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GROUND STATE SOLUTION OF A NONLOCAL BOUNDARY-VALUE PROBLEM

CYRIL JOEL BATKAM

Abstract. In this article, we apply the Nehari manifold method to study the Kirchhoff type equation

$$-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u)$$

subject to Dirichlet boundary conditions. Under a general 4-superlinear condition on the nonlinearity f, we prove the existence of a ground state solution, that is a nontrivial solution which has least energy among the set of nontrivial solutions. If f is odd with respect to the second variable, we also obtain the existence of infinitely many solutions. Under our assumptions the Nehari manifold does not need to be of class $\mathcal{C}^1$.

1. Introduction

In this paper, we are concerned with the nonlocal boundary-value problem

$$\begin{gathered}-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u), \quad \text{in } \Omega\\ u=0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary ($N=1,2,3$), $a>0$ and $b>0$, and the nonlinearity $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions.

- (F1) f is continuous and there exists a constant $c>0$ such that $|f(x,u)|\le c\big(1+|u|^{p-1}\big)$, where $p>4$ for $N=1,2$ and $4<p<2^\star:=2N/(N-2)$ for $N=3$.
- (F2) $f(x,u)=\circ(u)$ uniformly in x as $|u|\to 0$.
- (F3) $F(x,u)/u^4\to\infty$ uniformly in x as $|u|\to\infty$, where $F(x,u)=\int_0^u f(x,s)ds$.
- (F4) $u\mapsto f(x,u)/u^3$ is positive for $u\neq 0$, non-increasing on $(-\infty,0)$ and non-decreasing on $(0,\infty)$.

Usually, in the study of (1.1) the following Ambrosetti-Rabinowitz's type condition is used: There exist $\mu>4$ and $R>0$ such that

$$0<\mu F(x,u)\le uf(x,u)\quad \forall x\in\overline{\Omega},\ |u|\ge R. \tag{1.2}$$

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Integrating (1.2) yields the existence of constants $c_1, c_2 > 0$ such that $F(x,u) \geq c_1|u|^\mu - c_2$ for all u; therefore (1.2) is stronger that (F3). It is well known that (1.2) is mainly used to verify the boundedness of the Palais-Smale sequences of the energy functional, and without it the problem becomes more complicated. However, there are many functions which are 4-superlinear but do not satisfy (1.2). An example of f satisfying assumptions (F1)–(F4), which does not satisfy (1.2) is given at the end of this article.

We call problem (1.1) nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2 dx$, which implies that the first equation in (1.1) is no longer a pointwise equality. This causes some mathematical difficulties which make the study of such problems particularly interesting. On the other hand, for a physical point of view problem (1.1) is related to the stationary analogue of the hyperbolic equations

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u),$$

proposed by Kirchhoff [7] as an extension of the classical d'Alembert wave equations for free vibrations of elastic strings. The Kirchhoff's model takes into account the changing in length of the string produced by transverse vibrations. Problem (1.1) has been widely studied by variational methods since the paper of Lions [8], where an abstract framework to attack it was introduced. Perera and Zhang [9] considered (1.1) in the case that f is asymptotically linear at 0 and asymptotically 4-linear at infinity, and they obtained a nontrivial solution by using the Yang index and critical group. He and Zou [5], under condition (1.2) and without condition (1.2), obtained the existence of infinitely many solutions of (1.1) by using the fountain theorems. Alves et al. [1] considered (1.1) with a critical term and obtained a nontrivial solution of mountain pass type. In [3, 13, 6] the authors obtained some existence results for a Kirchhoff's type problem by using the Nehari manifold approach.

In this article, we also study (1.1) via a reduction on the Nehari manifold. We are firstly interested in the existence of a ground state solution of (1.1); that is a nontrivial solution which has least energy among the set of nontrivial solutions of (1.1). Let $X := H_0^1(\Omega)$ be the usual Sobolev space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$,

$$\langle u,v\rangle = \int_\Omega \nabla u \nabla v dx, \quad \|u\|^2 = \langle u,u\rangle.$$

Under assumption (F1), the solutions of (1.1) are critical points of the functional $\Phi \in \mathcal{C}^1(X,\mathbb{R})$,

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)dx. \tag{1.3}$$

We define the Nehari manifold

$$\mathcal{N} := \big\{u \in X\backslash\{0\} : \langle\Phi'(u),u\rangle = 0\big\}. \tag{1.4}$$

Then any nontrivial solution of (1.1) belongs to $\mathcal{N}$. We would like to show that $\inf_\mathcal{N}\Phi$ is attained at some $u_0 \in \mathcal{N}$ which is a critical point of Φ. Since under our assumptions on f above we do not know if the sub-manifold $\mathcal{N}$ of X is of class $\mathcal{C}^1$, we cannot apply the minimax theorems in [10, 11, 14] directly to $\mathcal{N}$ in order to extract the critical points of the functional Φ. To circumvent this difficulty we follow, as in [13, 6], an approach by Szulkin and Weth [12]. Our main result is the following theorem.

Theorem 1.1. *Let $a > 0$ and $b > 0$. If f satisfies* (F1)–(F4), *then* (1.1) *has a ground state solution. Moreover, if in addition*

(F5) $f(x, -u) = -f(x, u)$ *for all* $(x, u) \in \overline{\Omega} \times \mathbb{R}$,

then (1.1) *has infinitely many solutions.*

2. Preliminaries

Throughout this article, we denote by $|\cdot|_r$ the norm of the Lebesgue space $L^r(\Omega)$. We consider X, Φ and $\mathcal{N}$ as defined in the introduction. A standard argument shows the following lemma.

Lemma 2.1. *If* (F1) *is satisfied, then* $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ *and we have*

$$\langle \Phi'(u), v\rangle = \big(a + b\|u\|^2\big) \int_\Omega \nabla u \nabla v dx - \int_\Omega v f(x, u) dx. \tag{2.1}$$

It is well known that the Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the map $\alpha_u : [0, \infty) \to \mathbb{R}$ defined by

$$\alpha_u(t) := \Phi(tu), \tag{2.2}$$

where $u \in X$ is fixed. Such a map is known as a fibering map which was introduced by Drábek and Pohozaev in [4] and discussed in Brown and Zhang [2]. The following result shows that α_u has a unique maximum point if $u \neq 0$.

Lemma 2.2. *Assume that* (F1)–(F4) *are satisfied. Then for any $u \in X\backslash\{0\}$, there exists a unique $t_u > 0$ such that $\alpha_u'(t) > 0$ for every $t \in (0, t_u)$ and $\alpha_u'(t) < 0$ for every $t > t_u$.*

Proof. (F1) and (F2) imply that for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|f(x, u)| \le \varepsilon|u| + c_\varepsilon|u|^{p-1} \quad \text{and} \quad |F(x, u)| \le \varepsilon|u|^2 + c_\varepsilon|u|^p. \tag{2.3}$$

Then using (1.3), we deduce that

$$\alpha_u(t) \ge \Big(\frac{a}{2}\|u\|^2 - \varepsilon|u|_2^2\Big)t^2 + \frac{b}{4}\|u\|^4 t^4 - c_\varepsilon t^p |u|_p^p.$$

Since $u \neq 0$, we can choose ε in such a way that $\frac{a}{2}\|u\|^2 - \varepsilon|u|_2^2 > 0$. It then follows, since $p > 4$, that $\alpha_u(t) > 0$ for $t > 0$ sufficiently small.

On the other hand (F3) implies: For each $\delta > 0$ there exists $c_\delta > 0$ such that

$$F(x, u) \ge \delta|u|^4 - c_\delta. \tag{2.4}$$

This implies that

$$\alpha_u(t) \le \frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|u\|^4 - \delta t^4|u|_4^4 + c_\delta|\Omega|.$$

If we choose $\delta > 0$ big enough such that $\frac{b}{4}\|u\|^4 - \delta|u|_4^4 < 0$, we see that $\alpha_u(t) \to -\infty$ as $t \to \infty$. We deduce that α_u has a positive maximum.

Now noting that

$$\alpha_u'(t) = \langle \Phi'(tu), u\rangle = a\|u\|^2 t + b\|u\|^4 t^3 - \int_\Omega u f(x, tu) dx,$$

the equation $\alpha_u'(t) = 0$ is equivalent to

$$b\|u\|^4 = -\frac{a\|u\|^2}{t^2} + \frac{1}{t^3}\int_\Omega u f(x, tu) dx.$$

By (F4) the map $t \mapsto \frac{1}{t^3}\int_\Omega uf(x,tu)dx$ is increasing on $(0,\infty)$. It is then easy to deduce that the map $t \mapsto -\frac{a\|u\|^2}{t^2}+\frac{1}{t^3}\int_\Omega uf(x,tu)dx$ is strictly increasing on $(0,\infty)$. Hence the maximum point of α_u is unique. $\square$

The following lemma gives some properties of t_u. Let

$$S := \{u \in X : \|u\| = 1\}.$$

Lemma 2.3. *If* (F1)–(F4) *are satisfied then:*

1. *There exists $\delta > 0$ such that $t_u \geq \delta$ for every $u \in S$, where t_u is as in Lemma 2.2 above.*
2. *For any compact $K \subset S$, there exists a constant C_K such that $t_u \leq C_K$ for every $u \in K$.*

Proof. (1) Let $u \in S$ and recall that t_u is the unique point of maximum of the map $\alpha_u(t) = \Phi(tu)$. We deduce from (2.3) and the Sobolev embedding theorem that

$$\Phi(w) \geq \Big(\frac{a}{2} - \varepsilon c_1\Big)\|w\|^2 + \frac{b}{4}\|w\|^4 - c_\varepsilon c_2\|w\|^p, \quad \forall w \in X\backslash\{0\},$$

where $c_1 > 0$ and $c_2 > 0$ are constants. By choosing ε such that $\frac{a}{2} - \varepsilon c_1 \geq \frac{a}{4}$, we obtain

$$\Phi(w) \geq \frac{a}{4}\|w\|^2 + \frac{b}{4}\|w\|^4 - c_3\|w\|^p, \quad \forall w \in X\backslash\{0\}.$$

There then exists $\delta > 0$ sufficiently small such that setting $w = \delta u$ we obtain

$$\Phi(\delta u) \geq \frac{a}{4}\delta^2 + \frac{b}{4}\delta^4 - c_3\delta^p > 0, \quad \forall u \in S.$$

Since $t_u > 0$ is the unique point of maximum of the function α_u we have

$$\alpha_u(t_u) \geq \alpha_u(\delta) \geq \delta_\star := \frac{a}{4}\delta^2 + \frac{b}{4}\delta^4 - c_3\delta^p > 0, \quad \forall u \in S,$$

where $\delta_\star > 0$ does not depend on $u \in S$. We would like to show that $t_u \geq \gamma > 0$ for some $\gamma > 0$ and for all $u \in S$. Suppose, on the contrary, that there is a sequence (t_{u_j}, u_j), with $u_j \in S$ such that $t_{u_j} \to 0^+$. Since $u_j \in S$, we have $t_{u_j}u_j \to 0$ in X and so, in view of the continuity of Φ, we obtain

$$0 < \delta_\star \leq \alpha_{u_j}(t_{u_j}) = \Phi(t_{u_j}u_j) \to 0 = \Phi(0)$$

which is a contradiction. Hence, there is $\gamma > 0$ such that $t_u \geq \gamma > 0$ for all $u \in S$.

(2) Let K be a compact subset of S. Arguing by contradiction, we assume that there exists a sequence $(u_n) \subset K$ such that $t_{u_n} \to \infty$. We know that there exists $\delta > 0$ such that $\Phi(t_{u_n}u_n) \geq \Phi(\delta u_n) > 0$. Hence we have

$$\begin{aligned} 0 < \frac{\Phi(t_{u_n}u_n)}{t_{u_n}^4} &= \frac{1}{t_{u_n}^4}\Big[\frac{a}{2}\|t_{u_n}u_n\|^2 + \frac{b}{4}\|t_{u_n}u_n\|^4 - \int_\Omega F(x, t_{u_n}u_n)dx\Big] \\ &= \frac{a}{2t_{u_n}^2} + \frac{b}{4} - \int_\Omega \frac{F(x,t_{u_n}u_n)}{t_{u_n}^4}dx \\ &= \frac{a}{2t_{u_n}^2} + \frac{b}{4} - \int_\Omega |u_n|^4\frac{F(x,t_{u_n}u_n)}{|t_{u_n}u_n|^4}dx. \end{aligned} \tag{2.5}$$

Now since K is compact, the sequence (u_n) has a converging subsequence. We can then assume that $u_n \to u$ in X. By the Sobolev embedding theorem $u_n \to u$ in $L^2(\Omega)$, and up to a subsequence $u_n(x) \to u(x)$ a.e. in Ω. Clearly $\|u\| = 1$, and consequently $u \neq 0$ and $|t_{u_n}u_n| \to \infty$. We point out here that (F2) and (F4) imply $F(x,u) \geq 0$ for all $(x,u) \in \Omega \times \mathbb{R}$. Hence by using Fatou's lemma and (F3) we obtain, by passing to the limit $n \to \infty$ in (2.5), the contradiction $0 \leq -\infty$. Consequently, there exists $C_K > 0$ such that $t_u \leq C_K$ for every $u \in K$. □

Now we consider the following two mappings.

$$M : S \to \mathcal{N}, \quad M(u) := t_u u,$$
$$\Psi : S \to \mathbb{R}, \quad \Psi(u) := \Phi \circ M(u).$$

The next two lemmas are due to Szulkin and Weth [12]. Indeed, Lemmas 2.2 and 2.3 above show that the assumptions in [12] are satisfied.

Lemma 2.4 ([12, Proposition 8]). *The mapping M defined above is a homeomorphism between S and $\mathcal{N}$ whose inverse M^{-1} is given by*

$$M^{-1}(u) = \frac{u}{\|u\|}, \quad \forall u \in \mathcal{N}.$$

We recall that a sequence $(u_n) \subset X$ is said to be a Palais-Smale sequence for a functional $\varphi \in \mathcal{C}^1(X,\mathbb{R})$ if

$$\varphi'(u_n) \to 0 \quad \text{and} \quad \sup_n |\varphi(u_n)| < \infty.$$

If every such sequence has a convergent subsequence, then φ is said to satisfy the Palais-Smale condition.

Lemma 2.5 ([12, Corollary 10]).
(a) $\Psi \in \mathcal{C}^1(S,\mathbb{R})$ *and*

$$\langle \Psi'(u), v\rangle = \|M(u)\| \langle \Phi'(M(u)), v\rangle \quad \forall v \in T_u(S),$$

where $T_u(S)$ is the tangent space of S at u.
(b) *If (u_n) is a Palais-Smale sequence for Ψ, then $(M(u_n))$ is a Palais-Smale sequence for Φ. If $(u_n) \subset \mathcal{N}$ is a bounded Palais-Smale sequence for Φ, then $(M^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.*
(c) *u is a critical point of Ψ if and only if $M(u)$ is a nontrivial critical point of Φ. Moreover, the corresponding critical values coincide and $inf_S \Psi = inf_{\mathcal{N}} \Phi$.*
(d) *If Φ is even, then so is Ψ.*

Finally our multiplicity result will be deduced from the following lemma.

Lemma 2.6 ([11]). *Let X be an infinite dimensional Hilbert space and let $J \in \mathcal{C}^1(S,\mathbb{R})$ be even. If J is bounded below and satisfies the Palais-Smale condition, then it possesses infinitely many distinct pairs of critical points.*

3. Proof of the main result

We shall prove our main result by applying Lemma 2.5. First we verify the Palais-Smale condition.

Lemma 3.1. *The functional $\Phi|_{\mathcal{N}}$ satisfies the Palais-Smale condition; that is, every Palais-Smale sequence for $\Phi|_{\mathcal{N}}$ has a convergent subsequence.*

Proof. Let $(u_n) \subset \mathcal{N}$ such that $d := \sup_n \Phi(u_n) < \infty$ and $\Phi'(u_n) \to 0$. We want to show that the sequence (u_n) has a convergent subsequence.

First we show that (u_n) is bounded. Arguing by contradiction, we assume that (u_n) is unbounded. Hence, up to a subsequence we have $\|u_n\| \to \infty$ and $v_n := u_n/\|u_n\| \rightharpoonup v$. By definition of t_{v_n} we have for all $t > 0$

$$\Phi(t_{v_n} v_n) \geq \Phi(t v_n) = \frac{a}{2}t^2 + \frac{b}{4}t^4 - \int_\Omega F(x, t v_n)dx.$$

Since $v_n = M^{-1}(u_n)$, it follows that $u_n = t_{v_n} v_n$ and

$$d \geq \Phi(u_n) \geq \frac{a}{2}t^2 + \frac{b}{4}t^4 - \int_\Omega F(x, t v_n)dx. \tag{3.1}$$

If $v = 0$, then the Rellich-Kondrashov theorem implies that $v_n \to 0$ in $L^2(\Omega)$ and in $L^p(\Omega)$. By using (2.3) we deduce that for every $\varepsilon > 0$,

$$\int_\Omega F(x, t v_n)dx \leq \varepsilon t^2 |v_n|_2^2 + c_\varepsilon t^p |v_n|_p^p \to 0.$$

We then obtain by taking the limit $n \to \infty$ in (3.1)

$$d \geq \frac{a}{2}t^2 + \frac{b}{4}t^4.$$

But this leads to a contradiction if we take t sufficiently large. Consequently we have $v \neq 0$. By (1.3) and the definition of v_n we have

$$0 \leq \frac{\Phi(u_n)}{\|u_n\|^4} = \frac{a}{2\|u_n\|^2} + \frac{b}{4} - \int_\Omega |v_n|^4 \frac{F(x, \|u_n\| v_n)}{\big|\|u_n\| v_n\big|^4} dx.$$

Since $\big|\|u_n\| v_n\big| \to \infty$, we obtain by using one more time Fatou's lemma the contradiction $0 \leq -\infty$. The sequence (u_n) is then bounded.

Up to a subsequence we have $u_n \rightharpoonup u$ in X. By Rellich-Kondrashov theorem $u_n \to u$ in $L^p(\Omega)$. One can easily verify, using (1.3) and (2.1) that

$$\begin{aligned}
&\big(a + b\|u_n\|^2\big)\|u_n - u\|^2 \\
&= \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle - b\big(\|u_n\|^2 - \|u\|^2\big) \int_\Omega \nabla u \nabla (u_n - u)dx \\
&\quad + \int_\Omega (u_n - u)\big(f(x, u_n) - f(x, u)\big)dx.
\end{aligned}$$

Clearly we have

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0, \quad \big(\|u_n\|^2 - \|u\|^2\big) \int_\Omega \nabla u \nabla (u_n - u)dx \to 0.$$

By Hölder's inequality,

$$\Big|\int_\Omega (u_n - u)\big(f(x, u_n) - f(x, u)\big)dx\Big| \leq |u_n - u|_p \big|f(x, u_n) - f(x, u)\big|_{\frac{p}{p-1}}.$$

By (F1), f satisfies the assumptions of [14, Theorem A.2]. Hence $|f(x, u_n) - f(x, u)|_{\frac{p}{p-1}} \to 0$, and consequently

$$\big(a + b\|u_n\|^2\big)\|u_n - u\|^2 \to 0,$$

which implies that $u_n \to u$ in X. $\square$

Proof of Theorem 1.1. We know from Lemma 2.5-(a) that Ψ is of class $\mathcal{C}^1$ on S. Since Ψ is also bounded below on S, Ekeland's variational principle yields the existence of a sequence $(u_n) \subset S$ such that

$$\Psi(u_n) \to \inf_S \Psi \quad \text{and} \quad \Psi'(u_n) \to 0.$$

By Lemma 2.5 the sequence $\big(v_n := M(u_n)\big) \subset \mathcal{N}$ is a Palais-Smale sequence for Φ. By Lemma 3.1, we have $v_n \to v$ up to a subsequence. Since M is a homeomorphism we deduce that $u_n \to u := M^{-1}(v)$. Hence $\Psi'(u) = 0$ and $\Psi(u) = \inf_S \Psi$. By Lemma 2.5-(c), v is a nontrivial critical point of Φ, and

$$\Phi(v) = \Psi(u) = \inf_S \Psi = \inf_{\mathcal{N}} \Phi.$$

It follows that v is a ground state solution of (1.1).

Now (F5) implies that Φ is even. Hence by Lemma 2.5-(d), Ψ is also even. We have seen above that $\Psi \in \mathcal{C}^1(S, \mathbb{R})$ is bounded below and satisfies the Palais-Smale condition. It then follows from Lemma 2.6 that Ψ has infinite many distinct pairs of critical points. Hence Φ has infinitely many critical points by Lemma 2.5, and consequently (1.1) has infinitely many solutions. $\square$

Finally, we present an example to illustrate that there is a nonlinear function f which satisfies the conditions (F1)–(F5), but does not satisfy the condition (1.2).

Example 3.2. Let $f(x,u) = u^3 \ln(1+|u|)$. Integrating by parts we obtain

$$F(x,u) = \frac{1}{4}u^4 \ln\big(1+|u|\big) - \frac{1}{4}\Big(\frac{1}{4}u^4 - \frac{1}{3}|u|^3 + \frac{1}{2}|u|^2 - |u| - \ln\big(1+|u|\big)\Big).$$

It is readily seen that the assumptions (F1)–(F5) are satisfied. It is well known that integrating (1.2) yields the existence of a constant $c_1 > 0$ such that $F(x,u) \geq c_1 |u|^\mu$ for $|u|$ large. Therefore, if (1.2) is satisfied for our example above, then we have that for $|u|$ large,

$$\frac{1}{4}u^4 \ln\big(1+|u|\big) - \frac{1}{4}\Big(\frac{1}{4}u^4 - \frac{1}{3}|u|^3 + \frac{1}{2}|u|^2 - |u| - \ln\big(1+|u|\big)\Big) \geq c_1 |u|^\mu.$$

Dividing the two members of this inequality by $|u|^\mu$ and letting $|u| \to \infty$ we get, since $\mu > 4$, the contradiction $0 \geq c_1$. This shows that the condition (1.2) is not satisfied in our case.

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Cyril Joel Batkam
Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, J1K 2R1, Canada
E-mail address: cyril.joel.batkam@usherbrooke.ca