Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 259, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR FRACTIONAL HAMILTONIAN SYSTEMS

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ABSTRACT. In this work we prove the existence of solutions for the fractional differential equation

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t,u(t)), \quad u \in H^{\alpha}(\mathbb{R},\mathbb{R}^N). $$

where $\alpha \in (1/2, 1)$. Assuming L is coercive at infinity we show that this equation has at least one nontrivial solution.

1. INTRODUCTION

Fractional differential equations both ordinary and partial ones are applied in mathematical modeling of processes in physics, mechanics, control theory, biochemistry, bioengineering and economics. Therefore the theory of fractional differential equations is an area intensively developed during last decades [1, 11, 16, 22, 25]. The monographs [12, 17, 19], enclose a review of methods of solving fractional differential equations, which are an extension of processes from differential equations theory.

Recently, also equations including both - left and right fractional derivatives, are discussed. Let us point out that according to integration by parts formulas in fractional calculus, we obtain equations mixing left and right operators. Apart from their possible applications, equations with left and right derivatives are an interesting and new field in fractional differential equations theory. Some works in this topic can be founded in papers [3, 4, 13] and their references.

Recently Jiao and Zhou [14], for the first time, showed that the critical point theory is an effective approach for studying the existence for the following fractional boundary-value problem

$$ \begin{gathered} {}_tD^{\alpha}_{T}({}_0D^{\alpha}_t u(t)) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{1.1} $$

and obtained the existence of at least one nontrivial solution.

Motivated by this work, in this paper we consider a fractional differential equation with left and right fractional derivatives on $\mathbb{R}$, that is,

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t,u(t)) \tag{1.2} $$

2000 *Mathematics Subject Classification.* 26A33, 34C37, 35A15, 35B38.
Key words and phrases. Liouville-Weyl fractional derivative; fractional Hamiltonian systems; critical point; variational methods.

Submitted June 10, 2013. Published November 26, 2013.

where $\alpha \in (1/2, 1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric matrix-valued function and $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$; satisfies the following conditions:

(L1) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to +\infty$ as $t \to \infty$ and

$$(L(t)x, x) \geq l(t)|x|^2, \quad \text{for all } t \in \mathbb{R}\ x \in \mathbb{R}^n. \tag{1.3}$$

(W1) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and there exists a constant $\mu > 2$ such that

$$0 < \mu W(t, x) \leq (x, \nabla W(t, x)), \quad \text{for all } t \in \mathbb{R}\ x \in \mathbb{R}^n \setminus \{0\}.$$

(W2) $|\nabla W(t, x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(W3) There exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$|W(t, x)| + |\nabla W(t, x)| \leq |\overline{W(x)}| \quad \text{for every } x \in \mathbb{R}^n\ t \in \mathbb{R}.$$

In particular, if $\alpha = 1$, Equation (1.2) reduces to the standard second-order differential equation

$$u'' - L(t)u + \nabla W(t, u) = 0, \tag{1.4}$$

where $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given function and $\nabla W(t, u)$ is the gradient of W at u. The existence of homoclinic solution is one of the most important problems in the history of that kind of equations, and has been studied intensively by many mathematicians. Assuming that $L(t)$ and $W(t, u)$ are independent of t, or T-periodic in t, many authors have studied the existence of homoclinic solutions for (1.4) via critical point theory and variational methods. In this case, the existence of homoclinic solution can be obtained by going to the limit of periodic solutions of approximating problems.

If $L(t)$ and $W(t, u)$ are neither autonomous nor periodic in t, this problem is quite different from the ones just described, because the lack of compacteness of the Sobolev embedding. In [20] the authors considered (1.4) without periodicity assumptions on L and W and showed that (1.4) possesses one homoclinic solution by using a variant of the mountain pass theorem without the Palais-Smale contidion. In [18], under the same assumptions of [20], the authors, by employing a new compact embedding theorem, obtained the existence of homoclinic solution of (1.4).

Physical models containing left and right fractional differential operators have recently renewed attention from scientists which is mainly due to applications as models for physical phenomena exhibiting anomalous diffusion. A strong motivation for investigating the fractional differential equation (1.2) comes from symmetry fractional advection-dispersion equation (SADE for short). A fractional advection-dispersion equation (ADE for short) is a generalization of the classical ADE in which the second-order derivative is replaced with a fractional-order derivative. In contrast to the classical ADE, the fractional ADE has solutions that resemble the highly skewed and heavy-tailed breakthrough curves observed in field and laboratory studies [5], [7], in particular in contaminant transport of ground-water flow [6]. In [6], the authors state that solutes moving through a highly heterogeneous aquifer violations violates the basic assumptions of local second order theories because of large deviations from the stochastic process of Brownian motion.

According to [5], the one-dimensional form of the fractional ADE can be written as

$$\frac{\partial \mathcal{C}}{\partial t} = -v\frac{\partial \mathcal{C}}{\partial x} + \mathcal{D}j\frac{\partial^\gamma \mathcal{C}}{\partial^\gamma x} + \mathcal{D}(1-j)\frac{\partial^\gamma \mathcal{C}}{\partial(-x)^\gamma}, \tag{1.5}$$

where $\mathcal{C}$ is the expected concentration, t is time, v is a constant mean velocity, x is the distance in the direction of mean velocity, $\mathcal{D}$ is a constant dispersion coefficient, $0 \leq j \leq 1$ describes the skewness of the transport process, and γ is the order of left and right fractional differential operators. For discussions of this equation, see [6]

A special case of the fractional ADE (1.5) describes symmetric transitions, where $j = 1/2$. Defining the symmetric operator equivalent to the Riesz potential [23]

$$2\nabla^\gamma = D_+^\gamma + D_-^\gamma \tag{1.6}$$

gives the mass balance equation for advection and symmetric fractional dispersion

$$\frac{\partial \mathcal{C}}{\partial t} = -v\nabla\mathcal{C} + \mathcal{D}\nabla^\gamma\mathcal{C}. \tag{1.7}$$

The fractional ADE has been studied in one dimension ([6]), over infinite domains by using the Fourier transform of fractional differential operators to determine a classical solution. Variational methods, especially the Galerkin approximation has been investigated to find the solutions of fractional BVP [9] and fractional ADE [8] on a finite domain by establishing some suitable fractional derivative spaces.

Our goal in this paper is to show how variational methods based on Mountain pass theorem can be used to get existence results for (1.2). However, the direct application of the mountain pass theorem is not enough since the Palais-Smale sequences might lose compactness in the whole space $\mathbb{R}$. To overcome this difficulty we proof a version of compact embedding for fractional space following the ideas of [18]. Before stating our results let us introduce the main ingredients involved in our approach. We define

$$\|u\|^2_{I^\alpha_{-\infty}} = \int_{-\infty}^{\infty} |u(t)|^2 dt + \int_{-\infty}^{\infty} |{}_{-\infty}D_t^\alpha u(t)|^2 dt$$

and the space

$$I^\alpha_{-\infty}(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_\alpha}.$$

Now we say that $u \in I^\alpha_{-\infty}(\mathbb{R})$ is a weak solution of (1.2) if

$$\int_{-\infty}^{\infty} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t))]dt = \int_{-\infty}^{\infty} (\nabla W(t, u(t)), v(t))dt,$$

for all $v \in I^\alpha_{-\infty}(\mathbb{R})$. For $u \in I^\alpha_{-\infty}(\mathbb{R})$ we may define the functional

$$I(u) = \frac{1}{2}\int_{-\infty}^{\infty} [|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt - \int_{-\infty}^{\infty} W(t, u(t))dt. \tag{1.8}$$

which is of class C^1. We say that $u \in E^\alpha$ is a weak solution of (1.2) if u is a critical point of I.

Now we are in a position to state our main existence theorem.

Theorem 1.1. *Suppose that (L1), (W1)–(W3) hold. Then* (1.2) *possesses at least one nontrivial solution.*

The rest of the paper is organized as follows: in section 2, subsection 2.1, we describe the Liouville-Weyl fractional calculus; in subsection 2.2 we introduce the fractional space that we use in our work and some proposition are proven which will aid in our analysis. In section 3, we will prove Theorem 1.1.

2. Preliminary results

2.1. **Liouville-Weyl Fractional Calculus.** The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ are defined as

$$
{}_{-\infty}I_x^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}u(\xi)d\xi, \tag{2.1}
$$

$$
{}_xI_{\infty}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}u(\xi)d\xi\,. \tag{2.2}
$$

The Liouville-Weyl fractional derivative of order $0 < \alpha < 1$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals

$$
{}_{-\infty}D_x^{\alpha}u(x) = \frac{d}{dx}{}_{-\infty}I_x^{1-\alpha}u(x), \tag{2.3}
$$

$$
{}_xD_{\infty}^{\alpha}u(x) = -\frac{d}{dx}{}_xI_{\infty}^{1-\alpha}u(x)\,. \tag{2.4}
$$

The definitions (2.3) and (2.4) may be written in an alternative form:

$$
{}_{-\infty}D_x^{\alpha}u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}}d\xi, \tag{2.5}
$$

$$
{}_xD_{\infty}^{\alpha}u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}}d\xi\,. \tag{2.6}
$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$
\widehat{u}(w) = \int_{-\infty}^{\infty}e^{-ix.w}u(x)dx.
$$

Let $u(x)$ be defined on $(-\infty,\infty)$. Then the Fourier transform of the Liouville-Weyl integral and differential operator satisfies

$$
\widehat{{}_{-\infty}I_x^{\alpha}u(x)}(w) = (iw)^{-\alpha}\widehat{u}(w), \tag{2.7}
$$

$$
\widehat{{}_xI_{\infty}^{\alpha}u(x)}(w) = (-iw)^{-\alpha}\widehat{u}(w), \tag{2.8}
$$

$$
\widehat{{}_{-\infty}D_x^{\alpha}u(x)}(w) = (iw)^{\alpha}\widehat{u}(w), \tag{2.9}
$$

$$
\widehat{{}_xD_{\infty}^{\alpha}u(x)}(w) = (-iw)^{\alpha}\widehat{u}(w) \tag{2.10}
$$

2.2. **Fractional derivative spaces.** In this section we introduce some fractional spaces for more detail see [8]. Let $\alpha > 0$. Define the semi-norm

$$
|u|_{I_{-\infty}^{\alpha}} = \|{}_{-\infty}D_x^{\alpha}u\|_{L^2}
$$

and the norm

$$
\|u\|_{I_{-\infty}^{\alpha}} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^{\alpha}}^2\right)^{1/2}. \tag{2.11}
$$

Let

$$
I_{-\infty}^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^{\alpha}}}.
$$

Now we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the fourier transform. Let $0 < \alpha < 1$, let the semi-norm

$$
|u|_{\alpha} = \||w|^{\alpha}\widehat{u}\|_{L^2} \tag{2.12}
$$

and norm
$$\|u\|_\alpha = \left(\|u\|_{L^2}^2 + |u|_\alpha^2\right)^{1/2},$$
and let
$$H^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_\alpha}.$$
We note a function $u \in L^2(\mathbb{R})$ belongs to $I_{-\infty}^\alpha(\mathbb{R})$ if and only if
$$|w|^\alpha \widehat{u} \in L^2(\mathbb{R}). \tag{2.13}$$
Especially,
$$|u|_{I_{-\infty}^\alpha} = \||w|^\alpha \widehat{u}\|_{L^2}. \tag{2.14}$$
Therefore $I_{-\infty}^\alpha(\mathbb{R})$ and $H^\alpha(\mathbb{R})$ are equivalent with equivalent semi-norm and norm. Analogous to $I_{-\infty}^\alpha(\mathbb{R})$ we introduce $I_\infty^\alpha(\mathbb{R})$. Let the semi-norm
$$|u|_{I_\infty^\alpha} = \|{}_xD_\infty^\alpha u\|_{L^2}$$
and the norm
$$\|u\|_{I_\infty^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_\infty^\alpha}^2\right)^{1/2}. \tag{2.15}$$
Let
$$I_\infty^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_{I_\infty^\alpha}}.$$
Moreover $I_{-\infty}^\alpha(\mathbb{R})$ and $I_\infty^\alpha(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm [8].

Now we give the prove of the Sobolev lemma.

Theorem 2.1. *If $\alpha > 1/2$, then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_\alpha$ such that*
$$\sup_{x\in\mathbb{R}} |u(x)| \le C\|u\|_\alpha \tag{2.16}$$

Proof. By the Fourier inversion theorem, if $\widehat{u} \in L^1(\mathbb{R})$, then u is continuous and
$$\sup_{x\in\mathbb{R}} |u(x)| \le \|\widehat{u}\|_{L^1}.$$
Hence, to prove the theorem it is sufficient to prove that
$$\|\widehat{u}\|_{L^1} \le \|u\|_\alpha,$$
so by Schwarz inequality, we have
$$\begin{aligned}\int_{\mathbb{R}} |\widehat{u}(w)|dw &= \int_{\mathbb{R}} (1+|w|^2)^{\alpha/2}|\widehat{u}(w)|\frac{1}{(1+|w|^2)^{\alpha/2}}dw \\ &\le \Big(\int_{\mathbb{R}} (1+|w|^{2\alpha})|\widehat{u}(w)|^2 dw\Big)^{1/2}\Big(\int_{\mathbb{R}} (1+|w|^2)^{-\alpha}dw\Big)^{1/2}.\end{aligned}$$
The first integral on the right is $\|u\|_\alpha^2$, so the theorem depends on the fact that
$$\int_{\mathbb{R}} (1+|w|^2)^{-\alpha}dw < \infty$$
precisely when $\alpha > 1/2$. □

Remark 2.2. If $u \in H^\alpha(\mathbb{R})$, then $u \in L^q(\mathbb{R})$ for all $q \in [2,\infty]$, since
$$\int_{\mathbb{R}} |u(x)|^q dx \le \|u\|_\infty^{q-2}\|u\|_{L^2}^2.$$

Now we introduce a new fractional space. Let

$$X^\alpha = \left\{u \in H^\alpha(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t).u(t)dt < \infty\right\}.$$

The space X^α is a Hilbert space with the inner product

$$\langle u, v\rangle_{X^\alpha} = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t),\ {}_{-\infty}D_t^\alpha v(t)) + L(t)u(t).v(t)dt$$

and the corresponding norm

$$\|u\|^2_{X^\alpha} = \langle u, u\rangle_{X^\alpha}$$

Lemma 2.3. *Suppose L satisfies* (L1). *Then X^α is continuously embedded in $H^\alpha(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Since $l \in C(\mathbb{R}, (0, \infty))$ and l is coercive, then $l_{\min} = \min_{t\in\mathbb{R}} l(t)$ exists, so we have

$$(L(t)u(t), u(t)) \geq l(t)|u(t)|^2 \geq l_{\min}|u(t)|^2, \quad \forall t \in \mathbb{R}.$$

Then

$$\begin{aligned} l_{\min}\|u\|^2_\alpha &= l_{\min}\Big(\int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 + |u(t)|^2 dt\Big) \\ &\leq l_{\min}\int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 dt + \int_{\mathbb{R}} (L(t)u(t), u(t))dt \end{aligned}$$

So

$$\|u\|^2_\alpha \leq K\|u\|^2_{X^\alpha} \tag{2.17}$$

where $K = \max\{l_{\min}, 1\}/l_{\min}$. □

Lemma 2.4. *Suppose L satisfies* (L1). *Then the imbedding of X^α in $L^2(\mathbb{R})$ is compact.*

Proof. We note first that by Lemma 2.3 and Remark 2.2 we have

$$X^\alpha \hookrightarrow L^2(\mathbb{R}) \text{ is continuous.}$$

Now, let $(u_k) \in X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in X^α. We will show that $u_k \to u$ in $L^2(\mathbb{R})$. Suppose, without loss of generality, that $u_k \to 0$ in X^α. The Banach-Steinhaus theorem implies that

$$A = \sup_k \|u_k\|_{X^\alpha} < +\infty$$

Let $\epsilon > 0$; there is $T_0 < 0$ such that $\frac{1}{l(t)} \leq \epsilon$ for all t such that $t \leq T_0$. Similarly, there is $T_1 > 0$, such that $\frac{1}{l(t)} \leq \epsilon$ for all $t \geq T_1$. Sobolev's theorem (see e.g. [24]) implies that $u_k \to 0$ uniformly on $\overline{\Omega} = [T_0, T_1]$, so there is a k_0 such that

$$\int_\Omega |u_k(t)|^2 dt \leq \epsilon, \quad \text{for all } k \geq k_0. \tag{2.18}$$

Since $1/l(t) \leq \epsilon$ on $(-\infty, T_0]$ we have

$$\int_{-\infty}^{T_0} |u_k(t)|^2 dt \leq \epsilon \int_{-\infty}^{T_0} l(t)|u_k(t)|^2 dt \leq \epsilon A^2. \tag{2.19}$$

Similarly, since $1/l(t) \leq \epsilon$ on $[T_1, +\infty)$, we have

$$\int_{T_1}^{+\infty} |u_k(t)|^2 dt \leq \epsilon A^2. \tag{2.20}$$

Combining (2.18), (2.19) and (2.20) we obtain $u_k \to 0$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. $\square$

Lemma 2.5. *There are constants $c_1 > 0$ and $c_2 > 0$ such that*

$$W(t,u) \geq c_1|u|^\mu, \quad |u| \geq 1\,, \tag{2.21}$$

$$W(t,u) \leq c_2|u|^\mu, \quad |u| \leq 1\,. \tag{2.22}$$

Proof. By (W1) we note that

$$\mu W(t,\sigma u) \leq (\sigma u, \nabla W(t,\sigma u)).$$

Let $f(\sigma) = W(t,\sigma u)$, then

$$\frac{d}{d\sigma}\big(f(\sigma)\sigma^{-\mu}\big) \geq 0\,. \tag{2.23}$$

Now we consider two cases

Case 1. $|u| \leq 1$. In this case we integrate (2.23), from 1 to $1/|u|$ and we obtain

$$W(t,u) \leq W(t, \frac{u}{|u|})|u|^\mu. \tag{2.24}$$

Case 2. $|u| \geq 1$. In this case we integrate (2.23), from $1/|u|$ to 1 and we obtain

$$W(t,u) \geq |u|^\mu W(t, \frac{u}{|u|}). \tag{2.25}$$

Now, since $u \in \mathbb{R}^n$, $\frac{u}{|u|} \in B(0,1)$. So, since W is continuous and $B(0,1)$ is compact, there are $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq W(t,u) \leq c_2, \quad \text{for every } u \in B(0,1).$$

Therefore, we get the statement of the lemma. $\square$

Remark 2.6. By lemma 2.5, we have

$$W(t,u) = o(|u|^2) \quad \text{as } u \to 0 \text{ uniformly in } t \in \mathbb{R} \tag{2.26}$$

In addition, by (W2), for any $u \in \mathbb{R}^n$ such that $|u| \leq M_1$, there exists some constant $d > 0$ (dependent on M_1) such that

$$|\nabla W(t,u(t))| \leq d|u(t)|\,. \tag{2.27}$$

As in [18, lemma 2], we obtain the following result.

Lemma 2.7. *Suppose that* (L1), (W1)-(W2) *are satisfied. If $u_k \rightharpoonup u$ in X^α, then $\nabla W(t,u_k) \to \nabla W(t,u)$ in $L^2(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Assume that $u_k \rightharpoonup u$ in X^α. Then there exists a constant $d_1 > 0$ such that, by Banach-Steinhaus theorem and (2.16),

$$\sup_{k\in\mathbb{N}} \|u_k\|_\infty \leq d_1, \quad \|u\|_\infty \leq d_1.$$

By (W2), for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|u_k| < \delta \quad \text{implies} \quad |\nabla W(t,u_k)| \leq \epsilon|u_k|\,.$$

By (W3) there is $M > 0$ such that

$$|\nabla W(t,u_k)| \leq M, \ \text{ forall } \delta < u_k \leq d_1\,.$$

Therefore, there exists a constant $d_2 > 0$ (dependening on d_1) such that

$$|\nabla W(t,u_k(t))| \leq d_2|u_k(t)|, \ \ |\nabla W(t,u(t))| \leq d_2|u(t)|$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence,

$$|\nabla W(t, u_k(t)) - \nabla W(t, u(t))| \leq d_2(|u_k(t)| + |u(t)|) \leq d_2(|u_k(t) - u(t)| + 2|u(t)|)\,.$$

Since, by lemma 2.4, $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_{L^2} < \infty\,.$$

But this implies $u_k(t) \to u(t)$ almost everywhere $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |u_k(t) - u(t)| = v(t) \in L^2(\mathbb{R}, \mathbb{R}^n)\,.$$

Therefore,

$$|\nabla W(t, u_k(t)) - \nabla W(t, u(t))| \leq d_2(v(t) + 2|u(t)|)\,.$$

Then, using the Lebesgue's convergence theorem, the lemma is proved. $\square$

Now we introduce more symbols and some definitions. Let $\mathfrak{B}$ be a real Banach space, $I \in C^1(\mathfrak{B}, \mathbb{R})$, which means that I is a continuously Fréchet-differentiable functional defined on $\mathfrak{B}$. Recall that $I \in C^1(\mathfrak{B}, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_k\}_{k\in\mathbb{N}} \in \mathfrak{B}$, for which $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $\mathfrak{B}$.

Moreover, let B_r be the open ball in $\mathfrak{B}$ with the radius r and centered at 0 and ∂B_r denote its boundary. We obtain the existence of solutions to (1.2) by use of the following well-known Mountain Pass Theorems, see [21].

Theorem 2.8. *Let $\mathfrak{B}$ be a real Banach space and $I \in C^1(\mathfrak{B}, \mathbb{R})$ satisfying the (PS) condition. Suppose that $I(0) = 0$ and*

(i) *There are constants $\rho, \beta > 0$ such that $I|_{\partial B_\rho} \geq \beta$, and*
(ii) *There is and $e \in \mathfrak{B} \setminus \overline{B_\rho}$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \beta$. Moreover c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)),$$

where

$$\Gamma = \{\gamma \in C([0,1], \mathfrak{B}) : \gamma(0) = 0,\ \gamma(1) = e\}$$

3. Proof of Theorem 1.1

Now we establish the corresponding variational framework to obtain the existence of solutions for (1.2). Define the functional $I : X^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \Big[\frac{1}{2}|{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t))\Big]dt \\ &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, u(t))dt \end{aligned} \tag{3.1}$$

Lemma 3.1. *Under the conditions of Theorem 1.1, we have*

$$I'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]dt \tag{3.2}$$

for all $u, v \in X^\alpha$, which yields

$$I'(u)u = \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt. \tag{3.3}$$

Moreover, I is a continuously Fréchet-differentiable functional defined on X^α; i.e., $I \in C^1(X^\alpha, \mathbb{R})$.

Proof. We firstly show that $I : X^\alpha \to \mathbb{R}$. By (2.26), there is a $\delta > 0$ such that $|u| \leq \delta$ implies

$$W(t, u) \leq \epsilon |u|^2 \quad \text{for all } t \in \mathbb{R} \tag{3.4}$$

Let $u \in X^\alpha$, then $u \in C(\mathbb{R}, \mathbb{R}^n)$, the space of continuous function $u \in \mathbb{R}$ such that $u(t) \to 0$ as $|t| \to +\infty$. Therefore there is a constant $R > 0$ such that $|t| \geq R$ implies $|u(t)| \leq \delta$. Hence, by (3.4), we have

$$\int_{\mathbb{R}} W(t, u(t)) \leq \int_{-R}^{R} W(t, u(t))dt + \epsilon \int_{|t| \geq R} |u(t)|^2 dt < +\infty. \tag{3.5}$$

Combining (3.1) and (3.5), we show that $I : X^\alpha \to \mathbb{R}$.

Now we prove that $I \in C^1(X^\alpha, \mathbb{R})$. Rewrite I as follows

$$I = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2} \int_{\mathbb{R}} [|_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt, \quad I_2 = \int_{\mathbb{R}} W(t, u(t))dt$$

It is easy to check that $I_1 \in C^1(X^\alpha, \mathbb{R})$ and

$$I_1'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t))]\, dt. \tag{3.6}$$

Thus it is sufficient to show this is the case for I_2. In the process we will see that

$$I_2'(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt, \tag{3.7}$$

which is defined for all $u, v \in X^\alpha$. For any given $u \in X^\alpha$, let us define $J(u) : X^\alpha \to \mathbb{R}$ as follows

$$J(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt, \quad \forall v \in X^\alpha.$$

It is obvious that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u \in X^\alpha$, by (2.27), there is a constant $d_3 > 0$ such that

$$|\nabla W(t, u(t))| \leq d_3 |u(t)|,$$

which yields that, by Hölder's inequality and lemma 2.3,

$$\begin{aligned} |J(u)v| &= \Big| \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt \Big| \\ &\leq d_3 \int_{\mathbb{R}} |u(t)||v(t)|dt \leq \frac{d_3}{l_{\min}} \|u\|_{X^\alpha} \|v\|_{X^\alpha}. \end{aligned} \tag{3.8}$$

Moreover, for u and $v \in X^\alpha$, by mean value theorem, we have

$$\int_{\mathbb{R}} W(t, u(t) + v(t))dt - \int_{\mathbb{R}} W(t, u(t))dt = \int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)))dt,$$

where $h(t) \in (0,1)$. Therefore, by lemma 2.4 and Hölder's inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)), v(t))dt - \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt \\ &= \int_{\mathbb{R}} (\nabla W(t, u(t)) + h(t)v(t) - \nabla W(t, u(t)), v(t))dt \to 0 \end{aligned} \tag{3.9}$$

as $v \to 0$ in X^α. Combining (3.8) and (3.9), we see that (3.7) holds. It remains to prove that I_2' is continuous. Suppose that $u \to u_0$ in X^α and note that

$$\begin{aligned} \sup_{\|v\|_{X^\alpha}=1} |I_2'(u)v - I_2'(u_0)v| &= \sup_{\|v\|_{X^\alpha}=1} \Big| \int_{\mathbb{R}} (\nabla W(t, u(t)) - \nabla W(t, u_0(t)), v(t))dt \Big| \\ &\le \sup_{\|v\|_{X^\alpha}=1} \|\nabla W(., u(.)) - \nabla W(., u_0(.))\|_{L^2} \|v\|_{L^2} \\ &\le \frac{1}{\sqrt{l_{\min}}} \|\nabla W(., u(.)) - \nabla W(., u_0(.))\|_{L^2} \end{aligned}$$

By lemma 2.4, we obtain that $I_2'(u)v - I_2'(u_0)v \to 0$ as $\|u\|_{X^\alpha} \to \|u_0\|_{X^\alpha}$ uniformly with respect to v, which implies the continuity of I_2' and $I \in C^1(X^\alpha, \mathbb{R})$. ☐

Lemma 3.2. *Under conditions* (L1), (W1), (W2)*, I satisfies the (PS) condition.*

Proof. Assume that $(u_k)_{k\in\mathbb{N}} \in X^\alpha$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \le C_1, \quad \|I'(u_k)\|_{(X^\alpha)^*} \le C_1 \tag{3.10}$$

for every $k \in \mathbb{N}$. We firstly prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X^α. By (3.1), (3.3) and (W_1), we have

$$\begin{aligned} C_1 + \|u_k\|_{X^\alpha} &\ge I(u_k) - \frac{1}{\mu} I'(u_k)u_k \\ &= \Big(\frac{\mu}{2} - 1\Big) \|u_k\|^2_{X^\alpha} - \int_{\mathbb{R}} [W(t, u_k(t)) - \frac{1}{\mu}(\nabla W(t, u_k(t)), u_k(t))]dt \\ &\ge \Big(\frac{\mu}{2} - 1\Big) \|u_k\|^2_{X^\alpha}. \end{aligned} \tag{3.11}$$

Since $\mu > 2$, the inequality (3.11) shows that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X^α. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in X^α and hence, by lemma 2.4, $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. It follows from the definition of I that

$$\begin{aligned} &(I'(u_k) - I'(u))(u_k - u) \\ &= \|u_k - u\|^2_{X^\alpha} - \int_{\mathbb{R}} [\nabla W(t, u_k) - \nabla W(t, u)](u_k - u)dt. \end{aligned} \tag{3.12}$$

Since $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$, we have (see lemma 2.7) $\nabla W(t, u_k(t)) \to \nabla W(t, u(t))$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. Hence

$$\int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t))dt \to 0$$

as $k \to +\infty$. So (3.12) implies $\|u_k - u\|_{X^\alpha} \to 0$ as $k \to +\infty$. ☐

Now we are in the position to give the proof of Theorem 1.1. We divide the proof into several steps.

Proof of theorem 1.1. **Step 1.** It is clear that $I(0) = 0$ and $I \in C^1(X^\alpha, \mathbb{R})$ satisfies the (PS) condition by lemma 3.1 and 3.2.

Step 2. Now We show that there exist constant $\rho > 0$ and $\beta > 0$ such that I satisfies the condition (i) of theorem 2.8. By lemma 2.4, there is a $C_0 > 0$ such that

$$\|u\|_{L^2} \leq C_0 \|u\|_{X^\alpha}.$$

On the other hand by theorem 2.1, there is $C_\alpha > 0$ such that

$$\|u\|_\infty \leq C_\alpha \|u\|_{X^\alpha}.$$

By (2.26), for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$W(t, u(t)) \leq \epsilon |u(t)|^2 \quad \text{wherever } |u(t)| < \delta.$$

Let $\rho = \frac{\delta}{C_\alpha}$ and $\|u\|_{X^\alpha} \leq \rho$; we have $\|u\|_\infty \leq \frac{\delta}{C_\alpha}.C_\alpha = \delta$. Hence

$$|W(t, u(t))| \leq \epsilon |u(t)|^2 \quad \text{for all } t \in \mathbb{R}.$$

Integrating on $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}} W(t, u(t)) dt \leq \epsilon \|u\|_{L^2}^2 \leq \epsilon C_0^2 \|u\|_{X^\alpha}^2 .$$

So, if $\|u\|_{X^\alpha} = \rho$, then

$$I(u) = \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, u(t)) dt \geq (\frac{1}{2} - \epsilon C_0^2)\|u\|_{X^\alpha}^2 = (\frac{1}{2} - \epsilon C_0^2)\rho^2.$$

And it suffices to choose $\epsilon = \frac{1}{4C_0^2}$ to obtain

$$I(u) \geq \frac{\rho^2}{4C_0^2} = \beta > 0 . \tag{3.13}$$

Step 3. It remains to prove that there exists an $e \in X^\alpha$ such that $\|e\|_{X^\alpha} > \rho$ and $I(e) \leq 0$, where ρ is defined in Step 2. Consider

$$I(\sigma u) = \frac{\sigma^2}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, \sigma u(t)) dt$$

for all $\sigma \in \mathbb{R}$. By (2.21), there is $c_1 > 0$ such that

$$W(t, u(t)) \geq c_1 |u(t)|^\mu \quad \text{for all } |u(t)| \geq 1. \tag{3.14}$$

Take some $u \in X^\alpha$ such that $\|u\|_{X^\alpha} = 1$. Then there exists a subset Ω of positive measure of $\mathbb{R}$ such that $u(t) \neq 0$ for $t \in \Omega$. Take $\sigma > 0$ such that $\sigma|u(t)| \geq 1$ for $t \in \Omega$. Then by (3.14), we obtain

$$I(\sigma u) \leq \frac{\sigma^2}{2} - c_1 \sigma^\mu \int_\Omega |u(t)|^\mu dt. \tag{3.15}$$

Since $c_1 > 0$ and $\mu > 2$, (3.15) implies that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\sigma|u(t)| \geq 1$ for $t \in \Omega$ and $\|\sigma u\|_{X^\alpha} > \rho$, where ρ is defined in Step 2. By theorem 2.8, I possesses a critical value $c \geq \beta > 0$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], X^\alpha) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

Hence there is $u \in X^\alpha$ such that $I(u) = c$, $I'(u) = 0$. $\square$

Acknowledgements. The author was partially supported by grant 0607 from MECESUP.

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