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A NOTE ON $p(x)$-HARMONIC MAPS

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Abstract. This article is concerned with $L^{p(x)}$ estimates of the gradient of $p(x)$-harmonic maps. It is known that $p(x)$-harmonic maps are the weak solutions of a system with natural growth conditions, but it is difficult to use the classical elliptic techniques to find gradient estimates. In this article, we use the monotone inequality to show that the minimum $p(x)$-energy can be expressed by the $L^{p(x)}$ norm of a gradient of a function Φ, which is a weak solution of a single equation.

1. Introduction

Let $B=\{x\in\mathbb{R}^2:|x|<1\}$, $S^1=\{x\in\mathbb{R}^2:|x|=1\}$. Assume $g(x)=x$ on S^1, and $p(x)>1$ is a smooth function on $\overline{B}$. We are concerned with the gradient of $p(x)$-harmonic maps. A function u is called a $p(x)$-harmonic map, if it is a weak solution of

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)=u|\nabla u|^{p(x)}. \tag{1.1}$$

A function u_p is called a $p(x)$-energy minimizer if it is a solution of

$$\inf\Big\{\int_B\frac{1}{p(x)}|\nabla u|^{p(x)}dx:u\in W_g^{1,p(x)}(B,S^1)\Big\}, \tag{1.2}$$

where $p(x)\in(1,2)$. This minimum is also called the $p(x)$-energy minimum. It is not difficult to see that the $p(x)$-energy minimizer is a $p(x)$-harmonic map.

Partial regularity of $p(x)$-harmonic maps in the space $W^{1,p(x)}(B,\mathbb{R}^2)$ was given in [4]. When a $p(x)$-harmonic map u belongs to $W^{1,p(x)}(B,S^1)$ with S^1-valued boundary data g, it is more complicated to locate the singularities of u (cf. [9]).

When $p(x)\geq 2$, the class of function $W_g^{1,p(x)}(B,S^1)$ is empty (cf. Page xi in [3]), and problem (1.2) does not make sense. There are two penalized methods to investigate the $p(x)$-energy minimum, which are helpful to understand the local properties of $p(x)$-harmonic maps. First, the Ginzburg-Landau type functional

$$E_\varepsilon(u)=\int_B\frac{1}{p(x)}|\nabla u|^{p(x)}dx+\int_B\frac{1}{\varepsilon^{p(x)}}(1-|u|^2)^2dx$$

can be applied to study $p(x)$-harmonic maps. Since $W_g^{1,p(x)}(B,\mathbb{R}^2)$ is not empty, there exists a Ginzburg-Landau minimizer u_ε. The singularities of $p(x)$-harmonic

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maps are often viewed as the limit of zeros of u_ε (cf. [9]). Some papers studied the p-energy minimum by estimating $E_\varepsilon(u_\varepsilon)$ when p is a constant (cf. [1], [2, 7, 8, 10, 13]). Second, we can use the method of drilling holes which was introduced in [3, Page xii] to deal with the case of $p(x)$ non-constant. For example, we can consider the problem

$$\inf\Big\{\int_{B_\rho}\frac{1}{p(x)}|\nabla u|^{p(x)}dx : u\in W_g^{1,p(x)}(B_\rho,S^1); u|_{\partial B(0,\rho)}=\frac{x}{|x|}\Big\} \tag{1.3}$$

instead of the problem (1.2), since $W_g^{1,p(x)}(B_\rho,S^1)\neq\emptyset$. Here $B_\rho=B\setminus B(0,\rho)$.

In this paper, we investigate the $p(x)$-energy minimum with $p(x)>2$ by the second penalized method. This research is motivated from two aspects. On the one hand, the energy functional $\int_{B_\rho}|\nabla u|^{p(x)}dx$ can be used in the theories of phase transitions, such as the problems of superconductivity and superfluids. In the study of type-II superconductors, the vortices can be described by this hole $B(0,\rho)$ (cf. [3]). On the other hand, the energy functional $\int_{B_\rho}\frac{1}{p(x)}|\nabla u|^{p(x)}dx$ can be used in the partial regularity of $p(x)$-harmonic maps. In general, the $p(x)$-energy minimizer is a $p(x)$-harmonic map on the domain B_ρ. The singularities of $p(x)$-harmonic maps are also located in those holes.

Since (1.1) is a system with the natural growth condition, it seems difficult to estimate the weak solution by the classical elliptic technique. Now, we show that the $p(x)$-energy minimum can be expressed by the $L^{p(x)}$ norm of a gradient of a function Φ, which is a weak solution of a single equation. Then, the complicated partial regularity of $p(x)$-harmonic maps can be understood well by investigating the regularity of a weak solution of such a single equation.

Another problem is whether $x/|x|$ is a $p(x)$-harmonic map. In general, the solution of (1.3) exists and is also a $p(x)$-harmonic map when $p(x)$ is constant. However, a calculation shows the interesting result: if $p(x)=\tilde p(r,\theta)$ depends on θ, then $x/|x|$ is not $p(x)$-harmonic, and hence it does not minimize the $p(x)$-energy.

2. Main results and proofs

When $p(x)$ is constant and is in $(1,n)$, papers [5, 6, 11] show that $x/|x|$ is a p-energy minimizer, and hence is a p-harmonic map. The following result shows that if $p(x)$ is variable, then $x/|x|$ may be not a $p(x)$-harmonic map.

Theorem 2.1. *Let $p(x)>1$ be a $C^1(\overline{B})$ function. Assume $p(x)=\tilde p(r,\theta)$, then $x/|x|$ is a $p(x)$-harmonic map on $B\setminus\{0\}$ if and only if $\tilde p(r,\theta)$ is independent of θ. Namely, it is a C^1 function with one variable $r\in[0,1]$.*

Proof. In polar coordinates, $u(x)=x/|x|=(\cos\theta,\sin\theta)$. In this case, the $p(x)$-harmonic maps equation

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)=u|\nabla u|^{p(x)}$$

is equivalent to

$$-\operatorname{div}(|\nabla\theta|^{p(x)-2}\nabla\theta)=0.$$

Noting $\Delta\theta=0$, the equality above is true if and only if

$$-\nabla(|\nabla\theta|^{p(x)-2})\cdot\nabla\theta=0.$$

Namely,

$$r^{2-p(x)}\log r(\nabla p(x)\nabla\theta)+(p(x)-2)r^{1-p(x)}(\nabla r\nabla\theta)=0.$$

In view of $\nabla r\nabla\theta = 0$, the result above is equivalent to $\nabla p(x)\nabla\theta = 0$. In polar coordinates, $\nabla\theta = (0, \frac{1}{r})$, then $\nabla p(x)\nabla\theta = 0$ is equivalent to

$$\partial_\theta \tilde{p}(r,\theta) = 0,$$

which holds if and only if $\tilde{p}$ is independent of θ. The rest of the proof is not difficult to complete. $\square$

Hereafter, we assume $p(x)$ is independent of θ. We consider a more general class of functions than those in (1.3),

$$V_1 = \{v \in W^{1,p(x)}(B_\rho, S^1) : \deg(v, \partial B) = 1, \deg(v, \partial B(0,\rho)) = 1\}.$$

The main result in this paper, stated below, shows that the $p(x)$-energy minimum can be expressed by the $L^{p(x)}$ norm of the gradient of

$$\Phi(x) := \arctan\frac{x_2}{x_1}.$$

Theorem 2.2.

$$\min\Big\{\int_{B_\rho}\frac{1}{p(x)}|\nabla v|^{p(x)}dx, v \in V_1\Big\} = \int_{B_\rho}\frac{1}{p(x)}|\nabla\Phi|^{p(x)}dx. \tag{2.1}$$

Proof. **Step 1.** We claim $\Phi(x) = \arctan\frac{x_2}{x_1}$ solves the equation

$$-\operatorname{div}(|\nabla\phi|^{p(x)-2}\nabla\phi) = 0, \quad \text{in } B_\rho; \tag{2.2}$$

and there holds

$$\nabla\Phi(x)\cdot\tau = \frac{1}{|x|}. \tag{2.3}$$

In fact, by a simple calculation, we have $\nabla\Phi(x) = (-x_2, x_1)/|x|^2$. Therefore, (2.3) is true, and

$$\begin{aligned}
-\operatorname{div}(|\nabla\Phi|^{p(x)-2}\nabla\Phi) &= -\operatorname{div}[\frac{(-x_2,x_1)}{|x|^{p(x)}}] \\
&= (x_2, -x_1)\cdot[\frac{\log|x|}{|x|^{p(x)}}\nabla p(x) + p(x)\frac{x}{|x|^{p(x)+2}}].
\end{aligned}$$

Since $p(x)$ depends only on $|x|$, we have $(x_2, -x_1)\cdot\nabla p(x) = 0$. Thus, Φ solves (2.2).

Step 2. Let $v \in V_1$. Set

$$D = (-v\wedge v_{x_2} + \Phi_{x_2}, v\wedge v_{x_1} - \Phi_{x_1}),$$

then $\operatorname{div} D = 0$. On the other hand,

$$D\cdot\nu = -(v\wedge v_\tau) + \Phi_\tau,$$

where ν is a unit outward norm vector on the corresponding boundary, and τ is a unit tangent vector on the corresponding boundary. Noting (2.3), we have

$$\int_{\partial B(0,\rho)}\Phi_\tau ds = 2\pi.$$

In view of the definition of the degree

$$d = \frac{1}{2\pi}\int_{\partial B(0,\rho)} v\wedge v_\tau\, ds = 1,$$

we obtain
$$\int_{\partial B(0,\rho)} (D\cdot\nu)ds = 0,$$
Therefore, according to [3, Lemma I.1], there exists
$$H \in \{H; D = (H_{x_2}, -H_{x_1})\}.$$

Step 3. Set
$$V_3(v) = W_0^{1,p(x)}(B_\rho) \cap \{H; D = (H_{x_2}, -H_{x_1})\}.$$
Then $V_3(v) \neq \emptyset$ in veiw of $0 \in V_3(v)$.

If $V_3(v) = \{0\}$, then for any $v \in V_1$, $|\nabla v|^2 = |\nabla\Phi|^2$. Thus, (2.1) holds.

If $V_3(v) \setminus \{0\} \neq \emptyset$, then we can find
$$H \in V_3(v) \setminus \{0\} \tag{2.4}$$
such that
$$\begin{gathered} v \wedge v_{x_1} = \Phi_{x_1} - H_{x_1};\\ v \wedge v_{x_2} = \Phi_{x_2} - H_{x_2}. \end{gathered}$$
This means
$$|\nabla v|^{p(x)} = |\nabla(\Phi - H)|^{p(x)}.$$

Step 4. We claim that
$$|\nabla v|^{p(x)} \geq |\nabla\Phi|^{p(x)} - p(x)|\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla H. \tag{2.5}$$
To prove this inequality, we define the function
$$f(s,t) = |t-s|^{p(x)} - |t|^{p(x)} + p(x)|t|^{p(x)-2}(t\cdot s)$$
for two vectors s and t. According to the mean value theorem, there exists $\xi \in (0,1)$ such that
$$|t|^{p(x)} - |t-s|^{p(x)} = p(x)|t-\xi s|^{p(x)-2}(t-\xi s)\cdot s.$$
Hence, applying the monotone inequality [14, (2.11)]), we have
$$\begin{aligned} f(s,t) &= p(x)|t|^{p(x)-2}(t\cdot s) - p(x)|t-\xi s|^{p(x)-2}(t-\xi s)\cdot s\\ &= p(x)\xi^{-1}[|t|^{p(x)-2}t - |t-\xi s|^{p(x)-2}(t-\xi s)]\cdot[t-(t-\xi s)]\\ &\geq \gamma_0|s|^{p(x)} \geq 0. \end{aligned}$$
Here $\gamma_0 > 0$ only depends on $p(x)$. Taking $s = \nabla H$, $t = \nabla\Phi$, and by Step 3, we can see (2.5).

Step 5. For any $v \in V_1$, (2.5) implies that
$$\int_{B_\rho} \frac{1}{p(x)}|\nabla v|^{p(x)}dx \geq \int_{B_\rho} \frac{1}{p(x)}|\nabla\Phi|^{p(x)}dx - \int_{B_\rho} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla H dx. \tag{2.6}$$
Since $H \in W_0^{1,p(x)}(B_\rho)$, and Φ is a solution of (2.2), we see that the second term of the right-hand side of (2.6) is zero. Hence, (2.6) leads to
$$\int_{B_\rho} \frac{1}{p(x)}|\nabla v|^{p(x)}dx \geq \int_{B_\rho} \frac{1}{p(x)}|\nabla\Phi|^{p(x)}dx,$$
which implies
$$\inf_{V_1} \int_{B_\rho} \frac{1}{p(x)}|\nabla v|^{p(x)}dx \geq \int_{B_\rho} \frac{1}{p(x)}|\nabla\Phi|^{p(x)}dx. \tag{2.7}$$

Step 6. Let $u_* = (\cos\Phi, \sin\Phi)$ with $\Phi(x) = \arctan\frac{x_2}{x_1}$. Then, (2.3) implies $u_* \in V_1$. Clearly,

$$\int_{B_\rho} \frac{1}{p(x)} |\nabla u_*|^{p(x)} dx = \int_{B_\rho} \frac{1}{p(x)} |\nabla \Phi|^{p(x)} dx.$$

The $p(x)$-energy minimum attains $\int_{B_\rho} |\nabla\Phi|^{p(x)} dx$ at this function u_*. Combining with (2.7), we complete the proof. $\square$

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