Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 261, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND GLOBAL ASYMPTOTIC STABILITY OF POSITIVE PERIODIC SOLUTIONS OF A LOTKA-VOLTERRA TYPE COMPETITION SYSTEMS WITH DELAYS AND FEEDBACK CONTROLS

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Abstract. The existence of positive periodic solutions of a periodic Lotka-Volterra type competition system with delays and feedback controls is studied by applying the continuation theorem of coincidence degree theory. By contracting a suitable Liapunov functional, a set of sufficient conditions for the global asymptotic stability of the positive periodic solution of the system is given. A counterexample is given to show that the result on the existence of positive periodic solution in [4] is incorrect.

1. Introduction

In this paper, we consider the following non-autonomous Lotka-Volterra n-species competition system with delays and feedback controls

$$
\begin{aligned}
\dot{x}_i(t) = g_i(x_i(t))\Big[r_i(t) &- \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n}\sum_{k=1}^{m} b_{ijk}(t)x_j(t-\tau_{ijk}(t)) \\
&- \sum_{j=1}^{n}\sum_{k=1}^{m}\int_{-\infty}^{t} c_{ijk}(t,s)x_j(s)ds - d_i(t)u_i(t) \\
&- \sum_{k=1}^{m} e_{ik}(t)u_i(t-\sigma_{ik}(t)) - \int_{-\infty}^{t} f_i(t,s)u_i(s)ds\Big], \\
\dot{u}_i(t) = -\alpha_i(t)u_i(t) &+ \beta_i(t)x_i(t) + \sum_{k=1}^{m} p_{ik}(t)x_i(t-\gamma_{ik}(t)) \\
&+ \int_{-\infty}^{t} v_i(t,s)x_i(s)ds,
\end{aligned} \tag{1.1}
$$

where $i \in \{1, 2, \dots, n\}$, u_i denote indirect feedback control variables. For system (1.1), we introduce the following hypotheses

2000 *Mathematics Subject Classification.* 34K13, 92B05, 92D25, 93B52.
Key words and phrases. Lotka-Volterra competition system; time delay; feedback control; positive periodic solution; global asymptotic stability; continuation theorem.

Submitted January 8, 2013. Published November 26, 2013.

(H1) $r_i, \alpha_i \in C(\mathbb{R},\mathbb{R})$, $a_{ij}, b_{ijk}, d_i, e_{ik}, p_{ik}, \beta_i \in C(\mathbb{R},\mathbb{R}_+)$ are ω-periodic (ω is a fixed positive number) with $\int_0^\omega r_i(t)dt > 0$, $\int_0^\omega \alpha_i(t)dt > 0$, $i, j = 1, 2, \dots, n$; $k = 1, 2, \dots, m$.

(H2) $c_{ijk}, f_i, v_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ are ω-periodic functions; i.e.,

$$c_{ijk}(t+\omega, s+\omega) = c_{ijk}(t,s), f_i(t+\omega, s+\omega) = f_i(t,s), v_i(t+\omega, s+\omega) = v_i(t,s)$$

and $\int_{-\infty}^t c_{ijk}(t,s)ds$, $\int_{-\infty}^t f_i(t,s)ds$, $\int_{-\infty}^t v_i(t,s)ds$ are continuous with respect to t. Moreover

$$\int_0^{+\infty}\int_{-t}^0 c_{ijk}(s+t,s)\,ds\,dt < +\infty, \quad \int_0^{+\infty}\int_{-t}^0 f_i(s+t,s)\,ds\,dt < +\infty,$$
$$\int_0^{+\infty}\int_{-t}^0 v_i(s+t,s)\,ds\,dt < +\infty, \quad i,j = 1,2,\dots,n; k = 1,2,\dots,m.$$

(H3) $g_i \in C(\mathbb{R}_+,\mathbb{R}_+)$ is strictly increasing, $g_i(0) = 0$ and $\lim_{v\to 0^+} \frac{g_i(v)}{v}$ is a positive constant. Moreover, there are positive constants l and L such that $l \le \frac{g_i(v)}{v} \le L$ for all $v > 0$, $i = 1, 2, \dots, n$.

(H4) τ_{ijk}, σ_{ik}, $\gamma_{ik} \in C(\mathbb{R},\mathbb{R}_+)$ are ω-periodic for $i, j = 1, 2, \dots\ n, k = 1, 2, \dots, m$.

(H5) τ_{ijk}, σ_{ik}, $\gamma_{ik} \in C^1(\mathbb{R},\mathbb{R}_+)$ and $\dot\tau_{ijk}(t) < 1$, $\dot\sigma_{ik}(t) < 1$, $\dot\gamma_{ik}(t) < 1$ for all $t \in \mathbb{R}$, $i, j = 1, 2, \dots, n$, $k = 1, 2, \dots, m$.

We consider (1.1) with the initial conditions

$$\begin{aligned} x_i(s) = \phi_i(s), s \in (-\infty, 0], \quad \phi_i \in C((-\infty,0],\mathbb{R}_+), \phi_i(0) > 0 \\ u_i(s) = \psi_i(s), s \in (-\infty, 0], \quad \psi_i \in C((-\infty,0],\mathbb{R}_+), \quad \psi_i(0) > 0, \end{aligned} \tag{1.2}$$

for $i = 1, 2, \dots, n$. Throughout this paper, we use the following symbols: for an ω-periodic function $f \in C(\mathbb{R},\mathbb{R})$, we define

$$\begin{aligned} \bar f = \frac{1}{\omega}\int_0^\omega f(t)dt, \quad c_{ijk}^*(t) = \int_{-\infty}^t c_{ijk}(t,s)ds, \quad f_i^*(t) = \int_{-\infty}^t f_i(t,s)ds, \\ v_i^*(t) = \int_{-\infty}^t v_i(t,s)ds, \quad G_i(t,s) = \frac{\exp\{\int_t^s \alpha_i(v)dv\}}{\exp\{\int_0^\omega \alpha_i(v)dv\} - 1}, s \ge t, \\ P_i(t) = d_i(t)\int_t^{t+\omega} G_i(t,s)\Big[\beta_i(s) + \sum_{k=1}^m p_{ik}(s) + v_i^*(s)\Big]ds, \\ Q_i(t) = \sum_{j=1}^m e_{ij}(t)\int_t^{t+\omega} G_i(t,s)\Big[\beta_i(s) + \sum_{k=1}^m p_{ik}(s) + v_i^*(s)\Big]ds, \\ R_i(t) = \int_{-\infty}^t f_i(t,s)\int_s^{s+\omega} G_i(s,\tau)\Big[\beta_i(\tau) + \sum_{k=1}^m p_{ik}(\tau) + v_i^*(\tau)\Big]d\tau ds. \end{aligned} \tag{1.3}$$

Fan et al. [1] studied system (1.1) in the case $k = 1$, $g_i(v_i) = v_i$, $f_i(t,s) = v_i(t,s) = 0$ for $i = 1, 2, \dots, n$ and obtained several results for the existence and global asymptotically stability of positive periodic solutions of the system. Recently, Yan and Liu [4] considered system (1.1) in the case $e_{ik}(t) = p_{ik}(t) = 0$ and $f_i(t,s) =$

$v_i(t,s) = 0$ for $k = 2, 3, \dots, m$, $i = 1, 2, \dots, n$,

$$\begin{aligned} \dot{x}_i(t) = g_i(x_i(t))\Big[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n}\sum_{k=1}^{m} b_{ijk}(t)x_j(t-\tau_{ijk}(t)) \\ - \sum_{j=1}^{n}\sum_{k=1}^{m}\int_{-\infty}^{t} c_{ijk}(t,s)x_j(s)ds - d_i(t)u_i(t) - e_{i1}(t)u_i(t-\sigma_{i1}(t))\Big], \\ \dot{u}_i(t) = -\alpha_i(t)u_i(t) + \beta_i(t)x_i(t) + p_{i1}(t)x_i(t-\gamma_{i1}(t)), \quad i = 1, 2, \dots, n. \end{aligned} \tag{1.4}$$

By employing fixed point index theory on cones, Yan and Liu [4] established the following result.

Theorem 1.1 ([4]). *Assume that* (H1)–(H4) *hold. For system* (1.4)*, to have at least one positive ω-periodic solution, a necessary and sufficient condition is*

$$\min_{1\le i\leqslant n}\Big\{\sum_{j=1}^{n}\Big[\bar{a}_{ij} + \sum_{k=1}^{m}(\bar{b}_{ijk} + \bar{c}^*_{ijk})\Big] + \bar{P}_i + \bar{Q}_i\Big\} > 0 \tag{1.5}$$

and

$$\min_{1\le i\le n}\{\bar{\beta}_i + \bar{p}_{i1}\} > 0. \tag{1.6}$$

Unfortunately, the sufficient condition in the above theorem is incorrect, as shown by the following example.

Example. Consider the system

$$\begin{aligned} \dot{x}_1(t) = x_1(t)\Big[1 - x_1(t) - 3x_2(t) - 2x_2(t-\tau_1) - u_1(t) - u_1(t-\sigma_1)\Big], \\ \dot{x}_2(t) = x_1(t)\Big[2 - 3x_1(t) - x_2(t) - 2x_1(t-\tau_2) - u_2(t) - u_2(t-\sigma_2)\Big], \\ \dot{u}_1(t) = -u_1(t) + x_1(t) + x_1(t-\gamma_1),\ \dot{u}_2(t) = -u_2(t) + x_2(t) + x_2(t-\gamma_2), \end{aligned} \tag{1.7}$$

where $\tau_i, \sigma_i, \gamma_i$, $i = 1, 2$, are positive constants. It is easy to see that system (1.7) satisfies all hypotheses of Theorem 1.1 with $\omega = 1$. On the other hand, if $(x_1^*(t), x_2^*(t), u_1^*(t), u_2^*(t))$ is a positive 1-periodic solution of system (1.7), then

$$\begin{aligned} \frac{d}{dt}\ln x_1^*(t) = \Big[1 - x_1^*(t) - 3x_2^*(t) - 2x_2^*(t-\tau_1) - u_1^*(t) - u_1^*(t-\sigma_1)\Big], \\ \frac{d}{dt}\ln x_2^*(t) = \Big[2 - 3x_1^*(t) - x_2^*(t) - 2x_1^*(t-\tau_2) - u_1^*(t) - u_2^*(t-\sigma_2)\Big], \\ \frac{d}{dt}u_1^*(t) = -u_1^*(t) + x_1^*(t) + x_1^*(t-\gamma_1), \\ \frac{d}{dt}u_2^*(t) = -u_2^*(t) + x_2^*(t) + x_2^*(t-\gamma_2), \end{aligned} \tag{1.8}$$

Integrating (1.8) from 0 to 1 and simplifying, we obtain

$$\bar{x}_1^* + 5\bar{x}_2^* + 2\bar{u}_1^* = 1, \quad 5\bar{x}_1^* + \bar{x}_2^* + 2\bar{u}_2^* = 2, \quad \bar{u}_1^* = 2\bar{x}_1^*, \quad \bar{u}_2^* = 2x_2^*. \tag{1.9}$$

This implies

$$5\bar{x}_1^* + 5\bar{x}_2^* = 1, \quad 5\bar{x}_1^* + 5\bar{x}_2^* = 2,$$

which is impossible. Thus, system (1.7) has no positive 1-periodic solution and then the sufficient condition in Theorem 1.1 (given in [4]) is incorrect.

In the proof of Theorem 1.1 (see [4]), authors considered a map $\Phi : K \to K$, where

$$K = \{(x_1, \dots, x_n) \in E : x_i(t) \geqslant \delta_i\|x_i\|_0,\ i = 1, 2, \dots, n,\ t \in [0, \omega]\}$$

(with $\delta_i = \exp\{-\bar{r}_i\omega\}$) is a cone of the Banach space $E = \{x \in C(\mathbb{R}, \mathbb{R}^n) : x(t+\omega) = x(t)$ for all $t \in \mathbb{R}\}$ with the norm $\|x\|_0 = \sum_{i=1}^n \|x_i\|_0$ (where $\|x_i\|_0 = \max_{t\in[0,\omega]} |x_i(t)|$). By employing fixed point index theory on cone, it was proved in [4] that there exist positive constants r and R $(r < R)$ such that Φ has at least one fixed point x^* in $K_{r,R} = \{x \in K : r < \|x\|_0 \le R\}$, and then it was concluded that $(x^*(t), u^*(t))$ is positive ω-periodic solution of system (1.1). The mistake in the proof of Theorem 1.1 (see [4]) is that $x^* \in K_{r,R}$ does not imply that $x^*(t)$ is positive for $n \ge 2$.

Our purpose of this paper is by using the technique of coincidence degree theory developed by Gains and Mawhin in [2] to study the existence of positive periodic solutions of system (1.1). Moreover, by contracting a suitable Liapunov functional, we study the global asymptotic stability of the positive periodic solution of system (1.1). The remainder of this paper is organized as follows. Section 2 is preliminaries, in which we introduce the continuation theorem and some lemmas. Section 3 contains our main results on the existence and the global asymptotic stability of positive periodic solutions of system (1.1).

2. Preliminaries

Let Y and Z be two normed vector spaces, $L : \operatorname{Dom} L \subset Y \to Z$ be a linear mapping, and $N : Y \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : Y \to Y$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$, it follows that $L|_{\operatorname{Dom} L\cap\ker P} : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of Y, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to Y$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Lemma 2.1 (Continuation theorem [2]). *Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose that*

(i) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(ii) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \ker L$ and $\deg(JQN, \Omega \cap \ker L, 0) \neq 0$.*

Then the operator equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

Let us consider the system

$$
\begin{aligned}
\dot{x}_i(t) = g_i(x_i(t))\Big[r_i(t) &- \sum_{j=1}^n a_{ij}(t)x_j(t) - \sum_{j=1}^n\sum_{k=1}^m b_{ijk}(t)x_j(t-\tau_{ijk}(t)) \\
&- \sum_{j=1}^n\sum_{k=1}^m \int_{-\infty}^t c_{ijk}(t,s)x_j(s)ds - d_i(t)(V_ix_i)(t) \\
&- \sum_{k=1}^m e_{ik}(t)(V_ix_i)(t-\sigma_{ik}(t)) - \int_{-\infty}^t f_i(t,s)(V_ix_i)(s)ds\Big], \\
u_i(t) &= (V_ix_i)(t), \quad i = 1, 2, \dots, n,
\end{aligned} \tag{2.1}
$$

where

$$
\begin{aligned}
(V_i x_i)(t) = \int_t^{t+\omega} G_i(t,s)\Big[\beta_i(s)x_i(s) + \sum_{k=1}^m p_{ik}(s)x_i(s-\gamma_{ik}(s)) \\
+ \int_{-\infty}^s v_i(s,\tau)x_i(\tau)d\tau\Big]ds, \quad i=1,2,\dots,n.
\end{aligned} \tag{2.2}
$$

Lemma 2.2. *Assume* (H1)–(H4) *hold. Then* $(x_1(t),\dots,x_n(t),u_1(t),\dots,u_n(t))$ *is an ω-periodic solution of system* (1.1) *if and only if it is an ω-periodic solution of system* (2.1).

Proof. ($\Rightarrow$) Let $(x(t),u(t)) = (x_1(t),\dots,x_n(t),u_1(t),\dots,u_n(t))$ be an ω-periodic solution of system (1.1). It is easy to see from (1.1) that

$$
\begin{aligned}
u_i(\bar t) = \exp\{-\int_t^{\bar t}\alpha_i(\tau)d\tau\}\Big[u_i(t) + \int_t^{\bar t}\exp\{\int_t^\tau \alpha_i(\tau)d\tau\}\Big\{\beta_i(\tau)x_i(\tau) \\
+ \sum_{k=1}^m p_{ik}(\tau)x_i(t-\gamma_{ik}(\tau)) + \int_{-\infty}^\tau v_i(\tau,v)x_i(v)dv\Big\}d\tau\Big],
\end{aligned}
$$

for $\bar t \geqslant t$, $i=1,2,\dots,n$. Thus, since $u_i(t)=u_i(t+\omega)$ for $i=1,2,\dots,n$, it follows that

$$
\begin{aligned}
u_i(t) &= u_i(t+\omega) \\
&= \exp\{-\int_t^{t+\omega}\alpha_i(\tau)d\tau\}\Big[u_i(t) + \int_t^{t+\omega}\exp\{\int_t^\tau \alpha_i(\tau)d\tau\}\Big\{\beta_i(\tau)x_i(\tau) \\
&\quad + \sum_{k=1}^m p_{ik}(\tau)x_i(t-\gamma_{ik}(\tau)) + \int_{-\infty}^\tau v_i(\tau,v)x_i(v)dv\Big\}d\tau\Big],
\end{aligned}
$$

for $t\in\mathbb{R}$, $i=1,2,\dots,n$. Hence,

$$
\begin{aligned}
u_i(t) = \int_t^{t+\omega} G_i(t,\tau)\Big[\beta_i(\tau)x_i(\tau) + \sum_{k=1}^m p_{ik}(\tau)x_i(\tau-\gamma_{ik}(\tau)) \\
+ \int_{-\infty}^s v_i(\tau,v)x_i(v)dv\Big]d\tau = (V_i x_i)(t), \quad i=1,2,\dots,n,
\end{aligned}
$$

which implies that $(x(t),u(t))$ is an ω-periodic solution of system (2.1).

($\Leftarrow$) Let $(x_1(t),\dots,x_n(t),u_1(t),\dots,u_n(t))$ be an ω-periodic solution of system (2.1). It is easy to see from (2.2) that $(u_1(t),\dots,u_n(t))$ satisfies the system

$$
\dot u_i(t) = -\alpha_i(t)u_i(t) + \beta_i(t)x_i(t) + \sum_{k=1}^m p_{ik}(t)x_i(t-\gamma_{ik}(t)) + \int_{-\infty}^t v_i(t,s)x_i(s)ds,
$$

for $i=1,2,\dots,n$. Thus, $(x(t),u(t))$ is an ω-periodic solution of system (1.1). The proof is complete. ☐

Lemma 2.3 ([3]). *Suppose that $\nu\in C^1(\mathbb{R},\mathbb{R}_+)$ is ω-periodic and $\nu'(t)<1$ for all $t\in[0,\omega]$. Then the function $t-\nu(t)$ has a unique inverse $\eta(t)$ satisfying $\eta\in C(\mathbb{R},\mathbb{R})$ with $\eta(t+\omega)=\eta(t)$ for all $t\in\mathbb{R}$.*

Lemma 2.4. *Suppose that $q_{ij}>0,\delta_i>0$ for $i,j=1,2,\dots,n$. If*

$$
\delta_i - \sum_{j\neq i}^n q_{ij}\frac{\delta_j}{q_{jj}} > 0, i=1,2,\dots,n, \tag{2.3}
$$

then the system of algebraic equations

$$\delta_i - \sum_{j=1}^{n} q_{ij} x_j = 0, \quad i = 1, 2, \dots, n \tag{2.4}$$

has a unique solution $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$. *Moreover,* $x_i^* > 0$ *for* $i = 1, 2, \dots, n$.

Proof. Let $y_i = q_{ii} x_i$ for $i = 1, 2, \dots, n$, so that system (2.4) becomes

$$\delta_i - \sum_{j=1}^{n} q_{ij}^* y_j = 0, i = 1, 2, \dots, n, \tag{2.5}$$

where $q_{ij}^* = q_{ij}/q_{jj}$, $i, j = 1, 2, \dots, n$. Clearly, $q_{ii}^* = 1$ for $i = 1, 2, \dots, n$. By (2.3),

$$\delta_i - \sum_{j \neq i}^{n} q_{ij}^* \delta_j > 0, \quad i = 1, 2, \dots, n. \tag{2.6}$$

Let ϵ be a positive number such that $\epsilon < \min_{1 \le i \le n} \big[\delta_i - \sum_{j \neq i}^{n} q_{ij}^* \delta_j\big]$. Denote

$$\mathcal{D} = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : \epsilon \le y_i \le \delta_i, i = 1, 2, \dots, n\},$$

$F = (F_1, \dots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$, where

$$F_i(y) = \delta_i - \sum_{j=1}^{n} q_{ij}^* y_j, \quad i = 1, 2, \dots, n;$$

and $H = F + I$, where I is the identity operator on $\mathbb{R}^n$. It is easy to see that $\epsilon < H_i(y) < \delta_i$ for $i = 1, 2, \dots, n$, $y \in \mathcal{D}$; i.e., $H(y) \in \operatorname{int}(\mathcal{D})$-the interior of $\mathcal{D}$ for all $y \in \mathcal{D}$. Thus, $\deg(H - I, \operatorname{int}\mathcal{D}, 0) = \deg(F, \operatorname{int}\mathcal{D}, 0) = 1$. This implies that the equation $Fy = 0$ has at least one solution y^* in $\operatorname{int}\mathcal{D}$. Since $Fy \neq 0$ for all $y \in \partial\mathcal{D}$ and $Fy = 0$ is linear equation, it follows that y^* is the unique solution in $\mathbb{R}^n$ of the equation $Fy = 0$. Thus, equation (2.4) has a unique solution $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$. Moreover, $x_i^* > 0$ for $i = 1, 2, \dots, n$. The proof is complete. □

Definition 2.5. Let $(\tilde{x}(t), \tilde{u}(t)) = (\tilde{x}_1(t), \dots, \tilde{x}_n(t), \tilde{u}_1(t), \dots, \tilde{u}_n(t))$ be a positive ω-periodic solution of system (1.1). It is said to be globally asymptotically stable if any positive solution $(x(t), u(t)) = (x_1(t), \dots, x_n(t), u_1(t), \dots, u_n(t))$ of (1.1)-(1.2) satisfies

$$\lim_{t \to +\infty} \sum_{i=1}^{n} \Big\{|x_i(t) - \tilde{x}_i(t)| + |u_i(t) - \tilde{u}_i(t)|\Big\} = 0.$$

Remark 2.6. Let us put $y_i = h_i(x_i) := \int_1^{x_i} \frac{ds}{g_i(s)}$, $i = 1, 2, \dots, n$. By (H3), it is easy to see that $h_i : (0, +\infty) \to \mathbb{R}$, $x_i \mapsto y_i = h_i(x_i)$ has a unique inverse $\varphi_i : \mathbb{R} \to (0, +\infty)$, $y_i \mapsto x_i = \varphi_i(y_i)$. Moreover, $\varphi_i \in C^1(\mathbb{R}, (0, +\infty))$ and φ_i is strictly monotone increasing.

Remark 2.7. By (H5), Lemma 2.3 implies that the functions $t - \tau_{ijk}(t), t - \sigma_{ik}(t)$ and $t - \gamma_{ik}(t)$ have the unique inverses, respectively. Let $\mu_{ijk}(t)$, $\zeta_{ik}(t)$ and $\xi_{ik}(t)$ represent the inverses of functions $t - \tau_{ijk}(t), t - \sigma_{ik}(t)$ and $t - \gamma_{ik}(t)$, respectively. Obviously, $\mu_{ijk}, \zeta_{ik}, \xi_{ik} \in C(\mathbb{R}, \mathbb{R})$ and $\mu_{ijk}(t + \omega) = \mu_{ijk}(t), \zeta_{ik}(t + \omega) = \zeta_{ik}(t), \xi_{ik}(t + \omega) = \xi_{ik}(t)$ for all $t \in \mathbb{R}$.

Remark 2.8. It is easy from (H1)–(H4) to show that solutions of (1.1)-(1.2) are well defined for all $t \ge 0$ and satisfy $x_i(t) > 0$ and $u_i(t) > 0$ for all $t \ge 0$ and $i = 1, 2, \dots, n$.

3. Main Results

Theorem 3.1. *Assume that* (H1)–(H4) *hold. Let*

$$\int_0^{\omega}\Big[\beta_i(s)+\sum_{k=1}^{m}p_{ik}(s)+v_i^*(s)\Big]ds>0,\quad i=1,2,\dots,n, \tag{3.1}$$

$$A_i:=\bar{a}_{ii}+\sum_{k=1}^{m}\bar{b}_{iik}+\sum_{k=1}^{m}\bar{c}_{iik}^*+\bar{P}_i+\bar{Q}_i+\bar{R}_i>0,\quad i=1,2,\dots,n, \tag{3.2}$$

$$\bar{r}_i>\sum_{j\neq i}^{n}\Big(\bar{a}_{ij}+\sum_{k=1}^{m}\bar{b}_{ijk}+\sum_{k=1}^{m}\bar{c}_{ijk}^*+\bar{P}_j+\bar{Q}_j+\bar{R}_j\Big)\varphi_j(B_j),\quad i=1,2,\dots,n, \tag{3.3}$$

where $B_i=h_i(\bar{r}_i/A_i)+(\bar{r}_i+\overline{|r_i|})\omega$, $i=1,2,\dots,n$. *Then system* (1.1) *has at least one positive* ω*-periodic solution.*

Proof. Consider the system

$$\begin{aligned}\dot{y}_i(t)&=r_i(t)-\sum_{j=1}^{n}a_{ij}(t)\varphi_j(y_j(t))-\sum_{j=1}^{n}\sum_{k=1}^{m}b_{ijk}(t)\varphi_j(y_j(t-\tau_{ijk}(t)))\\&\quad-\sum_{j=1}^{n}\sum_{k=1}^{m}\int_{-\infty}^{t}c_{ijk}(t,s)\varphi_j(y_j(s))ds-d_i(t)(V_i\varphi_i(y_i))(t)\\&\quad-\sum_{k=1}^{m}e_{ik}(t)(V_i\varphi_i(y_i))(t-\sigma_{ik}(t))-\int_{-\infty}^{t}f_i(t,s)(V_i\varphi_i(y_i))(s)ds.\end{aligned} \tag{3.4}$$

By (1.3), (2.2) and (3.1), if system (3.4) has an ω-periodic solution $(y_1^*(t),\dots,y_n^*(t))$, then

$$\begin{aligned}u_i^*(t)&=(V_i\varphi_i(y_i^*))(t)\\&\geqslant\int_t^{t+\omega}\Big(\frac{1}{\exp\{\bar{\alpha}_i\omega\}-1}\min_{\tau\in[0,\omega]}\varphi_i(y_i^*(\tau))\Big)\Big[\beta_i(s)\\&\quad+\sum_{k=1}^{m}p_{ik}(s)+v_i^*(s)\Big]ds>0,\quad t\in\mathbb{R},\ i=1,2,\dots,n.\end{aligned}$$

Thus, $(x_1^*(t),\dots,x_n^*(t),u_1^*(t),\dots,u_n^*(t))$ with values $x_i^*(t)=\varphi_i(y_i^*(t))$ and $u_i^*(t)=(V_i\varphi_i(y_i^*))(t)$ for $i=1,2,\dots,n$ is a positive ω-periodic solution of system (2.1). So, by Lemma 2.2, we only need to show that system (3.4) has at least one ω-periodic solution in order to complete the proof. To apply the continuation theorem of coincidence degree theory to the existence of an ω-periodic solution of system (3.4), we take

$$Y=Z=\Big\{y(t)=(y_1(t),\dots,y_n(t))\in C(\mathbb{R},\mathbb{R}^n):y(t+\omega)=y(t)\text{ for all }t\in\mathbb{R}\Big\}.$$

Denote $\|y\|_0=\sum_{i=1}^{n}\|y_i\|_0$, where $\|y_i\|_0=\max_{t\in[0,\omega]}|y_i(t)|$. Then Y and Z are Banach spaces when they endowed with the norm $\|\cdot\|_0$.

We define $L : \operatorname{Dom} L \subset Y \to Z$ and $N : Y \to Z$ by setting $Ly = \dot{y}$ and $Ny = Fy = (F_1y, \dots, F_ny)$, where

$$\begin{aligned} F_iy(t) =& r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\varphi_j(y_j(t)) - \sum_{j=1}^{n}\sum_{k=1}^{m} b_{ijk}(t)\varphi_j(y_j(t-\tau_{ijk}(t))) \\ & - \sum_{j=1}^{n}\sum_{k=1}^{m}\int_{-\infty}^{t} c_{ijk}(t,s)\varphi_j(y_j(s))ds - d_i(t)(V_i\varphi_i(y_i))(t) \\ & - \sum_{k=1}^{m} e_{ik}(t)(V_i\varphi_i(y_i))(t-\sigma_{ik}(t)) - \int_{-\infty}^{t} f_i(t,s)(V_i\varphi_i(y_i))(s)ds. \end{aligned} \tag{3.5}$$

Further, we define continuous projectors $P : Y \to Y$ and $Q : Z \to Z$ as follows

$$Py = \frac{1}{\omega}\int_0^{\omega} y(s)ds, \quad Qz = \frac{1}{\omega}\int_0^{\omega} z(s)ds.$$

We easily see that $\operatorname{Im} L = \{z \in Z : \int_0^{\omega} z(s)ds = 0\}$ and $\ker L = \mathbb{R}^n$. So, $\operatorname{Im} L$ is closed in Z and $\dim\ker L = n = \operatorname{codim}\operatorname{Im} L$. Hence, L is a Fredholm mapping of index zero. Clearly that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I-Q)/$ Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \ker P \cap \operatorname{Dom} L$ has the form

$$K_Pz(t) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\,ds\,dt.$$

We know that

$$QNy(t) = \frac{1}{\omega}\int_0^{\omega} Fy(t)dt.$$

Thus,

$$\begin{aligned} K_P(I-Q)Ny(t) &= (K_PN - K_PQN)y(t) \\ &= \int_0^t Fy(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t Fy(s)\,ds\,dt + \Big(\frac{1}{2} - \frac{t}{\omega}\Big)\int_0^{\omega} Fy(s)ds. \end{aligned}$$

It is easy to see that QN and $K_p(I-Q)N$ are continuous. Furthermore, it can be verified that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset Y$ by using Arzela-Ascoli theorem and $QN(\bar{\Omega})$ is bounded. Therefore, N is L-compact on $\bar{\Omega}$ for any open bounded subset $\Omega \subset Y$. Now we are in a position to search for an appropriate open bounded subset Ω for the application of the continuation theorem (Lemma 2.1) to system (3.4).

Corresponding to the operator equation $Ly = \lambda Ny$ $(\lambda \in (0,1))$, we have

$$\begin{aligned} \dot{y}_i(t) = \lambda\Big[& r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\varphi_j(y_j(t)) - \sum_{j=1}^{n}\sum_{k=1}^{m} b_{ijk}(t)\varphi_j(y_j(t-\tau_{ijk}(t))) \\ & - \sum_{j=1}^{n}\sum_{k=1}^{m}\int_{-\infty}^{t} c_{ijk}(t,s)\varphi_j(y_j(s))ds - d_i(t)(V_i\varphi_i(y_i))(t) \\ & - \sum_{k=1}^{m} e_{ik}(t)(V_i\varphi_i(y_i))(t-\sigma_{ik}(t)) - \int_{-\infty}^{t} f_i(t,s)(V_i\varphi_i(y_i))(s)ds\Big]. \end{aligned} \tag{3.6}$$

Integrating (3.6) from 0 to ω and simplifying, we obtain

$$
\begin{aligned}
\bar{r}_i\omega = &\sum_{j=1}^{n}\int_0^\omega a_{ij}(t)\varphi_j(y_j(t))dt + \sum_{j=1}^{n}\sum_{k=1}^{m}\int_0^\omega b_{ijk}(t)\varphi_j(y_j(t-\tau_{ijk}(t)))dt \\
&+ \sum_{j=1}^{n}\sum_{k=1}^{m}\int_0^\omega\int_{-\infty}^{t} c_{ijk}(t,s)\varphi_j(y_j(s))\,ds\,dt + \int_0^\omega d_i(t)(V_i\varphi_i(y_i))(t)dt \\
&+ \sum_{k=1}^{m}\int_0^\omega e_{ik}(t)(V_i\varphi_i(y_i))(t-\sigma_{ik}(t))dt \\
&+ \int_0^\omega\int_{-\infty}^{t} f_i(t,s)(V_i\varphi_i(y_i))(s)\,ds\,dt.
\end{aligned}
\tag{3.7}
$$

Let

$$
y_i(\eta_i) = \max_{t\in[0,\omega]} y_i(t), y_i(\theta_i) = \min_{t\in[0,\omega]} y_i(t), \quad \eta_i,\theta_i\in[0,\omega],\ i=1,2,\dots,n. \tag{3.8}
$$

It is easy to see from (1.3), (2.2) and (3.8) that

$$
\begin{gathered}
\int_0^\omega d_i(t)(V_i\varphi_i(y_i))(t)dt \ge \bar{P}_i\omega\varphi_i(y_i(\theta_i)), \\
\sum_{k=1}^{m}\int_0^\omega e_{ik}(t)(V_i\varphi_i(y_i))(t-\sigma_{ik}(t))dt \ge \bar{Q}_i\omega\varphi_i(y_i(\theta_i)), \\
\int_0^\omega\int_{-\infty}^{t} f_i(t,s)(V_i\varphi_i(y_i))(s)\,ds\,dt \ge \bar{R}_i\omega\varphi_i(y_i(\theta_i))
\end{gathered}
\tag{3.9}
$$

and

$$
\begin{gathered}
\int_0^\omega d_i(t)(V_i\varphi_i(y_i))(t)dt \le \bar{P}_i\omega\varphi_i(y_i(\eta_i)), \\
\sum_{k=1}^{m}\int_0^\omega e_{ik}(t)(V_i\varphi_i(y_i))(t-\sigma_{ik}(t))dt \le \bar{Q}_i\omega\varphi_i(y_i(\eta_i)), \\
\int_0^\omega\int_{-\infty}^{t} f_i(t,s)(V_i\varphi_i(y_i))(s)\,ds\,dt \le \bar{R}_i\omega\varphi_i(y_i(\eta_i)).
\end{gathered}
\tag{3.10}
$$

It follows from (1.3), (3.7), (3.8) and (3.9) that

$$
\bar{r}_i \ge \Big(\bar{a}_{ii} + \sum_{k=1}^{m}\bar{b}_{iik} + \sum_{k=1}^{m}\bar{c}^*_{iik} + \bar{P}_i + \bar{Q}_i + \bar{R}_i\Big)\varphi_i(y_i(\theta_i)) = A_i\varphi_i(y_i(\theta_i)), \tag{3.11}
$$

for $i=1,2,\dots,n$. Thus, by (3.2) and Remark 2.6, we have

$$
y_i(\theta_i) \le h_i(\frac{\bar{r}_i}{A_i}), \quad i=1,2,\dots,n. \tag{3.12}
$$

From (3.6) and (3.7), we know that

$$
\int_0^\omega |\dot{y}_i(t)|dt \le (\bar{r}_i + \overline{|r_i|})\omega, \quad i=1,2,\dots,n, \tag{3.13}
$$

and thus, by (3.12),

$$
y_i(t) \le y_i(\theta_i) + \int_0^\omega |\dot{y}_i(t)|dt \le h_i(\frac{\bar{r}_i}{A_i}) + (\bar{r}_i + \overline{|r_i|})\omega = B_i, \quad t\in[0,\omega], \tag{3.14}
$$

for $i = 1, 2, \dots, n$. It is easy to see from (3.7), (3.8), (3.10) and (3.14) that

$$\begin{aligned} &\Big(\bar{a}_{ii} + \sum_{k=1}^{m} \bar{b}_{iik} + \sum_{k=1}^{m} \bar{c}^*_{iik} + \bar{P}_i + \bar{Q}_i + \bar{R}_i\Big)\varphi_i(y_i(\eta_i)) \\ &\geqslant \bar{r}_i - \sum_{j\neq i}^{n} \Big(\bar{a}_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}^*_{ijk} + \bar{P}_j + \bar{Q}_j + \bar{R}_j\Big)\varphi_j(y_j(\eta_j)), \end{aligned} \tag{3.15}$$

for $i = 1, 2, \dots, n$. Thus, by (3.3) and (3.14), we have

$$\varphi_i(y_i(\eta_i)) \geq \frac{\bar{r}_i - \sum_{j\neq i}^{n} \Big(\bar{a}_{ij} + \sum_{k=1}^{m} [\bar{b}_{ijk} + \bar{c}^*_{ijk}] + \bar{P}_j + \bar{Q}_j + \bar{R}_j\Big)\varphi_j(B_j)}{\bar{a}_{ii} + \sum_{k=1}^{m} [\bar{b}_{iik} + \bar{c}^*_{iik}] + \bar{P}_i + \bar{Q}_i + \bar{R}_i} =: C_i,$$

for $i = 1, 2, \dots, n$; or

$$y_i(\eta_i) \geq h_i(C_i), \quad i = 1, 2, \dots, n, \tag{3.16}$$

From (3.13) and (3.16), it follows that

$$y(t) \geq y_i(\eta_i) - \int_0^{\omega} |\dot{y}_i(t)| dt \geq h_i(C_i) - (\bar{r}_i + \overline{|r_i|})\omega =: D_i, t \in [0, \omega], \tag{3.17}$$

for $i = 1, 2, \dots, n$. From (3.12) and (3.16) we see that

$$\|y\|_0 \leq M := \sum_{i=1}^{n} (|B_i| + |D_i|). \tag{3.18}$$

By (3.3), Lemma 2.3 implies that the following system of algebraic equations

$$\bar{r}_i = \sum_{j=1}^{n} \Big[\bar{a}_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}^*_{ijk}\Big]\varphi_j(y_j) + (\bar{P}_i + \bar{Q}_i + \bar{R}_i)\varphi_i(y_i), \tag{3.19}$$

for $i = 1, 2, \dots, n$, has a unique solution $y^* = (y_1^*, \dots, y_n^*) \in \mathbb{R}^n$. Let $S = \|y^*\|_0 + M$. Evidently, S is independent of the choice of λ. Let $\Omega := \{y \in Y : \|y\|_1 < S\}$. It is clear that Ω satisfies the requirement (i) in Lemma 2.1. Moreover, $QNy \neq 0$ for any $y \in \partial\Omega \cap \ker L$. Let us take $J = I$, where I is the identity operator on $\mathbb{R}^n$. By straightforward computation, we have

$$\deg(JQN, \Omega \cap \ker L, 0) = \operatorname{sgn}\Big\{(-1)^n \Big[\det(w_{ij})\Big]\varphi_1(y_1^*) \dots \varphi_n(y_n^*)\Big\} \neq 0,$$

where $w_{ii} = A_i$,

$$w_{ij} = \bar{a}_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}^*_{ijk} \ (j \neq i), \quad i, j = 1, 2, \dots, n.$$

By Lemma 2.1, we conclude that the equation $Ly = Ny$ has at least one solution in Y. Therefore, system (1.1) has at least one positive ω-periodic solution. The proof is complete. □

Theorem 3.2. *Assume that* (H1)–(H5), (3.1) *and* (3.3) *hold. If there exist positive constants* $\nu_1, \nu_2, \dots, \nu_n$ *such that*

$$
\begin{aligned}
&\min_{t\in[0,\omega]} \Big\{\nu_i\Big[a_{ii}(t) - \beta_i(t) - \sum_{k=1}^{m} \frac{p_{ik}(\xi_{ik}(t))}{1-\dot{\gamma}_{ik}(\xi_{ik}(t))} - \int_0^{+\infty} v_i(t+\tau,t)d\tau\Big] \\
&- \sum_{j\neq i}^{n} \nu_j a_{ji}(t) - \sum_{j=1}^{n} \nu_j \sum_{k=1}^{m} \frac{b_{jik}(\mu_{jik}(t))}{1-\dot{\tau}_{jik}(\mu_{jik}(t))} \\
&- \sum_{j=1}^{n} \nu_j \sum_{k=1}^{m} \int_0^{+\infty} c_{jik}(t+\tau,t)d\tau\Big\} > 0, \\
&\min_{t\in[0,\omega]} \Big\{\alpha_i(t) - d_i(t) - \sum_{k=1}^{m} \frac{e_{ik}(\zeta_{ik}(t))}{1-\dot{\sigma}_{ik}(\zeta_{ik}(t))} - \int_0^{+\infty} f_i(t+\tau,t)d\tau\Big\} > 0,
\end{aligned} \tag{3.20}
$$

for $i = 1, 2, \dots, n$, *then system* (1.1) *has a unique positive* ω*-periodic solution which is globally asymptotically stable.*

Proof. From (3.20), we conclude that $\bar{a}_{ii} > 0$ for $i = 1, 2, \dots, n$, which means that (3.2) holds. By Theorem 3.1, system (1.1) has a positive ω-periodic solution $(\tilde{x}(t), \tilde{u}(t))$. Let $(x(t), u(t))$ be any other positive solution of (1.1) with initial condition (1.2). Consider the Liapunov functional $V(t) = V(\tilde{x}(t), \tilde{u}(t)), (x(t), u(t)))$ defined by

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{n} \nu_i \Big\{V_i^{(1)}(t) + V_i^{(2)}(t) + \sum_{j=1}^{n}\sum_{k=1}^{m} \Big[V_{ijk}^{(3)}(t) + V_{ijk}^{(4)}(t)\Big] \\
&\quad + \sum_{k=1}^{m} \Big[V_{ik}^{(5)}(t) + V_{ik}^{(6)}(t)\Big] + V_i^{(7)}(t) + V_i^{(8)}(t)\Big\},
\end{aligned} \tag{3.21}
$$

where, for $i, j = 1, 2, \dots, n$, $k = 1, 2, \dots, m$, we have

$$
\begin{aligned}
V_i^{(1)}(t) &= \Big|\int_{\tilde{x}_i(t)}^{x_i(t)} \frac{ds}{g_i(s)}\Big|, \quad V_i^{(2)}(t) = |u_i(t) - \tilde{u}_i(t)|, \\
V_{ijk}^{(3)}(t) &= \int_{t-\tau_{ijk}(t)}^{t} \frac{b_{ijk}(\mu_{ijk}(s))}{1-\dot{\tau}_{ijk}(\mu_{ijk}(s))} |x_j(s) - \tilde{x}_j(s)|\, ds, \\
V_{ijk}^{(4)}(t) &= \int_0^{+\infty}\int_{t-\tau}^{t} c_{ijk}(s+\tau,s)|x_j(s) - \tilde{x}_j(s)|\, ds\, d\tau, \\
V_{ik}^{(5)}(t) &= \int_{t-\sigma_{ik}(s)}^{t} \frac{e_{ik}(\zeta_{ik}(s))}{1-\dot{\sigma}_{ik}(\zeta_{ik}(s))} |u_i(s) - \tilde{u}_i(s)|\, ds, \\
V_{ik}^{(6)}(t) &= \int_{t-\gamma_{ik}(t)}^{t} \frac{p_{ik}(\xi_{ik}(s))}{1-\dot{\gamma}_{ik}(\xi_{ik}(s))} |x_i(s) - \tilde{x}_i(s)|\, ds, \\
V_i^{(7)}(t) &= \int_0^{+\infty}\int_{t-\tau}^{t} f_i(s+\tau,s)|u_i(s) - \tilde{u}_i(s)|\, ds\, d\tau, \\
V_i^{(8)}(t) &= \int_0^{+\infty}\int_{t-\tau}^{t} v_i(s+\tau,s)|u_i(s) - \tilde{u}_i(s)|\, ds\, d\tau
\end{aligned}
$$

Clearly $V(t)$ is continuous on $[0,+\infty)$. Calculating the upper right derivative of $V_i^{(1)}(t),\dots,V_i^{(8)}(t)$ along the solutions of system (1.1) for $t>0$, we obtain

$$\begin{aligned}
&D^+V_i^{(1)}\\
&=\Big[\frac{\dot{x}_i(t)}{g_i(x_i(t))}-\frac{\dot{\tilde{x}}_i(t)}{g_i(\tilde{x}_i(t))}\Big]\operatorname{sgn}[x_i(t)-\tilde{x}_i(t)]\\
&=\operatorname{sgn}[x_i(t)-\tilde{x}_i(t)]\Big[-\sum_{j=1}^n a_{ij}(x_j(t)-\tilde{x}_j(t))\\
&\quad-\sum_{j=1}^n\sum_{k=1}^m b_{ijk}(t)(x_j(t-\tau_{ijk}(t))-\tilde{x}_j(t-\tau_{ijk}(t)))\\
&\quad-\sum_{j=1}^n\sum_{k=1}^m\int_{-\infty}^t c_{ijk}(t,\tau)(x_j(\tau)-\tilde{x}_j(t))d\tau-d_i(t)(u_i(t)-\tilde{u}_i(t))\\
&\quad-\sum_{k=1}^m e_{ik}(t)(u_i(t-\sigma_{ik}(t))-\tilde{u}_i(t-\sigma_{ik}(t)))-\int_{-\infty}^t f_i(t,\tau)(u_i(\tau)-\tilde{u}_i(\tau))d\tau\Big];
\end{aligned}$$

and thus,

$$\begin{aligned}
D^+V_i^{(1)}&\leqslant -a_{ii}(t)|x_i(t)-\tilde{x}_i(t)|+\sum_{j\neq i}^n a_{ij}(t)|x_j(t)-\tilde{x}_j(t)|\\
&\quad+\sum_{j=1}^n\sum_{k=1}^m b_{ijk}(t)|x_j(t-\tau_{ijk}(t))-\tilde{x}_j(t-\tau_{ijk}(t))|\\
&\quad+\sum_{j=1}^n\sum_{k=1}^m\int_{-\infty}^t c_{ijk}(t,\tau)|x_j(\tau)-\tilde{x}_j(\tau)|d\tau-d_i(t)|u_i(t)-\tilde{u}_i(t)|\\
&\quad+\sum_{k=1}^m e_{ik}(t)|u_i(t-\sigma_{ik}(t))-\tilde{u}_i(t-\sigma_{ik}(t))|\\
&\quad+\int_{-\infty}^t f_i(t,\tau)|u_i(\tau)-\tilde{u}_i(\tau)|d\tau,
\end{aligned}\tag{3.22}$$

$$\begin{aligned}
D^+V_i^{(2)}&=[\dot{u}_i(t)-\dot{\tilde{u}}_i(t)]\operatorname{sgn}[u_i(t)-\tilde{u}_i(t)]\\
&=\operatorname{sgn}[u_i(t)-\tilde{u}_i(t)]\Big[-\alpha_i(t)(u_i(t)-\tilde{u}_i(t))+\beta_i(t)(x_i(t)-\tilde{x}_i(t))\\
&\quad+\sum_{k=1}^m p_{ik}(t)(x_i(t-\gamma_{ik}(t))-\tilde{x}_i(t-\gamma_{ik}(t)))\\
&\quad+\int_{-\infty}^t v_i(t,\tau)(x_i(\tau)-\tilde{x}_i(\tau))d\tau\Big]\\
&\le -\alpha_i(t)|u_i(t)-\tilde{u}_i(t)|+\beta_i(t)|x_i(t)-\tilde{x}_i(t)|\\
&\quad+\sum_{k=1}^m p_{ik}(t)|x_i(t-\gamma_{ik}(t))-\tilde{x}_i(t-\gamma_{ik}(t))|\\
&\quad+\int_{-\infty}^t v_i(t,\tau)|x_i(\tau)-\tilde{x}_i(\tau)|d\tau;
\end{aligned}\tag{3.23}$$

$$\begin{aligned}\dot V_{ijk}^{(3)} &= \frac{b_{ijk}(\mu_{ijk}(t))}{1-\dot\tau_{ijk}(\mu_{ijk}(t))}|x_j(t)-\tilde x_j(t)| \\ &\quad - b_{ijk}(t)|x_j(t-\tau_{ijk}(t))-\tilde x_j(t-\tau_{ijk}(t))|,\end{aligned} \tag{3.24}$$

$$\begin{aligned}\dot V_{ijk}^{(4)} &= \int_0^{+\infty} c_{ijk}(t+\tau,t)|x_j(t)-\tilde x_j(t)|d\tau \\ &\quad - \int_0^{+\infty} c_{ijk}(t,t-\tau)|x_j(t-\tau)-\tilde x_j(t-\tau)|d\tau;\end{aligned} \tag{3.25}$$

$$\dot V_{ik}^{(5)} = \frac{e_{ik}(\zeta_{ik}(t))}{1-\dot\sigma_{ik}(\zeta_{ik}(t))}|u_i(t)-\tilde u_i(t)| - e_{ik}(t)|u_i(t-\sigma_{ik}(t))-\tilde u_i(t-\sigma_{ik}(t))|; \tag{3.26}$$

$$\dot V_{ik}^{(6)} = \frac{p_{ik}(\xi_{ik}(t))}{1-\dot\gamma_{ik}(\xi_{ik}(t))}|x_i(t)-\tilde x_i(t)| - p_{ik}(t)|x_i(t-\gamma_{ik}(t))-\tilde x_i(t-\gamma_{ik}(t))|; \tag{3.27}$$

$$\begin{aligned}\dot V_{i}^{(7)} &= \int_0^{+\infty} f_i(t+\tau,t)|u_i(t)-\tilde u_i(t)|d\tau \\ &\quad - \int_0^{+\infty} f_i(t,t-\tau)|u_i(t-\tau)-\tilde u_i(t-\tau)|d\tau;\end{aligned} \tag{3.28}$$

$$\begin{aligned}\dot V_{i}^{(8)} &= \int_0^{+\infty} v_i(t+\tau,t)|x_i(t)-\tilde x_i(t)|d\tau \\ &\quad - \int_0^{+\infty} v_i(t,t-\tau)|x_i(t-\tau)-\tilde x_i(t-\tau)|d\tau.\end{aligned} \tag{3.29}$$

From (3.21)-(3.29) it follows that

$$\begin{aligned}D^+V &\le \sum_{i=1}^n \nu_i\Big\{\Big[-a_{ii}(t)+\beta_i(t)+\sum_{k=1}^m \frac{p_{ik}(\xi_{ik}(t))}{1-\dot\gamma_{ik}(\xi_{ik}(t))} \\ &\quad + \int_0^{+\infty} v_i(t+\tau,t)d\tau\Big]|x_i(t)-\tilde x_i(t)| \\ &\quad + \sum_{j\ne i}^n a_{ij}(t)|x_j(t)-\tilde x_j(t)| + \sum_{j=1}^n\sum_{k=1}^m \frac{b_{ijk}(\mu_{ijk}(t))}{1-\dot\tau_{ijk}(\mu_{ijk}(t))}|x_j(t)-\tilde x_j(t)| \\ &\quad + \sum_{j=1}^n\sum_{k=1}^m \Big[\int_0^{+\infty} c_{ijk}(t+\tau,t)d\tau\Big]|x_j(t)-\tilde x_j(t)|\Big\} + \sum_{i=1}^n \nu_i\Big\{-\alpha_i(t)+d_i(t) \\ &\quad + \sum_{k=1}^m \frac{e_{ik}(\zeta_{ik}(t))}{1-\dot\sigma_{ik}(\zeta_{ik}(t))} + \int_0^{+\infty} f_i(t+\tau,t)d\tau\Big\}|u_i(t)-\tilde u_i(t)| \\ &= \sum_{i=1}^n \Big\{\nu_i\Big[-a_{ii}(t)+\beta_i(t)+\sum_{k=1}^m \frac{p_{ik}(\xi_{ik}(t))}{1-\dot\gamma_{ik}(\xi_{ik}(t))} + \int_0^{+\infty} v_i(t+\tau,t)d\tau\Big] \\ &\quad + \sum_{j\ne i}^n \nu_j a_{ji}(t) + \sum_{j=1}^n \nu_j \sum_{k=1}^m \frac{b_{jik}(\mu_{jik}(t))}{1-\dot\tau_{jik}(\mu_{jik}(t))} \\ &\quad + \sum_{j=1}^n \nu_j \sum_{k=1}^m \int_0^{+\infty} c_{jik}(t+\tau,t)d\tau\Big\}|x_i(t)-\tilde x_i(t)|\end{aligned}$$

$$+\sum_{i=1}^{n}\Big\{\nu_i\Big[-\alpha_i(t)+d_i(t)+\sum_{k=1}^{m}\frac{e_{ik}(\zeta_{ik}(t))}{1-\dot\sigma_{ik}(\zeta_{ik}(t))} + \int_0^{+\infty} f_i(t+\tau,t)d\tau\Big]\Big\}|u_i(t)-\tilde u_i(t)|. \tag{3.30}$$

By (3.20), there exists positive number δ such that

$$\begin{aligned}\min_{t\in[0,\omega]}\Big\{\nu_i\Big[a_{ii}(t)-\beta_i(t)-\sum_{k=1}^{m}\frac{p_{ik}(\xi_{ik}(t))}{1-\dot\gamma_{ik}(\xi_{ik}(t))}-\int_0^{+\infty}v_i(t+\tau,t)d\tau\Big]\\ -\sum_{j\neq i}^{n}\nu_j a_{ji}(t)-\sum_{j=1}^{n}\nu_j\sum_{k=1}^{m}\frac{b_{jik}(\mu_{jik}(t))}{1-\dot\tau_{jik}(\mu_{jik}(t))}\\ -\sum_{j=1}^{n}\nu_j\sum_{k=1}^{m}\int_0^{+\infty}c_{jik}(t+\tau,t)d\tau\Big\}>\delta,\\ \min_{t\in[0,\omega]}\Big\{\nu_i\Big[\alpha_i(t)-d_i(t)-\sum_{k=1}^{m}\frac{e_{ik}(\zeta_{ik}(t))}{1-\dot\sigma_{ik}(\zeta_{ik}(t))}-\int_0^{+\infty}f_i(t+\tau,t)d\tau\Big]\Big\}>\delta,\end{aligned} \tag{3.31}$$

for $i=1,2,\dots,n$. In the view of (3.30) and (3.31) we have

$$D^+V(t)\le -\delta\sum_{i=1}^{n}\Big\{|x_i(t)-\tilde x_i(t)|+|u_i(t)-\tilde u_i(t)|\Big\},\quad t\ge 0. \tag{3.32}$$

Integrating both sides of (3.32) from 0 to t, we obtain

$$V(t)-V(0)\le -\delta\int_0^t\sum_{i=1}^{n}\Big\{|x_i(s)-\tilde x_i(s)|+|u_i(s)-\tilde u_i(s)|\Big\}ds, t\ge 0.$$

Thus,

$$\int_0^t\sum_{i=1}^{n}\Big\{|x_i(s)-\tilde x_i(s)|+|u_i(s)-\tilde u_i(s)|\Big\}ds\le\frac{V(0)}{\delta},\quad t\ge 0,$$

which implies

$$\int_0^{+\infty}\sum_{i=1}^{n}\Big\{|x_i(s)-\tilde x_i(s)|+|u_i(s)-\tilde u_i(s)|\Big\}ds\le\frac{V(0)}{\delta}. \tag{3.33}$$

Since

$$\nu_i\Big\{\Big|\int_{\tilde x_i(t)}^{x_i(t)}\frac{ds}{g_i(s)}\Big|+|u_i(t)-\tilde u_i(t)|\Big\}\le V(t)\le V(0),\quad t\ge 0,$$

it follows from (H3) that $x_i(t)$ and $u_i(t)$ are bounded on $[0,+\infty)$, and hence from (1.1) we can conclude that $\dot x_i(t)$ and $\dot u_i(t)$ are also bounded on $[0,+\infty)$. This implies that $x_i(t)$ and $u_i(t)$ are uniformly continuous on $[0,+\infty)$. Therefore, $\sum_{i=1}^{n}|x_i(t)-\tilde x_i(t)|+|u_i(t)-\tilde u_i(t)|$ is uniformly continuous on $[0,+\infty)$. Thus, (3.33) implies that

$$\lim_{t\to+\infty}\sum_{i=1}^{n}\Big\{|x_i(t)-\tilde x_i(t)|+|u_i(t)-\tilde u_i(t)|\Big\}=0$$

and $(\tilde x(t),\tilde u(t))$ is the unique positive ω-periodic solution of system (1.1). The proof is complete. □

As an example we consider system (1.1) with

$$\begin{gathered}
n=2, \quad m=1, \quad r_1(t)=r_2(t)\equiv\frac{\ln 5}{2}, \quad \alpha_1(t)=\alpha_2(t)\equiv 6,\\
a_{11}(t)=a_{22}(t)=21+\sin 2\pi t, \quad a_{12}(t)=a_{21}(t)=1+\sin 2\pi t,\\
b_{111}(t)=b_{121}(t)=b_{211}(t)=b_{221}(t)=1+2\sin\pi t,\\
c_{111}(t,s)=c_{121}(t,s)=c_{211}(t,s)=c_{221}(t,s)=v_1(t,s)=v_2(t,s)=\exp\{-(t-s)\},\\
e_{11}(t)=e_{21}(t)=1+\cos 2\pi t, \quad d_1(t)=d_2(t)=1-\sin 2\pi t,\\
\beta_1(t)=\beta_2(t)=1+\cos 2\pi t, \quad p_{11}(t)=p_{21}(t)=1-\cos 2\pi t,\\
\tau_{111}(t)=\tau_{121}(t)=\tau_{211}(t)=\tau_{221}(t)\equiv\tau^*>0,\\
\sigma_{11}(t)=\sigma_{21}(t)\equiv\sigma^*>0, \quad \gamma_{11}(t)=\gamma_{21}(t)\equiv\gamma^*>0,\ g_1(v)=g_2(v)=v.
\end{gathered}$$

It is easy to see that (H1)–(H5) hold. By straightforward computation, we have

$$\begin{gathered}
c^*_{ij1}(t)=f^*_i(t)=v^*_i(t)=1, \quad \bar{P}_i=\bar{Q}_i=\bar{R}_i=\frac{3}{5},\\
A_i=24.8, \quad B_i=\ln\frac{\ln 5}{49.6}+\ln 5, \quad \varphi_i(B_i)=\frac{5\ln 5}{49.6}, \quad i,j=1,2.
\end{gathered}$$

Let $\nu_1=\nu_2=1$. We can easily see that

$$\begin{gathered}
\int_0^1\Big[\beta_i(s)+p_{i1}(s)+v^*_i(s)\Big]ds=3>0,\\
\sum_{j\neq i}^2\Big(\bar{a}_{ij}+\bar{b}_{ij1}+\bar{c}^*_{ij1}+\bar{P}_j+\bar{Q}_j+\bar{R}_j\Big)\varphi_j(B_j)=\frac{30}{31}\times\frac{\ln 5}{2}<\frac{\ln 5}{2}=\bar{r}_i,
\end{gathered}$$

$$\begin{aligned}
&\min_{t\in[0,2\pi]}\Big\{\nu_i\Big[a_{ii}(t)-\beta_i(t)-\frac{p_{i1}(\xi_{i1}(t))}{1-\dot{\gamma}_{i1}(\xi_{i1}(t))}-\int_0^{+\infty}v_i(t+\tau,t)d\tau\Big]-\sum_{j\neq i}^2\nu_j a_{ji}(t)\\
&-\sum_{j=1}^2\nu_j\frac{b_{ji1}(\mu_{ji1}(t))}{1-\dot{\tau}_{ji1}(\mu_{ji1}(t))}-\sum_{j=1}^2\nu_j\int_0^{+\infty}c_{ji1}(t+\tau,t)d\tau\Big\}\\
&=\min_{t\in[0,2\pi]}[14-\cos 2\pi(t+\gamma^*)-2\sin 2\pi(t+\tau^*)]>0,
\end{aligned}$$

$$\begin{aligned}
&\min_{t\in[0,2\pi]}\Big\{\alpha_i(t)-d_i(t)-\frac{e_{i1}(\zeta_{i1}(t))}{1-\dot{\sigma}_{i1}(\zeta_{i1}(t))}-\int_0^{+\infty}f_i(t+\tau,t)d\tau\Big\}\\
&=\min_{t\in[0,2\pi]}[3+\sin 2\pi t-\cos 2\pi(t+\sigma^*)]>0, \quad i=1,2
\end{aligned}$$

Thuerefore, conditions (3.1), (3.3) and (3.30) hold. Hence, by Theorem 3.2, the system has a unique positive 1-periodic solution which is globally asymptotically stable.

Acknowledgements. The authors would like to thank the editors and the anonymous reviewers for their constructive comments. This work was supported by Hanoi National University of Education and the Minister of Education and Training of Vietnam.

References

[1] Fan, M.; Wang, K.; Wong, P.J.Y.; Agarwal, R.P.; *Periodicity and stability in periodic n-species Lotka-Volterra competition systems with feedback controls and deviating arguments.* Acta Mathematica Sinica, Vol. 19, 2003, pp. 801 - 822.

[2] Gains, R.E.; Mawhin, J.K.; *Coincidence Degree and Nonlinear Differential Equations.* Springer-Verlag, Berlin. 1977.

[3] Liu, Z.J.; Tan, R.H.; Chen, L.S.; *Global stability in a periodic delayed predator-prey system.* Applied Mathematics and Computation, Vol. 186, 2007, pp. 389 - 403.

[4] Yan, J.; Liu, G.; *Periodicity and stability for a Lotka-Volterra type competition with feedback controls and deviating arguments.* Nonlinear Analysis, Vol. 74, 2011, pp. 2916 - 2928.

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