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MULTIPLE SOLUTIONS FOR PERTURBED NON-LOCAL FRACTIONAL LAPLACIAN EQUATIONS

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Abstract. In article we consider problems modeled by the non-local fractional Laplacian equation

$$(-\Delta)^s u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where $s \in (0,1)$ is fixed, $(-\Delta)^s$ is the fractional Laplace operator, λ, μ are real parameters, Ω is an open bounded subset of $\mathbb{R}^n$ $(n > 2s)$ with Lipschitz boundary $\partial\Omega$ and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two suitable Carathéodory functions. By using variational methods in an appropriate abstract framework developed by Servadei and Valdinoci [17] we prove the existence of at least three weak solutions for certain values of the parameters.

1. Introduction

There are a lot interesting problems in the standard framework of the Laplacian (and, more generally, of uniformly elliptic operators), widely studied in the literature. A natural question is whether or not the existence results got in this classical context can be extended to the non-local framework of the fractional Laplacian type operators.

In this spirit, we study the existence of weak solutions for the following general non-local equation depending on two real parameters λ and μ given by

$$\begin{gathered} -\mathcal{L}_K u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega\,. \end{gathered} \tag{1.1}$$

More precisely, in our setting Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, (where $s \in (0,1)$) with smooth boundary $\partial\Omega$, while $\mathcal{L}_K$ is the integrodifferential operator defined as

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)\, dy, \quad x \in \mathbb{R}^n, \tag{1.2}$$

with the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ such that

$$mK \in L^1(\mathbb{R}^n), \quad \text{where } m(x) = \min\{|x|^2, 1\}; \tag{1.3}$$

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$$\text{there exists } \theta > 0 \text{ such that } K(x) \geq \theta |x|^{-(n+2s)} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}; \tag{1.4}$$

$$K(x) = K(-x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.5}$$

A model for K is given by the singular kernel $K(x) := |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n. \tag{1.6}$$

The homogeneous Dirichlet datum in (1.1) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on the boundary $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the non-local nature of the operator $\mathcal{L}_K$. Moreover, we will assume that $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two suitable Carathéodory functions with subcritical growth.

Recently, a great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. This type of operators arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves.

In this work, motivated by this large interest, we prove the existence of nontrivial weak solutions of problem (1.1) using variational and topological methods.

Our variational approach is realizable checking that the associated energy functional verifies the assumptions requested by a general critical point theorem obtained by Ricceri in [13, Theorem 1.1] (see Theorem 3.3 below) and thanks to a suitable variational setting developed by Servadei and Valdinoci in [17]. Indeed, the nonlocal analysis that we perform here in order to use Theorem is quite general and successfully exploited for other goals in several recent contributions; see [18, 19] and [5] for an elementary introduction to this topic and for a list of related references.

By a weak solutions of (1.1) we mean a solution of the problem

$$\begin{aligned} &\int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y)\, dx\, dy \\ &= \lambda \int_\Omega f(x, u(x))\varphi(x) dx + \mu \int_\Omega g(x, u(x))\varphi(x) dx, \quad \forall \varphi \in X_0,\ \forall u \in X_0. \end{aligned} \tag{1.7}$$

Hence, Problem (1.7) represents the weak formulation of (1.1). Note that, to write such a weak formulation, we need to assume condition (1.5).

The space X_0 in which we set problem (1.7) is a functional space, inspired by, but not equivalent to, the usual fractional Sobolev space. This new space was introduced in [16]; see also [17]. The choice of this space is motivated by the fact that it allows us to correctly encode the Dirichlet boundary datum in the weak formulation. We will recall its definition in Section 2, to make the present paper self-contained.

In the current literature [2, 3, 4, 6, 10, 11, 15, 17, 18, 19, 20, 21] the authors studied non-local fractional Laplacian equations with superlinear and subcritical or critical nonlinearities, while in [7, 8] the asymptotically linear case was exploited.

In this paper we are interested in equations depending on two real parameters. In many mathematical problems deriving from applications the presence of one (or more) parameter is a relevant feature, and the study of how solutions depend on parameters is an important topic.

Here, exploiting an abstract critical point theorem recalled in Theorem 3.1 and by using Proposition 3.2, we prove in the main result (see Theorem 3.3) the existence of an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number ρ with the following property: for each $\lambda \in \Lambda$ and each subcritical Carathéodory function g there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem (1.1) has at least three weak solutions whose norms in X_0 are less than ρ. We would like to stress that we do not assume any condition (only the natural subcritical growth condition) on the perturbation term g.

As an application of Theorem 3.3, we consider the model problem

$$
\begin{gathered}
(-\Delta)^s u = \lambda f(u) + \mu(1 + |u|^\beta) \quad \text{in } \Omega \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{gathered}
$$

where $\beta \in (0, 2^* - 1)$. Here the exponent $2^* := 2n/(n - 2s)$ is the fractional critical Sobolev exponent. Notice that when $s = 1$ the above exponent reduces to the classical critical Sobolev exponent $2_* := 2n/(n - 2)$. In this framework, the cited Theorem 3.3 reduces to the following result.

Theorem 1.1. *Let $s \in (0, 1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Further, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$
\sup_{s \in \mathbb{R}} \frac{|f(s)|}{1 + |s|^{\sigma - 1}} < +\infty,
$$

for some $1 < \sigma < 2^*$. Assume that there exist three positive real constants c, d and $1 < \gamma < 2$, such that

(I1') *$F(s) := \int_0^s f(t)dt > 0$, for every $s \in]0, c]$;*
(I2') *There exists $2 < \alpha < 2^*$, such that*

$$
\limsup_{s \to 0} \frac{F(s)}{|s|^\alpha} < +\infty;
$$

(I3') *$|F(s)| \le d(1 + |s|^\gamma)$ for every $s \in \mathbb{R}$.*

Then, there exist an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number ρ with the following property: For each $\lambda \in \Lambda$ there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem

$$
\begin{gathered}
(-\Delta)^s u = \lambda f(u) + \mu(1 + |u|^\beta) \quad \text{in } \Omega \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{gathered}
\tag{1.8}
$$

where $\beta \in (0, 2^* - 1)$, has at least three weak solutions $u_i \in H^s(\mathbb{R}^n)$ (with $i = 1, 2, 3$) such that $u_i = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and whose norms are less than ρ.

The article is organized as follows. In Section 2 we recall the definition of the functional space we work in and we give some notations. In Section 3 we prove our main result (Theorem 3.3) and its consequence (Theorem 1.1) for fractional Laplacian equations; see Remark 3.5. A direct application of our result, studying a non-local equation driven by the fractional Laplacian, is then presented; see Example 3.6.

2. Preliminaries

This section we introduce the notation used, and give some preliminary results which will be useful in the sequel.

2.1. **The functional space X_0.** We briefly recall the definition of the functional space X_0, firstly introduced in [16]. The reader familiar with this topic may skip this section and go directly to the next one.

The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map $(x,y)\mapsto (g(x)-g(y))\sqrt{K(x-y)}$ is in $L^2\big((\mathbb{R}^n\times\mathbb{R}^n)\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega), dxdy\big)$ (here $\mathcal{C}\Omega := \mathbb{R}^n\setminus\Omega$). Also, we denote by X_0 the linear subspace of X,

$$X_0 := \big\{g\in X : g=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\big\}.$$

We remark that X and X_0 are non-empty, since $C_0^2(\Omega)\subseteq X_0$ by [16, Lemma 11]. Moreover, the space X is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2K(x-y)dx\,dy\Big)^{1/2}, \tag{2.1}$$

where $Q=(\mathbb{R}^n\times\mathbb{R}^n)\setminus\mathcal{O}$ and $\mathcal{O}=(\mathcal{C}\Omega)\times(\mathcal{C}\Omega)\subset\mathbb{R}^n\times\mathbb{R}^n$. It is easily seen that $\|\cdot\|_X$ is a norm on X (see, for instance, [17] for a proof).

By [17, Lemmas 6 and 7] in the sequel we can take the function

$$X_0\ni v\mapsto \|v\|_{X_0} = \Big(\int_Q |v(x)-v(y)|^2K(x-y)\,dx\,dy\Big)^{1/2} \tag{2.2}$$

as norm on X_0. Also $(X_0,\|\cdot\|_{X_0})$ is a Hilbert space (for this see [17, Lemmas 7]), with scalar product

$$\langle u,v\rangle_{X_0} := \int_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,K(x-y)\,dx\,dy. \tag{2.3}$$

Note that in (2.2) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n\times\mathbb{R}^n$, since $v\in X_0$ (and so $v=0$ a.e. in $\mathbb{R}^n\setminus\Omega$). While for a general kernel K satisfying conditions (1.3)–(1.5) we have that $X_0\subset H^s(\mathbb{R}^n)$, in the model case $K(x):=|x|^{-(n+2s)}$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish almost everywhere outside Ω (see [19, Lemma 7]).

Here $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}. \tag{2.4}$$

Before concluding this subsection, we recall the embedding properties of X_0 into the usual Lebesgue spaces (see [17, Lemma 8]). The embedding $j: X_0\hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu\in[1,2^*]$, while it is compact whenever $\nu\in[1,2^*)$. Hence, for any $\nu\in[1,2^*]$ there exists a positive constant c_ν such that

$$\|v\|_{L^\nu(\mathbb{R}^n)}\le c_\nu\|v\|_{X_0}, \tag{2.5}$$

for any $v\in X_0$.

For further details on the fractional Sobolev spaces we refer the reader to [5] and to the references therein, while for other details on X and X_0 we refer to [16], where these functional spaces were introduced and various properties of these spaces were proved.

3. Main Results

3.1. **Multiple critical points.** For the reader's convenience, we recall the revised form of Ricceri's three critical points theorem (see [13, Theorem 1]) and [12, Proposition 3.1].

Theorem 3.1 ([13, Theorem 1]). *Let E be a reflexive real Banach space, $\Phi\colon E \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on E^* and Φ is bounded on each bounded subset of E; $\Psi\colon E \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval. Assume that*

$$\lim_{\|u\|_E \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty \tag{3.1}$$

for all $\lambda \in I$, and that there exists $h \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{u\in E} (\Phi(u) + \lambda(\Psi(u) + h)) < \inf_{u\in E} \sup_{\lambda\in I} (\Phi(u) + \lambda(\Psi(u) + h)). \tag{3.2}$$

Then, there exists an open interval $\Lambda \subseteq I$ and a positive real number ρ with the following property:

For every $\lambda \in \Lambda$ and every C^1 functional $J\colon E \mapsto \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0,$$

has at least three solutions in E whose norms are less than ρ.

Proposition 3.2 ([12, Proposition 3.1]). *Let E be a non-empty set and Φ, Ψ two real functions on E. Assume that there are $r > 0$ and $u_0, u_1 \in E$ such that*

$$\Phi(u_0) = -\Psi(u_0) = 0, \quad \Phi(u_1) > r, \quad \sup_{u\in\Phi^{-1}(]-\infty,r])} (-\Psi(u)) < r\frac{-\Psi(u_1)}{\Phi(u_1)}.$$

Then, for each h satisfying

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-\Psi(u)) < h < r\frac{-\Psi(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\geq 0} \inf_{u\in E} (\Phi(u) + \lambda(h + \Psi(u))) < \inf_{u\in E} \sup_{\lambda\geq 0} (\Phi(u) + \lambda(h + \Psi(u))).$$

For several related topics and a careful analysis of the abstract framework we refer the reader to the recent monograph [9].

3.2. **Three weak solutions.** Let us denote

$$F(x, s) := \int_0^s f(x, t)dt,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. For the rest of this article, the nonlinearity f in (1.1) is a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following growth conditions: There exist $a_1, a_2 \geq 0$ and $\sigma \in (1, 2^*)$ such that

$$|f(x, t)| \leq a_1 + a_2|t|^{\sigma-1} \quad \text{a.e. } x \in \Omega,\ t \in \mathbb{R}. \tag{3.3}$$

This assumption says that the function f has a subcritical growth. Our main result is the following multiplicity theorem.

Theorem 3.3. *Let $s \in (0,1)$, $n > 2s$, Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.3)–(1.5). *Further, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory such that condition* (3.3) *holds. In addition, assume that there exist three positive real constants c, d and $1 < \gamma < 2$, such that*

- (I1) *$F(x,s) > 0$ for a.e. $x \in \Omega$ and every $s \in]0,c]$;*
- (I2) *There exists $2 < \alpha < 2^*$, such that*
$$\limsup_{s\to 0} \frac{\sup_{x\in\Omega} F(x,s)}{|s|^\alpha} < +\infty;$$
- (I3) *$|F(x,s)| \le d(1+|s|^\gamma)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.*

Then, there exist an open interval $\Lambda \subseteq (0,+\infty)$ and a positive real number ρ with the following property:

For each $\lambda \in \Lambda$ and each Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying
$$\sup_{(x,t)\in\Omega\times\mathbb{R}} \frac{|g(x,t)|}{1+|t|^{\beta-1}} < +\infty,$$
for some $1 < \beta < 2^*$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, problem (1.1) *has at least three weak solutions whose norms in X_0 are less than ρ.*

Proof. The idea of the proof consists in applying Theorem 3.1 and Proposition 3.2. The energy functional corresponding to problem (1.1) is defined on $E := X_0$ as
$$H(u) := \Phi(u) + \lambda\Psi(u) + \mu J(u), \tag{3.4}$$
where
$$\Phi(u) := \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy, \tag{3.5}$$
$$\Psi(u) := -\int_\Omega F(x,u(x))dx, \tag{3.6}$$
$$J(u) := -\int_\Omega G(x,u(x))dx, \tag{3.7}$$
where
$$G(x,s) := \int_0^s g(x,t)dt,$$
for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

Of course, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on E^*, moreover, Ψ and J are continuously Gâteaux differentiable functionals whose Gâteaux derivative is compact. Obviously, Φ is bounded on each bounded subset of E under our assumptions.

Taking into account (3.5), (3.6) and (3.7), for each u and $v \in E$, one has
$$\langle \Phi'(u), v\rangle = \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy,$$
$$\langle \Psi'(u), v\rangle = -\int_\Omega f(x,u(x))v(x)\,dx,$$

$$\langle J'(u), v\rangle = -\int_\Omega g(x,u(x))v(x)\,dx.$$

Now, from (I3) and (2.5) it follows that

$$\begin{aligned}\lambda\Psi(u) &= -\lambda\int_\Omega F(x,u(x))dx\\ &\geq -\lambda d\int_\Omega (1+|u(x)|^\gamma)dx\\ &\geq -\lambda d(\text{meas}(\Omega)+\|u\|^\gamma_{L^\gamma(\Omega)})\\ &\geq -C_\lambda(1+\|u\|^\gamma_{X_0}).\end{aligned}$$

for a suitable positive constant C_λ. Thus, we obtain

$$\Phi(u)+\lambda\Psi(u) \geq \frac{1}{2}\|u\|^2_{X_0} - C_\lambda(1+\|u\|^\gamma_{X_0}),$$

Since $\gamma < 2$, it follows that

$$\lim_{\|u\|_{X_0}\to+\infty}(\Phi(u)+\lambda\Psi(u)) = +\infty \quad \forall u\in E,\ \lambda\in(0,+\infty).$$

Then, assumption (3.1) of Theorem 3.1 is satisfied.

Next, we prove that

$$\sup_{\lambda\in I}\inf_{u\in E}(\Phi(u)+\lambda(\Psi(u)+h)) < \inf_{u\in E}\sup_{\lambda\in I}(\Phi(u)+\lambda(\Psi(u)+h)),$$

for some $h\in\mathbb{R}$. This means that assumption (3.2) is also satisfied. For our goal it suffices to verify the conditions of Proposition 3.2. Hence, let us take $u_0\equiv 0$ and observe that

$$\Phi(u_0) = -\Psi(u_0) = 0.$$

At this point, we claim that there exist $r>0$ and $u_1\in X$ such that $\Phi(u_1)>r$ and (3.2) is satisfied. By (I2) there exist $\eta\in[0,1]$ and $C_1>0$, such that

$$F(x,s) < C_1|s|^\alpha, \quad \forall s\in[-\eta,\eta],\ \text{a.e. } x\in\Omega.$$

Then, by using (I3), we can find a constant M such that

$$F(x,s) < M|s|^\alpha,$$

for every $s\in\mathbb{R}$ and a.e. $x\in\Omega$.

Consequently, fixing a constant $r>0$, by the Sobolev embedding theorem, for a suitable positive constant C_2 we have

$$-\Psi(u) = \int_\Omega F(x,u(x))dx < M\int_\Omega |u(x)|^\alpha dx \leq M\|u\|^\alpha_{L^\alpha(\Omega)} \leq C_2 r^{\alpha/2},$$

for every $u\in\Phi^{-1}(]-\infty,r])$. Hence, being $\alpha>2$, it follows that

$$\lim_{r\to0^+}\frac{\sup_{\|u\|^2_{X_0}\leq 2r}(-\Psi(u))}{r} = 0. \tag{3.8}$$

Let $u_1\in C^2_0(\Omega)$ be a function positive in Ω, with $u|_{\partial\Omega}=0$ and

$$\max_{x\in\overline{\Omega}} u(x) \leq c.$$

Note that $C_0^2(\Omega) \subseteq X_0$; see [16, Lemma 11]. Then, of course, $u_1 \in E$ and in addition $\Phi(u_1) > 0$. In view of (I1) we also have

$$-\Psi(u_1) = \int_\Omega F(x, u_1(x))dx > 0.$$

Therefore, from (3.8), we can find $r \in (0, \Phi(u_1))$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} (-\Psi(u)) = \sup_{\|u\|_{X_0}^2 \le 2r} (-\Psi(u)) < r \frac{-\Psi(u_1)}{\Phi(u_1)}.$$

The proof is complete. □

The following is special case of Theorem 3.3.

Corollary 3.4. *Let $s \in (0,1)$, $n > 2s$, Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying conditions* (1.3)–(1.5)*.*

Further, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\sup_{s \in \mathbb{R}} \frac{|f(s)|}{1 + |s|^{\sigma - 1}} < +\infty,$$

for some $1 < \sigma < 2^*$. In addition, assume that there exist three positive real constants c, d and $1 < \gamma < 2$, such that conditions (I1')–(I3') *hold. Then, there are an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$ the problem*

$$\begin{gathered} -\mathcal{L}_K u = \lambda f(u) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{gathered}$$

has at least three weak solutions whose norms in X_0 are less than ρ.

Remark 3.5. In the model case in which $K(x) := |x|^{-(n+2s)}$ the space X_0 can be characterized as follows

$$X_0 = \{v \in H^s(\mathbb{R}^n) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\};$$

for details see [19, Lemma 7-b]. Then, is immediate to observe that Theorem 1.1 in the Introduction is a direct consequence of Theorem 3.3.

Example 3.6. Let Ω be an open bounded set of $\mathbb{R}^3$ with Lipschitz boundary. A simple example of application of Theorem 1.1 can be produced considering the real function

$$f(t) := \begin{cases} 0 & \text{if } t < 0 \\ t^{\alpha - 1} & \text{if } 0 \le t \le 1 \\ t^{\gamma - 1} & \text{if } t > 1, \end{cases}$$

where $2 < \alpha < 3$ and $1 < \gamma < 2$. In such a case there exist an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number ρ with the following property:

For each $\lambda \in \Lambda$ there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the Dirichlet problem

$$\begin{gathered} (-\Delta)^{1/2} u = \lambda f(u) + \mu(1 + |u|^\beta) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{gathered}$$

where $\beta \in (0, 2)$, has at least three non-trivial weak solutions $u_i \in H^{1/2}(\mathbb{R}^3)$ (with $i = 1, 2, 3$) such that $u_i = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and whose norms are less than ρ.

Remark 3.7. We point out that, recently, Teng in [22], by using a non-smooth critical point theorem due to Arcoya and Carmona [1], studied the existence of two non-trivial weak solutions for a parametric non-local hemivariational inequalities with the Dirichlet boundary condition. More precisely, they proved, for suitable value of the real parameters λ and μ, the existence of two non-trivial weak solutions for the Dirichlet problem

$$\begin{gathered} -\mathcal{L}_K u \in \lambda(\partial j(x,u) + \mu \partial k(x,u)) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{gathered}$$

where $j,k : \Omega\times\mathbb{R} \to \mathbb{R}$ are suitable measurable functions such that, for almost every $x \in \Omega$, one has that $j(x,\cdot)$ and $k(x,\cdot)$ are locally Lipschitz continuous functions. Here, the expressions $\partial j(x,\cdot)$ and $\partial k(x,\cdot)$ denote the generalized subdifferential in the sense of Clarke. We just observe that Theorem 3.3 and the smooth version of the above mentioned result are independent.

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