphi \in \widetilde{L}^2_{0,0}$ with $s = \lambda = 0$ in (2.5). Therefore, $\varphi \in W^{1,2}_{0,0}(\Omega)$ with $r = 1$, $p = 2$, $s = \lambda = 0$ in (2.6) because with $s = \lambda = 0 = |\alpha|$, $r = 1$ there holds

$$\|\varphi\|_{\widetilde{L}^2_{-1,0}}(\Omega) = \|\delta^{-1}\varphi\|_{L^2(\Omega)} \le c\|\frac{\varphi}{\sqrt{1+x^2}}\|_{L^2(\Omega)} < \infty$$

by taking δ proportional to $\sqrt{1+x^2}$. □

3. A priori estimate for the electrical field

Next we give an existence and uniqueness result for the solution of (1.14)-(1.15). The proof uses an a priori estimate. The ideas of this section are based on those of Peron [3, 17], but using compactness results for the embedding of weighted spaces with unbounded domains. This is a crucial difference of our proof compared to Peron's proof. An alternative proof may be obtained using [10, Theorem 2.1].

For the rest of this article we assume the following condition.

Spectral hypothesis: We assume κ^2 is not an eigenvalue of the limit problem. That is, we assume that if $\mathbf{E}_0 \in \mathbf{W}(\mathrm{curl}, \Omega^{\rm is})$ is such that for all $\mathbf{E}' \in \mathbf{W}(\mathrm{curl}, \Omega^{\rm is})$,

$$\int_{\Omega^{\rm is}} (\mathrm{curl}\,\mathbf{E}_0 \cdot \mathrm{curl}\,\overline{\mathbf{E}'} - \kappa^2 \mathbf{E}_0 \cdot \overline{\mathbf{E}'})dx = 0, \quad \mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Sigma, \tag{3.1}$$

then $\mathbf{E}_0 = 0$.

Now, we can formulate our main theorem of this section.

Theorem 3.1. *Under the spectral hypothesis* (3.1)*, there exists a constant* $\rho_0 > 0$*, such that for all* $\rho > \rho_0$*, problem* (1.14)-(1.15) *admits an unique solution* $\mathbf{E}_\rho \in \mathbb{X}_{TN}(\mathbb{R}^3, \rho)$ *for* $\mathbf{F} \in \mathbf{W}_0(\mathrm{div}, \mathbb{R}^3)$*, satisfying*

$$\begin{aligned} &\|\mathrm{curl}\,\mathbf{E}_\rho\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathrm{div}(\varepsilon(\rho)\mathbf{E}_\rho)\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\mathbf{E}_\rho\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \rho\|\mathbf{E}_\rho\|_{\mathbf{L}^2(\Omega^{\rm cd})} \\ &\le C\|\mathbf{F}\|_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)}, \end{aligned} \tag{3.2}$$

with a constant $C > 0$*, independent of* ρ*.*

The proof of the above theorem is given in various steps, below. The estimate (3.2) is based on the a priori estimate (3.3).

Theorem 3.2. *If* (3.1) *holds, then there exists a constant* $\rho_0 > 0$*, such that, for all* $\rho > \rho_0$*, if* $\mathbf{E}_\rho \in \mathbb{X}_{TN}(\mathbb{R}^3, \rho)$ *satisfies* (1.14)-(1.15) *for* $\mathbf{F} \in \mathbf{W}_0(\mathrm{div}, \mathbb{R}^3)$*, then*

$$\|\mathbf{E}_\rho\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} \le C\|\mathbf{F}\|_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)}, \tag{3.3}$$

where $C > 0$ *is a constant independent of* ρ*.*

Proof. The proof is similar to the one given by Peron [17, Theorem 2.35]. Here we use a compact embedding of $\mathbf{PH}^1(\mathbb{R}^3)$ into $\mathbf{L}^2_{0,-1}(\mathbb{R}^3)$ where

$$\mathbf{PH}^1(\mathbb{R}^3) = \{\varphi : \varphi^{\rm is} \in (\mathbb{W}^1_0(\Omega^{\rm is}))^3,\ \varphi^{\rm cd} \in (H^1(\Omega^{\rm cd}))^3\}.$$

Let $\mathbf{E}_\rho \in \mathbb{X}_{TN}(\mathbb{R}^3,\rho)$, be a solution of (1.14)-(1.15). For $\Phi \in \mathbb{X}_{TN}(\mathbb{R}^3,\rho)$,

$$\begin{aligned} &\int_{\mathbb{R}^3}\Big(\frac{1}{\mu}\operatorname{curl}\mathbf{E}_\rho\cdot\operatorname{curl}\overline{\Phi}+\alpha\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\operatorname{div}(\overline{\varepsilon(\rho)\Phi})-\frac{\kappa^2}{\mu}\mathbf{E}_\rho\overline{\Phi}\Big)dx\\ &-\frac{1}{\mu}i\rho^2\int_{\Omega^{\rm cd}}\mathbf{E}_\rho\overline{\Phi}dx \qquad (3.4)\\ &=\int_{\mathbb{R}^3}\Big(\mathbf{F}\cdot\overline{\Phi}-\frac{\alpha}{\kappa^2}\operatorname{div}\mathbf{F}\cdot\operatorname{div}(\overline{\varepsilon(\rho)\Phi})\Big)dx. \end{aligned}$$

By Theorem 1.5 there holds

$$\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)+\frac{1}{\kappa^2}\operatorname{div}\mathbf{F}=0,\quad \text{in } \mathbb{R}^3.$$

For $\Phi \in \mathbb{X}_{TN}(\mathbb{R}^3,\rho)$,

$$\int_{\mathbb{R}^3}(\operatorname{curl}\mathbf{E}_\rho\cdot\operatorname{curl}\overline{\Phi}-\kappa^2\mathbf{E}_\rho\overline{\Phi})dx-i\rho^2\int_{\Omega^{\rm cd}}\mathbf{E}_\rho\overline{\Phi}dx=\mu\int_{\mathbb{R}^3}\mathbf{F}\cdot\overline{\Phi}dx, \qquad (3.5)$$

for all $\Phi \in \mathbb{X}_{TN}(\mathbb{R}^3,\rho)$. Just as in Peron [17] we prove the theorem by contradiction argument, but we crucially apply a compactness result for the embedding in weighted Sobolev spaces by Avantaggiati and Troisi [1].

Since Peron [17] considers only bounded domains, he can, in contrary, apply standard embedding arguments (Rellich's theorem). Suppose that exists a sequence $\{\mathbf{F}_{\rho_n}\}_{n\ge1}$ in $\mathbf{W}(\operatorname{div},\mathbb{R}^3)$ with $\rho_n\to\infty$, $\|\mathbf{F}_{\rho_n}\|_{\mathbf{W}(\operatorname{div},\mathbb{R}^3)}=1$, $\mathbf{F}_{\rho_n}\cdot\mathbf{n}=0$ in Σ, and such that for the corresponding solutions $\mathbf{E}_{\rho_n}\in\mathbb{X}_{TN}(\mathbb{R}^3,\rho_n)$ satisfy

$$\lim_{n\to\infty}\|\mathbf{E}_{\rho_n}\|_{\mathbb{X}_{TN}(\mathbb{R}^3,\rho_n)}=\infty.$$

Letting $\widetilde{\mathbf{E}}_{\rho_n}=(\|\mathbf{E}_{\rho_n}\|_{\mathbf{W}(\operatorname{div},\mathbb{R}^3)})^{-1}\mathbf{E}_{\rho_n}$ we have

$$\|\widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{W}(\operatorname{div},\mathbb{R}^3)}=1,\quad \lim_{n\to\infty}\|\widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{W}(\operatorname{div},\mathbb{R}^3)}=0. \qquad (3.6)$$

With $\Phi=\widetilde{\mathbf{E}}_{\rho_n}$, equality (3.5) becomes

$$\|\operatorname{curl}\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}-\kappa^2\|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}-i\rho_n^2\|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})}=\mu(\widetilde{\mathbf{F}}_{\rho_n},\widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \qquad (3.7)$$

Taking imaginary parts we have

$$\rho_n^2\|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2(\Omega^{\rm cd})}=-\mu\operatorname{Im}(\widetilde{\mathbf{F}}_{\rho_n},\widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \qquad (3.8)$$

By the Cauchy-Schwartz inequality we obtain,

$$|\operatorname{Im}(\widetilde{\mathbf{F}}_{\rho_n},\widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}|\le\|\widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{W}(\operatorname{div},\mathbb{R}^3)}\|\widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{W}(\operatorname{div},\mathbb{R}^3)}.$$

Hence (3.6) yields

$$\lim_{n\to\infty}\|\widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{L}^2(\Omega^{\rm cd})}=0. \qquad (3.9)$$

Also, taking real parts in (3.7),

$$\|\operatorname{curl}\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}-\kappa^2\|\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}=\mu\operatorname{Re}(\widetilde{\mathbf{F}}_{\rho_n},\widetilde{\mathbf{E}}_{\rho_n})_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \qquad (3.10)$$

Hence due to Cauchy-Schwartz inequality and (3.6), there are constants C_1 and C_2 independent of n, such that

$$\|\operatorname{curl}\widetilde{\mathbf{E}}_{\rho_n}\|^2_{\mathbf{L}^2(\mathbb{R}^3)}\le C_1+C_2\|\widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \qquad (3.11)$$

Therefore, $\{\operatorname{curl}\widetilde{\mathbf{E}}_{\rho_n}\}_{n\ge1}$ is bounded in $\mathbf{L}^2_{0,-1}(\mathbb{R}^3)$.

Let $(\mathbf{w}_{\rho_n}, \varphi_{\rho_n}) \in \mathbb{W}_0^1(\mathbb{R}^3) \times \mathcal{V}$, (for definition of $\mathcal{V}$ see [16, Chapter 2] and [15]), be given by Girault [6, Theorems 3.2 and 5.1], such that

$$\widetilde{\mathbf{E}}_{\rho_n} = \widetilde{\mathbf{w}}_{\rho_n} + \nabla\widetilde{\varphi}_{\rho_n}, \quad \operatorname{div}\widetilde{\mathbf{w}}_{\rho_n} = 0, \quad \text{in } \mathbb{R}^3,$$

and

$$\|\widetilde{\mathbf{w}}_{\rho_n}\|_{\mathbb{W}_0^1(\mathbb{R}^3)} \le C\|\operatorname{curl}\widetilde{\mathbf{E}}_{\rho_n}\|_{\mathbf{L}^2(\mathbb{R}^3)}, \tag{3.12}$$

where $C > 0$ is a constant independent of n. Therefore, $\{\widetilde{\mathbf{w}}_{\rho_n}\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{W}_0^1(\mathbb{R}^3)$. According to Lemma 1.7 and (1.16), $\widetilde{\varphi}_{\rho_n}$ satisfies

$$\int_{\mathbb{R}^3} \varepsilon(\rho)\nabla\widetilde{\varphi}_{\rho_n} \cdot \overline{\nabla\psi}dx = \frac{1}{\kappa^2}\int_{\mathbb{R}^3} \operatorname{div}\widetilde{\mathbf{F}}_{\rho_n} \cdot \overline{\psi}dx + \frac{1}{\mu}i\rho^2\int_\Sigma \widetilde{\mathbf{w}}_{\rho_n} \cdot \mathbf{n}|_\Sigma\overline{\psi}ds. \tag{3.13}$$

for all $\psi \in \mathcal{V}$.

Let $\rho_0 > 0$ and the constant $C_{\rho_0} > 0$ be given by [16, Theorem 3] and [15, Teorema 1]. We set $\delta_n = 1 + i\rho_n^2$. Then there exists $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$ we have $|\delta_n| \ge \rho_0$. Note that $\operatorname{div}\widetilde{\mathbf{F}}_{\rho_n}$ and $\widetilde{\mathbf{w}}_{\rho_n} \cdot \mathbf{n}$ verify the hypotheses of [16, Theorem 3] and [15, Teorema 1]. Also, problem (3.13) is coercive on $\mathcal{V}$. Hence the solution of (3.13) belongs to $PH^2(\mathbb{R}^3)$ and there holds

$$\|\widetilde{\varphi}_{\rho_n}^{\rm cd}\|_{H^2(\Omega^{\rm cd})} + \|\widetilde{\varphi}_{\rho_n}^{\rm is}\|_{\mathbb{W}_1^2(\Omega^{\rm is})} \le C_{\delta_0}\Big(\|\operatorname{div}\widetilde{\mathbf{F}}_{\rho_n}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} + \|\widetilde{\mathbf{w}}_{\rho_n} \cdot \mathbf{n}\|_{H^{1/2}(\Sigma)}\Big).$$

for any $n \ge n_0$. Thus $\{\nabla\widetilde{\varphi}_{\rho_n}\}_{n\ge1}$ is bounded in $\mathbf{PH}^1(\mathbb{R}^3)$, and $\{\widetilde{\mathbf{E}}_{\rho_n}\}_{n\ge1}$ is bounded in $\mathbf{H}^1(\Omega^{\rm cd}) \cup \big(\mathbb{W}_0^1(\Omega^{\rm is})\big)^3$.

According to Lemma 2.3, the embedding of $\mathbf{PH}^1(\mathbb{R}^3)$ in $\mathbf{L}^2_{0,-1}(\mathbb{R}^3)$ is compact. This implies that there exists a subsequence $\{\widetilde{\mathbf{E}}_{\rho_n}\}_{n\ge1}$ and $\widetilde{\mathbf{E}} \in \mathbf{L}^2_{0,-1}(\mathbb{R}^3)$, such that

$$\widetilde{\mathbf{E}}_{\rho_n} \rightharpoonup \widetilde{\mathbf{E}} \quad \text{in } \big(\mathbf{PH}^1(\mathbb{R}^3)\big)^3, \quad \widetilde{\mathbf{E}}_{\rho_n} \to \widetilde{\mathbf{E}} \quad \text{in } \mathbf{L}^2_{0,-1}(\mathbb{R}^3). \tag{3.14}$$

By (3.6), we have

$$\|\widetilde{\mathbf{E}}\|_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} = 1. \tag{3.15}$$

To obtain a contradiction, we show that $\widetilde{\mathbf{E}} = 0$ in $\Omega^{\rm is} \cup \Omega^{\rm cd}$. Due to (3.9), $\|\widetilde{\mathbf{E}}\|_{\mathbf{L}^2(\Omega^{\rm cd})} = 0$. Hence

$$\widetilde{\mathbf{E}} = 0, \quad \text{in } \Omega^{\rm cd}. \tag{3.16}$$

Next, we take $\Phi \in \mathbb{X}_{TN}(\mathbb{R}^3, \rho)$ with support in $\Omega^{\rm is}$. Then $\mathbf{n}\cdot\Phi = 0$, $\mathbf{n}\times\Phi = 0$ on Σ and due to (3.5), we have

$$(\operatorname{curl}\widetilde{\mathbf{E}}_{\rho_n}, \operatorname{curl}\Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{\rm is})} - \kappa^2(\widetilde{\mathbf{E}}_{\rho_n}, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{\rm is})} = \mu(\widetilde{\mathbf{F}}_{\rho_n}, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{\rm is})}. \tag{3.17}$$

Letting $n \to \infty$ in (3.17) and using (3.14) we obtain

$$(\operatorname{curl}\widetilde{\mathbf{E}}, \operatorname{curl}\Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{\rm is})} - \kappa^2(\widetilde{\mathbf{E}}, \Phi)_{\mathbf{L}^2_{0,-1}(\Omega^{\rm is})} = 0. \tag{3.18}$$

Now (3.1) gives $\widetilde{\mathbf{E}} = 0$, in $\Omega^{\rm is}$, and therefore $\widetilde{\mathbf{E}} = 0$, in $\mathbb{R}^3$, which is a contradiction to (3.15) and therefore (3.3) holds. □

Now with the help of Theorem 3.2 we can prove Theorem 3.1.

Proof of Theorem 3.1. Let $\rho_0 > 0$ be given by Theorem 3.2. Let us assume $\mathbf{E}_\rho$ satisfies (1.14)-(1.15). Then $\mathbf{E}_\rho$ satisfies (3.5) and taking $\Phi = \mathbf{E}_\rho$ we obtain

$$\|\operatorname{curl}\mathbf{E}_\rho\|^2_{\mathbf{L}^2(\mathbb{R}^3)} - \kappa^2\|\mathbf{E}_\rho\|^2_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)} - i\rho^2\|\mathbf{E}_\rho\|^2_{\mathbf{L}^2(\Omega^{\rm cd})} = \mu(\mathbf{F}, \mathbf{E}_\rho)_{\mathbf{L}^2_{0,-1}(\mathbb{R}^3)}. \tag{3.19}$$

Taking real and imaginary parts as in the proof of Theorem 3.2 we obtain the a priori estimate (3.2) from (1.16) and

$$\rho\|\mathbf{E}_\rho\|_{\mathbf{L}^2(\Omega^{\mathrm{cd}})} \le C_1\|\mathbf{F}\|_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)}, \tag{3.20}$$

$$\|\operatorname{curl}\mathbf{E}_\rho\|_{\mathbf{L}^2(\mathbb{R}^3)} \le C_2\|\mathbf{F}\|_{\mathbf{W}(\mathrm{div},\mathbb{R}^3)}. \tag{3.21}$$

Next, note that the a priori estimate (3.2) implies the injectivity of the solution operator to the variational problem (1.14). Therefore to show existence of the solution it suffices to demonstrate that this operator is surjective. We introduce the sesquilinear form c_ρ defined by

$$c_\rho(\mathbf{E}_\rho,\mathbf{E}'_\rho) = \int_{\mathbb{R}^3}\Big(\frac{1}{\mu}\operatorname{curl}\mathbf{E}_\rho\cdot\operatorname{curl}\overline{\mathbf{E}'_\rho} + \alpha\operatorname{div}(\varepsilon(\rho)\mathbf{E}_\rho)\cdot\operatorname{div}(\overline{\varepsilon(\rho)\mathbf{E}'_\rho})\Big)dx. \tag{3.22}$$

for all $\mathbf{E}_\rho,\mathbf{E}'_\rho \in \mathbb{X}_T(\mathbb{R}^3,\rho)$. The bilinear form c_ρ is coercive on $\mathbb{X}_T(\mathbb{R}^3,\rho)$. By the Lax-Milgram Theorem there exist a bounded linear operator $\mathbf{M}$ such that $c_\rho(\mathbf{E}_\rho,\mathbf{E}'_\rho) = \langle\mathbf{M}\mathbf{E}_\rho,\mathbf{E}'_\rho\rangle$. Since the embedding $I_\rho(\mathbf{E}_\rho) = \varepsilon(\rho)\mathbf{E}_\rho$ for $\mathbf{E}_\rho \in \mathbb{X}_T(\mathbb{R}^3,\rho)$ from $\mathbb{X}_T(\mathbb{R}^3,\rho)$ into $\mathbb{X}_T(\mathbb{R}^3,\rho)'$ is compact. Hence $\mathbf{M}-\kappa^2 I_\rho$ is a Fredholm operator. In particular, it is surjective if and only if his adjoint $\mathbf{M}^* - \kappa^2 I^*_\rho$ is injective where $I^*_\rho = \overline{\varepsilon(\rho)}I_\rho$. Let c^*_ρ be the sesquilinear form associated with the operator c_ρ; i.e.,

$$c^*_\rho(\mathbf{E}_\rho,\mathbf{E}'_\rho) = \int_{\mathbb{R}^3}\Big(\frac{1}{\mu}\operatorname{curl}\mathbf{E}_\rho\cdot\operatorname{curl}\overline{\mathbf{E}'_\rho} + \alpha\operatorname{div}(\overline{\varepsilon(\rho)}\mathbf{E}_\rho)\cdot\operatorname{div}(\varepsilon(\rho)\overline{\mathbf{E}'_\rho})\Big)dx, \tag{3.23}$$

for all $\mathbf{E}_\rho,\mathbf{E}'_\rho \in \mathbb{X}_T(\mathbb{R}^3,\rho)$.

As in Theorem 3.1, an a priori estimate for $\mathbf{M}^* - \kappa^2 I^*_\rho$ is proven, yielding its injectivity. Hence $\mathbf{M}-\kappa^2 I_\rho$ is a subjectivity of the operator. Proving the Theorem. □

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