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BLOWUP AND EXISTENCE OF GLOBAL SOLUTIONS TO NONLINEAR PARABOLIC EQUATIONS WITH DEGENERATE DIFFUSION

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Abstract. In this article, we consider the degenerate parabolic equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda u^m + \mu|\nabla u|^q$$

on a smoothly bounded domain $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$), with homogeneous Dirichlet boundary conditions. The values of $p > 2$, q, m, λ and μ will vary in different circumstances, and the solutions will have different behaviors. Our main goal is to present the sufficient conditions for L^∞ blowup, for gradient blowup, and for the existence of global solutions. A general comparison principle is also established.

1. Introduction

In this article, we study the initial-boundary value problem of p-Laplacian equation

$$\begin{gathered} u_t - \Delta_p u = \lambda u^m + \mu|\nabla u|^q, \quad x \in \Omega,\ t > 0, \\ u = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smoothly bounded domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 2$, and $m, q \geq 1$ will be decided later. λ and μ satisfy: $\lambda = 0$, $\mu > 0$ or $\mu = 0$, $\lambda > 0$ or $\lambda\mu < 0$. We also assume that the initial data satisfies

$$u_0 \in W_0^{1,\infty}(\Omega), \quad u_0 \geq 0.$$

When $p = 2$ and $\mu = 0$, the equation in (1.1) is called the semilinear reaction-diffusion one and is studied by many mathematicians. For various blowup properties of its solutions under different initial-boundary conditions, we refer the readers to [27] and the references therein.

When $p = q = 2$ and $\lambda = 0$, the equation becomes the well-known Kardar-Parisi-Zhang (KPZ) equation describing the the profile of a growing interface in certain physical models (see [15]). The case of $q \geq 1$ was a general one which was developed by Krug and Spohn aiming at studying the effect of the nonlinear gradient term to the properties of solutions (see [17]). The general case of Kardar-Parisi-Zhang

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equations are also the viscosity approximation of Hamilton-Jacobi type equations from stochastic control theory (see [24]). If $q \geq 1$, it's well known that under certain conditions the gradient of the solution will become infinity when t approaches to a finite time T while the solution itself keeps uniformly bounded by the maximum principle, i.e.

$$T < \infty, \quad \sup_{\Omega \times [0,T)} |u| < \infty, \quad \lim_{t \to T^-} \|\nabla u\|_{L^\infty} = +\infty,$$

this phenomenon is called *gradient blowup*. See [14, 27, 29] for examples. Other properties about the solution such as blowup profile, blowup set, blowup rate and so on had also been studied in [13, 21] and the references therein. Zhang and Hu [35] studied the equation $u_t = u_{xx} + x^m |u_x|^p$ in $[0,1] \times [0,\infty)$. They proved that gradient blowup will occur under suitable initial and boundary conditions. They also obtained the gradient blowup rate upper and lower bounds. It was shown in [39] that the solution of equation $u_t - \Delta u = a(x)|\nabla u|^p + h(x)$ will also exhibit gradient blowup phenomenon under certain conditions. Zhang and Li [39] also studied the gradient blowup rate estimates. Besides, solutions of equation of the form $u_t - \Delta u = \mathrm{e}^{|\nabla u|}$ with homogeneous Dirichlet boundary condition and suitably large initial data will also exhibit gradient blowup phenomenon. The blowup criterion, blowup rate and other properties of solutions of this equation can be found in [36, 37, 38, 42].

When $p = 2$, $\lambda\mu < 0$, the properties of solutions become more complicated. If $\lambda > 0$, $\mu < 0$, then the solutions will blow up with the L^∞ norm in a finite time, i.e.

$$T < \infty, \quad \lim_{t \to T^-} \|u\|_{L^\infty} = +\infty,$$

provided that $m > q$. While if $m \leq q$, the global existence can be obtained. See [7, 9, 16, 25, 26, 27, 28, 29, 30, 31, 32] for examples. We also need to state that gradient blowup phenomenon cannot occur in this case. If $\lambda < 0$, $\mu > 0$, then gradient blowup will occur given $q > m$ or $q = m$ and $\mu \gg |\lambda|$, see [14, 27, 29] for examples. However, the properties of solutions in the case of $q < m$ have not been resolved. For the related results, we refer the readers to [27, 29] for details. Besides, some general growth conditions of the nonlinear terms were also obtained, see [6, 10, 16, 27, 29, 32] for examples.

When $p > 2$, the equation (1.1) is degenerate on points where $|\nabla u| = 0$. In this case, the classical maximum principle will be invalid to p-Laplacian equations and the existence of classical solutions cannot be obtained generally. However, we can obtain the weak solutions by means of approximation with regular solutions, see [2, 40] for examples. For the solutions of degenerate equations, the L^∞ blowup and gradient blowup had been studied when $\mu = 0$, $\lambda > 0$ and $\lambda = 0$, $\mu > 0$, respectively. More precisely, when $\lambda > 0$ and $\mu = 0$, the L^∞ blowup will occur under given conditions such as the initial data is large enough if $m > p - 1$, or the coefficient of the nonlinear term is large enough if $m = p - 1$, see [22] for details. Some other related results can be seen in [20, 40]. Besides, the blowup rate had also been studied in [41]. If $\lambda = 0$ and $\mu > 0$, the solutions will exhibit gradient blowup phenomenon under certain conditions, see [2, 3, 19] for examples. Other properties of solutions can be found in [3, 4, 19]. However, there has no other results on the specific gradient blowup rate so far except for [34]. When $\lambda\mu < 0$, the properties of solutions has not been studied except for [1] in which the L^∞ blowup and some other asymptotic behaviors of the equation $u_t - \operatorname{div}(u^{m-1}|Du|^{\lambda-1}Du) = -\epsilon|Du^\nu|^q + \delta u^p$ were studied in $\mathbb{R}^N$. The main goals of this paper are to study the properties of solutions of (1.1) and give some sufficient conditions about L^∞ blowup, gradient blowup and global existence when Ω is a smoothly bounded domain.

First, we give the following definition of weak solutions for (1.1).

Definition 1.1. Set $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\partial Q_T = S_T \cup \{\overline{\Omega} \times \{0\}\}$, $s = \max\{p,q,m\}$. A nonnegative function $u(x,t)$ is called a weak super- (sub-) solution of (1.1) on Q_T if it satisfies

$$\begin{gathered} u \in C\big(\overline{\Omega} \times [0,T)\big) \cap L^s\big((0,T); W^{1,s}(\Omega)\big), \quad \partial_t u \in L^2\big((0,T); L^2(\Omega)\big), \\ u(x,0) \geq u(\leq) u_0, \quad u|_{\partial\Omega} \geq (\leq) 0, \\ \iint_{Q_T} \partial_t u\phi + |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \geq (\leq) \iint_{Q_T} (\lambda u^m + \mu|\nabla u|^q)\phi \,\mathrm{d}x\,\mathrm{d}t. \end{gathered} \tag{1.2}$$

Here, $\phi \in C(\overline{Q_T}) \cap L^p\big((0,T); W^{1,p}(\Omega)\big)$, and $\phi \geq 0$, $\phi|_{S_T} = 0$. u is a weak solution if it's a super-solution and a sub-solution. Here and after, we denote by $T_{\max}$ the maximal existence time.

Remark 1.2. The existence of local solutions for (1.1) can been found in [2, 40] and in [11, Section 2] when $\lambda = 0$ or $\mu = 0$. For the general case, we can consider the approximate problem

$$\begin{gathered} \frac{\partial u_\varepsilon}{\partial t} - \Delta_p^\varepsilon u_\varepsilon = \lambda u_\varepsilon^m + \mu\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{q/2} - \mu\varepsilon^{q/2}, \quad x \in \Omega, t > 0, \\ u_\varepsilon(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.3}$$

where $\Delta_p^\varepsilon u_\varepsilon$ is defined as follows

$$\Delta_p^\varepsilon u_\varepsilon := \operatorname{div}\Big(\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{\frac{p-2}{2}} \nabla u_\varepsilon\Big).$$

The existence of local solutions which belong to $C^{1+\alpha,(1+\alpha)/2}(Q_T)$ for (1.3) can be obtained as in [2] in the case $\lambda < 0$, $\mu > 0$, and in [40] in the case $\lambda > 0$, $\mu < 0$. Then we can obtain the existence of local solutions $u \in L^\infty_{\rm loc}([0,T); W_0^{1,\infty}(\Omega))$ for (1.1) by letting $\varepsilon \to 0$. However, if $\lambda > 0$, $\mu > 0$, there has no $C^{1+\alpha,(1+\alpha)/2}$ estimate so far, which is used to obtain the local existence of the weak solution of (1.1).

Before giving our main results, we use the following two figures to state intuitionally how the relation between q (≥ 1) and m (≥ 1) affects the properties of the solution of (1.1).

In figures 1 and 2, we did not point out which domain the boundary lines and the coordinate axis belong to as the properties of the solution of (1.1) there is somewhat complicated. For more details, we will introduce in our main theorems below (For convenience, the statement of some known results may be different from the original ones).

Theorem 1.3. *For $\lambda > 0$, $\mu < 0$, assume that $u_0 = \eta\psi$, $\psi \geq 0$, $\psi \not\equiv 0$, then there exists $\eta_0(p,q,m,\lambda,\mu,\Omega) > 0$, such that when $\eta > \eta_0$, $T_{\max} < \infty$, if $m > \max\{q, p-1\}$ and $q \geq \frac{p}{2}$. Moreover, if $q \leq p-1$, then L^∞ blowup occurs.*

Theorem 1.4. *Assume that p,q,m,λ and μ satisfy one of the following conditions:*

(i) *$\lambda < 0$, $\mu > 0$, $q > \max\{p,m\}$;*

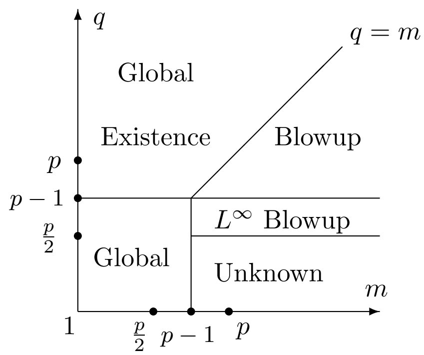

Figure 1. $\lambda > 0, \mu < 0$

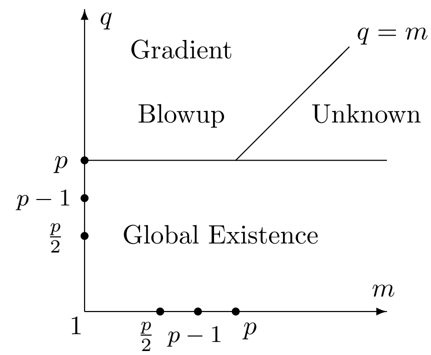

Figure 2. $\lambda < 0, \mu > 0$

(ii) *$q = m > p$, $\lambda < 0$, $\mu > 0$, and $\mu \gg |\lambda|$.*
Set $r = q/(q-p)$, if there exists $k > 0$, such that $\int_\Omega u_0^{r+1}\,\mathrm{d}x > k$, then gradient blowup occurs.

Remark 1.5. (a) It can be seen from Theorem 1.4 that the relation $q = m$ is critical for gradient blowup to occur, and the solutions will exhibit different asymptotic behavior in the critical case.

(b) If $m > q > p$ and $\lambda < 0$, $\mu > 0$, we do not know whether gradient blowup occurs or not even when $p = 2$. Noticing that the source term is an absorption term, and its influence is stronger than the gradient term to the properties of the solutions, we may conjecture that gradient blowup may not occur in this case. We leave this question to the interested readers.

When $\mu = 0$ and $\lambda > 0$, or $\lambda = 0$ and $\mu > 0$, there are known results about L^∞ blowup or gradient blowup for the solution of (1.1), see Theorems 1.6 and 1.8 below. For the details, we refer the readers to [22, Theorem 4.1] and [19, Proposition 5.3] respectively.

Theorem 1.6. *Set $\mu = 0$, $\lambda > 0$*

(i) *Assume that $m > p - 1 > 1$. Given a nonnegative, nontrivial initial datum $u_0 \in C_0(\overline{\Omega})$, there exists $\eta_0 > 0$ (depending only on u_0) such that for all $\eta > \eta_0$, the unique weak solution $u(\cdot, t)$ of Problem (1.1) with initial data ηu_0 blows up in a finite time T^*. Moreover, there is some $C(u_0) > 0$ such that*

$$T^*(\eta u_0) \geq \frac{C(u_0)}{\eta^{p-1}}, \quad \eta \gg 1.$$

(ii) *For $m = p - 1 > 1$, the unique weak solution of (1.1) with nontrivial, nonnegative $u_0 \in C_0(\overline{\Omega})$ blows up in finite time provided that $\lambda > \lambda_1$.*

Remark 1.7. (a) In Theorem 1.6(ii), λ_1 denotes the first eigenvalue of the Dirichlet problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda|\psi|^{p-2}\psi, \quad x \in \Omega, \\ \psi = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.4}$$

(b) It was also proved in [40] that L^∞ blowup occurs when Ω is a large ball and global solution exists if the measure of Ω is small enough. The results obtained by Zhao was proved in the case that the nonlinear terms are replaced by $f(u)$ satisfying: $f(s)$ is odd, $f \geq 0$ on $\mathbb{R}^+$, and

$$\lim_{u\to\infty} \frac{\int_0^u f(s)\,\mathrm{d}s}{|u|^p} = +\infty.$$

Theorem 1.8. *Assume that $\lambda = 0, \mu > 0$ and $q > p > 2$. Define $r = q/(q-p)$ as in Theorem 1.4. There exists a positive real number κ depending on μ, Ω, p, q such that, if $\|u_0\|_{r+1} > \kappa$, then (1.1) has no global solution, i.e. gradient blowup occurs.*

Remark 1.9. The gradient blowup of solution for (1.1) when $\lambda = 0$ was also proved in [2] under an inhomogeneous Dirichlet boundary condition. The proof there depends on the first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition.

Theorem 1.10. *Let $\lambda > 0$, $\mu < 0$, assume that u is nondecreasing in time, then u exists globally in time if $q \geq m$ and $q > p - 1$.*

Theorem 1.11. (i) *If $\lambda\mu \neq 0$, $p - 1 > \max\{q, m\}$, then the solution of (1.1) is global in time.*

(ii) *If $\lambda\mu \neq 0$, $q = p - 1$, $m \leq p - 1$, or $q \leq p - 1$, $m = p - 1$ and the measure of Ω is small enough, then the solution of (1.1) is global in time. In particular, the smallness for Ω is unnecessary if $\mu > 0, \lambda < 0$.*

(iii) *If $\mu = 0$, $\lambda > 0$, $m > p-1 > 1$, then there exists $\eta > 0$ such that the solution of (1.1) exists globally provided that $\|u_0\|_\infty < \eta$.*

(iv) *If $\mu = 0$, $\lambda > 0$, $1 < m < p - 1$. Then the solution of (1.1) exists globally for any initial data.*

(v) *Assume that $\lambda = 0$, $\mu > 0$, if $p \geq q > p - 1 > 1$, then the solution of (1.1) exists globally for any initial data; if $q > p$, then the solution exists globally if u_0 is small enough; if $q \leq p - 1$, then the solution of (1.1) is global in time.*

Remark 1.12. Statements (i) and (ii) in the above theorem are direct consequences of [40, Theorem 3.1] in which the author considered a more general situation; i.e. the nonlinear terms are replaced by $f(\nabla u, u, x, t)$ which satisfies suitable growth conditions. We also point out that the assumption for the size of Ω is unnecessary if $\lambda < 0$, $\mu > 0$ as the uniform boundedness for u_n in [40] can be directly obtained by the maximum principle. Statements (iii) and (iv) can be found in [22, Theorems 4.2 and 4.3]. The first two results of (v) are simplified forms of [19, Theorem 1.4] in which Laurençot and Stinner gave a specific condition on initial data u_0 and studied the asymptotic behavior of u as $t \to \infty$. The third one is a special case of [40, Theorem 3.1] without the condition that Ω is small enough as we can obtain the uniform boundedness of the approximate solution by the maximum principle.

Theorem 1.13. *Let $\lambda < 0$, $\mu > 0$ and $q \leq p$, $m \geq 1$. Then the solution u exists globally.*

The rest of this article is organized as follows. In Section 2, we establish a general comparison principle and some gradient estimates. The proofs of our main results are included in Section 3. In Section 4, we discuss the cases when $\lambda \leq 0$, $\mu \leq 0$ and $\lambda > 0$, $\mu > 0$. We also assume that λ, μ are constants without specific statement in this paper. Thus, we can assume that $|\lambda| = |\mu| = 1$ in some cases.

2. Preliminaries

Due to the degeneracy of the p-Laplacian operator, the classical maximum principle for the nondegenerate operators may not apply. However, we can prove the following more general comparison principle for (1.1). We also need to point out that the comparison principle below can be extended to a more general case under the condition that $\lambda u^m + \mu|\nabla u|^q$ is replaced by $B(u, \nabla u)$ which is locally Lipschitz continuous with respect to u.

Proposition 2.1. *Assume that $u, v \in L^\infty_{\rm loc}((0,T); W^{1,\infty}(\Omega))$ are sub- and super-solution of (1.1) respectively. If $q \ge \frac{p}{2}$ and $m \ge 1$, then $u \le v$ on Q_T.*

The proof of the comparison principle relies on the following algebraic lemma (see [2, 23]).

Lemma 2.2. *Let $\sigma > 1$. For all $\vec{a}, \vec{b} \in \mathbb{R}^N$, then*

$$\big\langle |\vec{a}|^{\sigma-2}\vec{a} - |\vec{b}|^{\sigma-2}\vec{b}, \vec{a} - \vec{b}\big\rangle \ge \frac{4}{\sigma^2}\big||\vec{a}|^{\frac{\sigma-2}{2}}\vec{a} - |\vec{b}|^{\frac{\sigma-2}{2}}\vec{b}\big|^2.$$

Proof of Proposition 2.1. Without loss of generality, we assume that $\lambda, \mu > 0$. Choose $\phi = (u-v)^+$ as the test function. Obviously, $\phi|_{S_T} = 0$, $\phi(x,0) = 0$. Then for any $\tau \in (0,T)$, we have

$$\begin{aligned}
\int_0^\tau \int_\Omega \partial_t\phi\phi \,\mathrm{d}x\,\mathrm{d}t \le & -\underbrace{\int_0^\tau \int_{\{\phi(\cdot,t)>0\}} [|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v]\cdot\nabla\phi \,\mathrm{d}x\,\mathrm{d}t}_{\mathcal{M}} \\
& + \mu \underbrace{\int_0^\tau \int_{\{\phi(\cdot,t)>0\}} [|\nabla u|^q - |\nabla v|^q]\phi \,\mathrm{d}x\,\mathrm{d}t}_{\mathcal{G}} \\
& + \lambda \underbrace{\int_0^\tau \int_{\{\phi(\cdot,t)>0\}} (u^m - v^m)\phi \,\mathrm{d}x\,\mathrm{d}t}_{\mathcal{S}}.
\end{aligned} \tag{2.1}$$

Then by Lemma 2.2, we have

$$\mathcal{M} \ge \frac{4}{p^2}\int_0^\tau \int_{\{\phi(\cdot,t)>0\}} \Big||\nabla u|^{\frac{p-2}{2}}\nabla u - |\nabla v|^{\frac{p-2}{2}}\nabla v\Big|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.2}$$

For the term $\mathcal{G}$, as in [2, Proposition 2.1], we have

$$\begin{aligned}
\mathcal{G} \le & \, C\varepsilon \int_0^\tau \int_{\{\phi(\cdot,t)>0\}} \Big||\nabla u|^{\frac{p-2}{2}}\nabla u - |\nabla v|^{\frac{p-2}{2}}\nabla v\Big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
& + C(\varepsilon)\int_0^\tau \int_{\{\phi(\cdot,t)>0\}} \phi^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.3}$$

Here, C is a constant which depends on p, q and $\max\{|\nabla u|^{p/2}, |\nabla v|^{p/2}\}$.

By the mean value theorem, we derive

$$\mathcal{S} \le m\|u\|^{m-1}_{L^\infty} \int_0^\tau \int_{\{\phi(\cdot,t)>0\}} \phi^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.4}$$

Choosing $0 < \varepsilon < 4/(\mu C p^2)$, combining the estimates (2.2)-(2.4) and integrating by parts, we obtain

$$\int_\Omega \phi^2(\tau)\,\mathrm{d}x \le C\big(\lambda,\mu,\varepsilon,m,p,q,\|u\|_{L^\infty},\max\{|\nabla u|^{p/2},|\nabla v|^{p/2}\}\big)\int_0^\tau\int_\Omega \phi^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.5}$$

Then the conclusion follows from the Gronwall lemma. □

Remark 2.3. (i) If $\lambda = 0$ or $\mu = 0$, the corresponding comparison principle had been studied by many researchers. See [2, Proposition 2.1], [22, Theorem 2.5], and [33, Lemma 2.1] for examples.

(ii) We can see from the proof above that the boundedness for the sub-solution can be removed if $m = 1$ or $\lambda \le 0$.

Next, we give some results concerning the gradient estimates under the following condition:

(H1) There exists a constant M, such that $\|u\|_{L^\infty(\bar\Omega\times[0,T_{\max}])} \le M$ and u is nondecreasing in t.

For any fixed $x_0 \in \Omega$, we choose R such that $B(x_0,R) \subset \Omega$. Let $\alpha \in (0,1), R' = 3R/4$, we select a cut-off function which will be used later satisfying:

(i) $\eta \in C^2(\bar B(x_0,R')), 0 < \eta < 1, \eta(x_0) = 1$ and $\eta = 0$ for $|x - x_0| = R'$.
(ii) $|\nabla\eta| \le CR^{-1}\eta^\alpha$ and $|D^2\eta| + \eta^{-1}|\nabla\eta|^2 \le CR^{-2}\eta^\alpha$ for $|x-x_0| < R'$ and $C = C(\alpha) > 0$.

Proposition 2.4. *Assume that $\lambda > 0$, $\mu < 0$, $q > p-1$, $m \ge 1$ and that (H1) is satisfied, then the unique weak solution of* (1.1) *satisfies the gradient estimate*

$$|\nabla u| \le C_1\delta(x)^{-\frac{1}{q-p+1}} + C_2 \quad \text{in } \Omega\times(0,T_{\max}), \tag{2.6}$$

where $C_1 = C_1(p,q,m,N) > 0$, $C_2 = C_2(p,q,m,\Omega,M,u_0) > 0$.

Proof. Since the proof is similar to that in [2, Theorem 1.4], we just give an outline, and refer the readers to [2, Section 3]. For convenience, we assume that $\lambda = -\mu = 1$. Let $w = |\nabla u|^2$, then w satisfies

$$\mathcal{L}w = -2w^{\frac{p-2}{2}}\left|D^2u\right|^2 + 2mu^{m-1}w, \tag{2.7}$$

where

$$\begin{gathered}\mathcal{L}w = w_t - \mathcal{A}w - \vec{J}\cdot\nabla w,\\ \mathcal{A}w = |\nabla u|^{p-2}\Delta w + (p-2)|\nabla u|^{p-4}(\nabla u)^T D^2 w\nabla u,\\ \begin{aligned}\vec{J} := \big[(p-2)w^{(p-4)/2}\Delta u + \frac{(p-2)(p-4)}{2}w^{\frac{p-6}{2}}\nabla u\cdot\nabla w - qw^{(q-2)/2}\big]\nabla u\\ + \frac{p-2}{2}w^{(p-4)/2}\nabla w.\end{aligned}\end{gathered} \tag{2.8}$$

Letting $z = \eta w$, we have

$$\mathcal{L}z = \eta\mathcal{L}w + w\mathcal{L}\eta - 2w^{\frac{p-2}{2}}\nabla\eta\cdot\nabla w - 2(p-2)w^{(p-4)/2}(\nabla\eta\cdot\nabla u)(\nabla w\cdot\nabla u). \tag{2.9}$$

Using Young's inequality and the properties of η, we derive

$$\mathcal{L}z + \eta w^{\frac{p-2}{2}}\left|D^2u\right|^2 \le C(p,q,N)R^{-2}\eta^\alpha w^{\frac{p}{2}} + C'(p,q,N)R^{-1}\eta^\alpha w^{\frac{q+1}{2}} + 2mu^{m-1}z. \tag{2.10}$$

Using the fact that u is nondecreasing in t and that u is uniformly bounded, we have

$$
\begin{aligned}
|\nabla u(x_1,t_1)|^q &= -u_t + \Delta_p u + u^m \\
&\le (p-2+\sqrt{N})|\nabla u|^{p-2}\left|D^2u(x_1,t_1)\right| + C(m,M),
\end{aligned}
\tag{2.11}
$$

where $(x_1,t_1)\in B(x_0,R')\times(t_0,T)$ satisfies $|\nabla u(x_1,t_1)|>0$. Thus, we can derive

$$
\frac{1}{C(N,p)}|\nabla u(x_1,t_1)|^{2q-p+2} \le C(p,q,m,M,N) + w^{\frac{p-2}{2}}|D^2u(x_1,t_1)|^2.
$$

Hence

$$
\begin{aligned}
&\mathcal{L}z + \frac{1}{C(p,N)}\eta w^{\frac{2q-p+2}{2}} \\
&\le C(p,q,m,M,N) + CR^{-2}\eta^\alpha w^{\frac{p}{2}} + C'R^{-1}\eta^\alpha w^{\frac{q+1}{2}} + C(m,M)\eta w.
\end{aligned}
$$

Similar as in [2], we set $\alpha=(q+1)/(2q-p+2)$. By Young's inequality, we have

$$
2mu^{m-1}z \le C(p,q,m,M)\eta^{\frac{2q-p}{2q-p+2}} + \frac{1}{4C(N,p)}\eta w^{\frac{2q-p+2}{2}}.
$$

The estimates for $R^{-2}\eta^\alpha w^{\frac{p}{2}}$ and $R^{-1}\eta^\alpha w^{\frac{q+1}{2}}$ are the same as in [2]. Thus, we have

$$
\mathcal{L}z + \frac{1}{2C(N,p)}z^{\frac{2q-p+2}{2}} \le C'(p,q,m,M,N) + CR^{-\frac{2q-p+2}{q-p+1}}. \tag{2.12}
$$

Then following the same argument as in [2], we obtain the desired estimate. □

Remark 2.5. Following the same procedure, we can assert that the estimate (2.6) is still valid without the condition (H1) if $\lambda<0,\mu>0$. In fact, in this case, we can easily obtain the uniform boundedness for u by the comparison principle as $M=\|u_0\|_{L^\infty}$ is a super-solution. Then following the same manner as in [2], we can still have

$$
u_t \le \frac{1}{p-2}\frac{\|u_0\|_{L^\infty}}{t} \quad \text{in } \mathcal{D}'(\Omega), \text{ a.e. } t>0, \tag{2.13}
$$

which can be used to prove the gradient estimate (2.6) without the assumption that u is nondecreasing in t.

Note that the gradient estimate above implies that there is no interior gradient blowup.

Remark 2.6. The assumption that u is nondecreasing in time is reasonable, as we can assume that u_0 satisfies:

$$
\operatorname{div}\Big(\big(|\nabla u_0|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla u_0\Big) + \lambda u_0^m + \mu|\nabla u_0|^q \ge 0
$$

and that u_ε is uniformly bounded. Then differentiating the approximate equation in (1.3) with respect to t, we have

$$
w_t - \sum_{i,j=1}^N a_{ij}w_{ij} = m\lambda u_\varepsilon^{m-1}w + \vec{d}\cdot\nabla w, \tag{2.14}
$$

where, $w = \frac{\partial u_\varepsilon}{\partial t}$, $w_{ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}$ and

$$
\begin{gathered}
a_{ij} = (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \Big(\delta_{ij} + \frac{p-2}{|\nabla u_\varepsilon|^2 + \varepsilon} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \Big), \\
\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \\
\begin{aligned}
\vec{d} = (p-2)(|\nabla u_\varepsilon|^2 + \varepsilon)^{(p-4)/2} \Big(& \Delta u_\varepsilon \nabla u_\varepsilon + \nabla(|\nabla u_\varepsilon|^2) \\
& + (p-4)\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{-1} (\nabla u_\varepsilon)^T D^2 u_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \Big) \\
& + q\mu \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{(q-2)/2} \nabla u_\varepsilon.
\end{aligned}
\end{gathered}
\tag{2.15}
$$

It is easy to know that the equation (2.14) is uniformly parabolic as we can prove that the matrix $(a_{ij})_1^N$ is positive definite for any fixed $\varepsilon > 0$. Combining this with the assumption for u_0 and u_ε, using the maximum principle, we can assert that $w \geq 0$ which implies that u is nondecreasing in time.

Next, we will give an example for u_0 to show that the assumption above is reasonable. For convenience, we assume that $0 \in \Omega, B_1(0) \subset \Omega$.

If $q > m$, we assume that $\lambda = -\mu = 1$, define $u_0(x)$ as

$$
u_0(x) = \begin{cases} \beta^{-\beta} \Big(\frac{\beta-1}{\beta+N-2} \Big)^{\beta}, & 0 \leq |x| \leq \frac{N-1}{\beta+N-2}, \\ \beta^{-\beta}(1-|x|)^{\beta}, & \frac{N-1}{\beta+N-2} < |x| \leq 1, \\ 0, & x \in \Omega \backslash \overline{B_1(0)}, \end{cases} \tag{2.16}
$$

where $\beta = q/(q-m)$. A direct computation shows that

$$
\nabla u_0(x) = \begin{cases} -\beta^{-(\beta-1)}(1-|x|)^{\beta-1} \frac{x}{|x|}, & \frac{N-1}{\beta+N-2} < |x| \leq 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.17}
$$

If $\frac{N-1}{\beta+N-2} < |x| \leq 1$, then $u_0^m = \beta^{-m\beta}(1-|x|)^{m\beta} = |\nabla u_0|^q$. A further computation shows that

$$
\Delta u_0 = \beta^{-(\beta-1)}(1-|x|)^{\beta-2} \Big(\beta - 1 - \frac{N-1}{|x|}(1-|x|) \Big) \geq 0 \tag{2.18}
$$

and that

$$
(\nabla u_0)^T D^2 u_0 \nabla u_0 = \beta^{-3(\beta-1)}(\beta-1)(1-|x|)^{3(\beta-1)-1} \geq 0 \tag{2.19}
$$

in the case that $\frac{N-1}{\beta+N-2} < |x| \leq 1$. While if $0 \leq |x| < \frac{N-1}{\beta+N-2}$ or $x \in \Omega \backslash \overline{B_1(0)}$, $\operatorname{div}\big((|\nabla u_0|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_0\big) + u_0^m - |\nabla u_0|^q = u_0^m \geq 0$. Thus, $u_0(x)$ satisfies the desired assumption in $\mathcal{D}'(\Omega)$ if $q > m$.

If $q = m > p-1$, we assume that $\mu = -1$ and $\beta > 1$. Similarly, we can define

$$
u_0(x) = \begin{cases} \big(\frac{\beta-1}{\beta+N-2} \big)^{\beta}, & 0 \leq |x| \leq \frac{N-1}{\beta+N-2}, \\ (1-|x|)^{\beta}, & \frac{N-1}{\beta+N-2} < |x| \leq 1, \\ 0, & x \in \Omega \backslash \overline{B_1(0)}. \end{cases} \tag{2.20}
$$

We can see from definition (2.20) that

$$
\nabla u_0(x) = \begin{cases} -\beta(1-|x|)^{\beta-1} \frac{x}{|x|}, & \frac{N-1}{\beta+N-2} < |x| \leq 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.21}
$$

Let us now consider the case $\frac{N-1}{\beta+N-2} < |x| \le 1$. A further computation shows that

$$\Delta u_0 = \beta(1-|x|)^{\beta-2}\Big(\beta - 1 - \frac{N-1}{|x|}(1-|x|)\Big) \ge 0, \tag{2.22}$$

and that, if $(p-3)(\beta-1) \ge 1$,

$$\begin{aligned} &(p-2)\left(|\nabla u_0|^2+\varepsilon\right)^{(p-4)/2}(\nabla u_0)^T D^2 u_0 \nabla u_0 \\ &= (p-2)\beta^3(\beta-1)\left(|\nabla u_0|^2+\varepsilon\right)^{(p-4)/2}(1-|x|)^{3(\beta-1)-1} \\ &= (p-2)\beta^3(\beta-1)|\nabla u_0|^{p-4}(1-|x|)^{3(\beta-1)-1} + O(\varepsilon) \\ &= (p-2)\beta^{p-1}(\beta-1)(1-|x|)^{(p-1)(\beta-1)-1} + O(\varepsilon). \end{aligned} \tag{2.23}$$

Then, for any fixed $\beta > 1$, we have

$$(p-2)\beta^{p-1}(\beta-1)(1-|x|)^{(p-1)(\beta-1)-1} > \beta^m(1-|x|)^{m(\beta-1)}, \text{if} \quad r_1 \le |x| \le 1, \tag{2.24}$$

where r_1 is a constant close to 1. For the fixed β and r_1, let $\lambda \ge \beta^m(1-r_1)^{-m}$. Then we have

$$\begin{aligned} \lambda u_0^m - |\nabla u_0|^m &= (1-|x|)^{m\beta}(\lambda - \beta^m(1-|x|)^{-m}) \\ &\ge (1-r_1)^{m\beta}(\lambda - \beta^m(1-r_1)^{-m}) \ge 0, \end{aligned} \tag{2.25}$$

if $\frac{N-1}{\beta+N-2} < |x| \le r_1$. If $0 \le |x| < \frac{N-1}{\beta+N-2}$ or $x \in \Omega\backslash\overline{B_1(0)}$,

$$\operatorname{div}\Big((|\nabla u_0|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_0\Big) + u_0^m - |\nabla u_0|^q = u_0^m \ge 0.$$

Thus, $u_0(x)$ satisfies the desired assumption in $\mathcal{D}'(\Omega)$ if $q = m$.

3. Proof of main results

In this section, we give the proofs of Theorems 1.3, 1.4, 1.10 and 1.13.

3.1. Proof of L^∞ blowup.

The proof of Theorem 1.3 is based on the construction of a self-similar sub-solution which was used in [32], the similar results can also be found in [22, 27].

Proof of Theorem 1.3. In this case, we may assume that $\lambda = -\mu = 1$. Set

$$v(x,t) = \frac{1}{(1-\delta t)^k} V\Big(\frac{|x|}{(1-\delta t)^r}\Big), \quad t_0 \le t < \frac{1}{\delta}, \tag{3.1}$$

where

$$V(y) = 1 + \frac{A}{\sigma} - \frac{y^\sigma}{\sigma A^{\sigma-1}}, \quad y \ge 0, \ \sigma = \frac{p}{p-1}. \tag{3.2}$$

The parameters k, r, A, δ satisfy

$$k = \frac{1}{m-1}, \quad 0 < r < \min\Big\{\frac{m-p+1}{p(m-1)}, \frac{m-q}{q(m-1)}\Big\}, \quad A > \frac{k}{r}, \quad \delta < \frac{1}{k(1+\frac{A}{\sigma})}.$$

A direct calculation shows that $km = k+1 > (p-2)(k+r)+k+2r$, $k+1 > (k+r)q$, and the auxiliary function $V(y)$ satisfies

$$\begin{gathered} 1 \le V(y) \le 1+\frac{A}{\sigma}, \quad -1 \le V'(y) \le 0, \quad \text{if } 0 \le y \le A, \\ 0 \le V(y) \le 1, \quad -\big(\frac{R}{A}\big)^{\sigma-1} \le V'(y) \le -1, \quad \text{if } A \le y \le R, \\ (p-1)|V'(y)|^{p-2}V''(y) + \frac{N-1}{y}|V'(y)|^{p-2}V'(y) = -\frac{N}{A}, \quad \text{if } 0 < y < R. \end{gathered} \tag{3.3}$$

Here $R = (A^{\sigma-1}(A+\sigma))^{1/\sigma}$ is the zero of $V(y)$. If we denote

$$D := \Big\{(x,t) : t_0 \le t < \frac{1}{\delta}, |x| < R(1-\delta t)^r\Big\}, \tag{3.4}$$

then $v(x,t) > 0$ if and only if $(x,t) \in D$, and $v(x,t)$ is smooth in D. Next, we will show that $v(x,t)$ is a sub-solution of (1.1). Let $y = \frac{|x|}{(1-\delta t)^r}$, then we have

$$\begin{aligned} \mathcal{L}_p v &= v_t - \Delta_p v - v^m + |\nabla v|^q \\ &= \frac{\delta(kV(y)+ryV'(y))}{(1-\delta t)^{k+1}} - \frac{(p-1)|V'(y)|^{p-2}V''(y) + \frac{N-1}{y}|V'(y)|^{p-2}V'(y)}{(1-\delta t)^{(p-2)(k+r)+(k+2r)}} \\ &\quad - \frac{V^m(y)}{(1-\delta t)^{mk}} + \frac{|V'(y)|^q}{(1-\delta t)^{q(k+r)}}. \end{aligned}$$

If $0 \le y \le A$, then we can choose $t_0(p,q,m,\delta,N,A)$ close to $\frac{1}{\delta}$ such that

$$\begin{aligned} \mathcal{L}_p v \le \frac{1}{(1-\delta t)^{k+1}} \Big(& \delta k(1+\frac{A}{\sigma}) - 1 + \frac{N}{A}(1-\delta t_0)^{1-2r-(p-2)(k+r)} \\ & + (1-\delta t_0)^{k+1-q(k+r)}\Big) \le 0. \end{aligned} \tag{3.5}$$

Similarly, we can obtain the estimate

$$\begin{aligned} \mathcal{L}_p v \le \frac{1}{(1-\delta t)^{k+1}} \Big(& \delta(k-rA) + \frac{N}{A}(1-\delta t_0)^{1-2r-(p-2)(k+r)} \\ & + \Big(\frac{R}{A}\Big)^{q(\sigma-1)}(1-\delta t_0)^{k+1-q(k+r)}\Big) \le 0 \end{aligned} \tag{3.6}$$

when $A \le y < R$. Combining (3.5) with (3.6), we conclude that $\mathcal{L}_p v \le 0$ in D. To show that v is a sub-solution, we also need to estimate the initial boundary value. By translation, we can assume without loss of generality that $0 \in \Omega$ and $\psi \ge \gamma > 0$ in $B(0,\rho)$ for some $\delta, \rho > 0$. We can also choose suitable t_0 such that $B(0, R(1-\delta t)^r) \subset \Omega$. Besides, we need $\eta > \eta_0$ be large enough such that $u_0 \ge v(\cdot, t_0)$ for $x \in B(0, R(1-\delta t_0)^r)$. Since $v > 0$ if and only if $(x,t) \in D$, we have $u_0 \ge v(\cdot, t_0)$ in $\overline{\Omega}$. It is obviously that $v \le 0$ when $(x,t) \in \partial\Omega \times (t_0, \frac{1}{\delta})$. By the comparison principle, we can deduce that

$$u(x,t) \ge v(x, t+t_0), \quad (x,t) \in D. \tag{3.7}$$

Since $\lim_{t\to 1/\delta} v(0,t) \to \infty$, we conclude that $T_{\max} \le \frac{1}{\delta} - t_0 < \infty$.

If $q \le p-1$, then as in [40], we can conclude that if u is uniformly bounded, then ∇u is Hölder continuous in its existence time which implies that gradient blowup cannot occur in this case. So, L^∞ blowup occurs. $\square$

3.2. **Proof of gradient blowup.** The uniform boundedness of the solution can be easily obtained by the fact that $M = \|u_0\|_{L^\infty}$ is a super-solution and 0 is a sub-solution. So it's enough to prove that the maximal existence time of (1.1) is finite if we want to show that the gradient blowup occurs.

Proof of Theorem 1.4. Assume that $T_{\max} = \infty$, let $y(t) = \frac{1}{r+1}\int_\Omega u^{r+1}\mathrm{d}x$. We also point out that C_1, C_2 denote constants which may vary from line to line. If $q > m$, we have

$$\begin{aligned} y'(t) &= \mu \int_\Omega u^r |\nabla u|^q \,\mathrm{d}x - r\int_\Omega u^{r-1}|\nabla u|^p \,\mathrm{d}x - |\lambda| \int_\Omega u^{m+r}\,\mathrm{d}x \\ &= \mu \int_\Omega u^r |\nabla u|^q \,\mathrm{d}x - r\int_\Omega (u^r|\nabla u|^q)^{p/q} \,\mathrm{d}x - |\lambda| \int_\Omega u^{m+r}\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

Here we used the fact that $r-1 = \frac{p}{q-p} = \frac{pr}{q}$. By Hölder's and Young's inequalities, we derive

$$\begin{aligned} \int_\Omega (u^r|\nabla u|^q)^{p/q}\mathrm{d}x &\le \Big(\int_\Omega u^r|\nabla u|^q \mathrm{d}x\Big)^{p/q} |\Omega|^{(q-p)/q} \\ &\le \epsilon\frac{p}{q}\int_\Omega u^r|\nabla u|^q\mathrm{d}x + C(\epsilon)\frac{q-p}{q}|\Omega| \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \int_\Omega u^{m+r}\,\mathrm{d}x = \int_\Omega (u^{q+r})^{\frac{m+r}{q+r}}\,\mathrm{d}x &\le \Big(\int_\Omega u^{q+r}\,\mathrm{d}x\Big)^{\frac{m+r}{q+r}} |\Omega|^{\frac{q-m}{q+r}} \\ &\le \varepsilon\frac{m+r}{q+r}\int_\Omega u^{q+r}\,\mathrm{d}x + C(\varepsilon)\frac{q-m}{q+r}|\Omega|. \end{aligned} \tag{3.10}$$

Then

$$\begin{aligned} y'(t) &\ge \frac{\mu q - \epsilon p}{q}\int_\Omega u^r|\nabla u|^q\,\mathrm{d}x - |\lambda|\varepsilon\frac{m+r}{q+r}\int_\Omega u^{q+r}\,\mathrm{d}x - C \\ &= \frac{\mu q - \epsilon p}{q}\Big(\frac{q}{q+r}\Big)^q\int_\Omega |\nabla u^{\frac{q+r}{q}}|^q\,\mathrm{d}x - |\lambda|\varepsilon\frac{m+r}{q+r}\int_\Omega u^{q+r}\,\mathrm{d}x - C \\ &\ge \Big(\frac{\mu q - \epsilon p}{q}\big(\frac{q}{q+r}\big)^q C' - |\lambda|\varepsilon\frac{m+r}{q+r}\Big)\int_\Omega u^{q+r}\,\mathrm{d}x - C \\ &= C_1\int_\Omega u^{q+r}\,\mathrm{d}x - C, \end{aligned} \tag{3.11}$$

here we used Poincaré's inequality. Also, applying the reverse Hölder's inequality, we have

$$y'(t) \ge C_1\Big(\int_\Omega u^{r+1}\,\mathrm{d}x\Big)^{\frac{q+r}{r+1}}|\Omega|^{\frac{1-q}{r+1}} - C_2 \ge C_1\Big(\int_\Omega u^{r+1}\,\mathrm{d}x\Big)^{\frac{q+r}{r+1}} - C_2. \tag{3.12}$$

Then we have

$$y'(t) \ge C_1 y^{\frac{q+r}{r+1}}(t) - C_2, \tag{3.13}$$

where $C_1(p,q,m,\lambda,\mu,\epsilon,\varepsilon,\Omega)$, $C_2(p,q,m,\lambda,\mu,\epsilon,\varepsilon,\Omega) > 0$ with suitable ϵ and ε. Set

$$k > \big(\frac{2C_2}{C_1}\big)^{\frac{r+1}{q+r}}, \tag{3.14}$$

then if $y(0) > k$, we have

$$y'(t) \ge \frac{C_1 y^{\frac{q+r}{r+1}}(t)}{2}. \tag{3.15}$$

A contradiction then follows by integrating (3.15). Therefore, $T_{\max} < \infty$, i.e. gradient blowup occurs.

If $q = m$, the proof above is still valid for $\mu \gg |\lambda|$. $\square$

Remark 3.1. We can also assume that $\int_\Omega u_0\varphi_1^\alpha \,\mathrm{d}x$ is large enough if we set $y(t) = \int_\Omega u\varphi_1^\alpha \,\mathrm{d}x$. Here, $\frac{p-1}{q-p+1} < \alpha < q-1$, φ_1 is the first eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition. Then combining the fact that $l < 1$ implies $\int_\Omega \varphi_1^{-l}\mathrm{d}x < \infty$ (see [29, Lemma 5.1]) with Hölder's, Young and Poincaré's inequalities, we can obtain the blowup inequality $y'(t) \geq C_1 y^q(t)$.

3.3. **Proof of global existence.** In this part, we will give a proof of Theorem 1.10 based on constructing a super-solution.

Proof of Theorem 1.10. For convenience, we assume that $\lambda = -\mu = 1$. Denote by $\rho(\Omega)$ the diameter of Ω. Then the boundedness of Ω implies that $\rho(\Omega) < \infty$. Let $\varepsilon \in (0,1)$ such that there exists a ball with radius ε which belongs to $B(\cdot, \rho(\Omega)+1)\cap\Omega^c$. For any $a \in \Omega$, let x_a satisfy

$$B(x_a,\varepsilon) \subseteq B(x_a, \rho(\Omega)+1)\cap\Omega^c, \quad |x_a - a| < \rho(\Omega)+1. \tag{3.16}$$

If $q > m$, we define

$$V(x,t) = \frac{K}{\sigma} r^\sigma, \quad \sigma = \frac{p}{p-1}, \quad r = |x - x_a|, \quad x \in \Omega. \tag{3.17}$$

Obviously, $\varepsilon \leq r < \rho(\Omega)+1$. Let us now look for a suitable K such that $V(x,t)$ is a super-solution of (1.1). A direct calculation shows that

$$\mathcal{L}_p V = V_t - \Delta_p V - V^m + |\nabla V|^q = -NK^{p-1} + K^q r^{\frac{q}{p-1}} - \big(\frac{K}{\sigma}\big)^m r^{\frac{mp}{p-1}}. \tag{3.18}$$

Then

$$\mathcal{L}_p V \geq 0 \iff K^q r^{\frac{q}{p-1}} \geq NK^{p-1} + \Big(\frac{K}{\sigma}\Big)^m r^{\frac{mp}{p-1}}. \tag{3.19}$$

Thus, we just need to choose K such that

$$K^q r^{\frac{q}{p-1}} \geq 2NK^{p-1}, \tag{3.20}$$

$$K^q r^{\frac{q}{p-1}} \geq 2\big(\frac{K}{\sigma}\big)^m r^{\frac{mp}{p-1}}. \tag{3.21}$$

Inequality (3.20) is satisfied if we choose

$$K \geq \Big(\frac{2N}{\varepsilon^{\frac{q}{p-1}}}\Big)^{\frac{1}{q-p+1}}, \tag{3.22}$$

provided that $q > p-1$. Dividing inequality (3.21) by $K^m r^{\frac{q}{p-1}}$, we derive

$$K^{q-m} \geq \frac{2}{\sigma^m} r^{\frac{mp-q}{p-1}}. \tag{3.23}$$

If $mp \geq q$, then we can set

$$K \geq \Big(\frac{2}{\sigma^m}\Big)^{\frac{1}{q-m}} (\rho(\Omega)+1)^{\frac{mp-q}{(p-1)(q-m)}}, \tag{3.24}$$

while when $mp < q$, we can set

$$K \geq \Big(\frac{2}{\sigma^m}\Big)^{\frac{1}{q-m}} \varepsilon^{\frac{mp-q}{(p-1)(q-m)}}. \tag{3.25}$$

To ensure that $V(x,0)\ge u_0$, we also need that $K\ge \frac{\sigma\|u_0\|_{L^\infty}}{\varepsilon^\sigma}$. Thus, letting

$$
\begin{aligned}
K\ge \max\Big\{&\frac{\sigma\|u_0\|_{L^\infty}}{\varepsilon^\sigma},\Big(\frac{2N}{\varepsilon^{\frac{q}{p-1}}}\Big)^{\frac{1}{q-p+1}},\Big(\frac{2}{\sigma^m}\Big)^{\frac{1}{q-m}}(\rho(\Omega)+1)^{\frac{mp-q}{(p-1)(q-m)}},\\
&\Big(\frac{2\lambda}{\sigma^m}\Big)^{\frac{1}{q-m}}\varepsilon^{\frac{mp-q}{(p-1)(q-m)}}\Big\}.
\end{aligned}
\tag{3.26}
$$

we obtain

$$
\mathcal{L}_pV=V_t-\Delta_pV-V^m+|\nabla V|^q\ge 0,\quad V(x,0)\ge u_0. \tag{3.27}
$$

It is obvious that $V(x,t)\ge 0=u(x,t)$ on $\partial\Omega$. Therefore, we conclude that $V(x,t)$ is a super-solution of (1.1). The comparison principle implies that

$$
0\le u(x,t)\le \frac{K(\rho(\Omega)+1)^{\frac{p}{p-1}}}{\sigma}<\infty; \tag{3.28}
$$

i.e. u is uniformly bounded in its time existence. If $q=m$, we need to modify the super-solution a little. Let

$$
\alpha\ge \max\{1,2^{1/q}(\rho(\Omega)+1)\}, \tag{3.29}
$$

and

$$
K\ge \max\Big\{\varepsilon^{-\alpha}\|u_0\|_{L^\infty},\Big(\frac{2((p-1)(\alpha-1)+N-1)}{\varepsilon^{(q-p+1)(\alpha-1)+1}}\Big)^{\frac{1}{q-p+1}}\Big\}, \tag{3.30}
$$

then the function $V(x,t)=Kr^\alpha$ is a super-solution of (1.1). We can also obtain the uniform boundedness of $u(x,t)$ by the same procedure.

To obtain the global existence, we need also to exclude the possibility of gradient blowup. By Proposition 2.4, we just need to show that $\frac{\partial u}{\partial n}$ is bounded on the boundary of Ω. Define: $\phi(x)=\min\{V(x),M\mathrm{dist}(x,\partial\Omega)\}$ for some sufficiently large M. Then we obtain a super-solution for u. Moreover, we can derive the boundedness of $\frac{\partial u}{\partial n}$. Thus, we complete the proof of Theorem 1.10. $\square$

Remark 3.2. We can also choose the super-solution as in [27, Theorem 36.4(i)] (where Quittner and Souplet proved the similar result when $p=2$) of the form $V(x,t)=Ke^{\alpha r}$, where

$$
\alpha=\big(\frac{2\lambda}{|\mu|}\big)^{1/q},\quad K\ge \max\Big\{\|u_0\|_{L^\infty},1,\Big(\frac{(p-1)\alpha^p+\frac{N-1}{\varepsilon}\alpha^{p-1}}{\lambda}\Big)^{\frac{1}{q-p+1}}\Big\}. \tag{3.31}
$$

Proof of Theorem 1.13. By Theorem 1.11 parts (i) and (ii), we just need to consider the case $m>p-1$.

If $q\le p-1$, by the maximum principle, we know that the approximate solution u_ε is uniformly bounded as $M:=\|u_0\|_{L^\infty}$ is a super-solution for any ε. Then the same manner as in [40, Theorem 3.1] shows that u is global in time.

If $p-1\le q\le p$, by Proposition 2.1, we know that $0\le u\le v$, where v is the solution of (1.1) with $\lambda=0$. Moreover, Theorem 1.11 (v) implies that v exists globally in time. Combining this with the fact that $u=v=0$ on $\partial\Omega$ and that there is no interior gradient blowup by Remark 2.5, we can obtain the global existence for u. $\square$

4. Extensions

As was shown in the previous sections that when $\lambda\mu < 0$, the following two cases occur.

(i) If $\lambda > 0$ and $\mu < 0$, then either L^∞ blowup or global existence occurs.
(ii) If $\lambda < 0$ and $\mu > 0$, then either gradient blowup or global existence occurs.

We also need to notice that the cases that $\lambda > 0$, $\mu > 0$ and $\lambda \le 0$, $\mu \le 0$ are necessary to investigate.

In the latter case, the properties of the solution of (1.1) are simple. As both terms in the right-hand side of (1.1) are non-positive, then neither gradient blowup nor L^∞ blowup can occur. Moreover, the solutions may become zero in finite time or infinite time under some suitable assumptions for the initial data, boundary condition and p, q, m.

If $\lambda = 0$, $\mu < 0$, there has no result concerning this case when Ω is a bounded domain. To our knowledge, the main results are about the Cauchy problem, i.e. $\Omega = \mathbb{R}^N$. For this problem, the solution itself and its gradient will become zero in infinite time under suitable conditions. For more details, we refer the readers to a latest paper [5] and the references therein.

If $\lambda < 0$, $\mu = 0$, then the term λu^m is an absorption term. In this case, there have some relative results concerning the extinction phenomenon, see [12] for an example. We also point out that the solution will become zero in $\Omega' \subsetneqq \Omega$ and be positive in the other part of Ω. This phenomenon is called *dead-core* which had been studied for the p-Laplacian operator by Diaz (see [8] and the references therein).

If $\lambda < 0$, $\mu < 0$, then the solution may also become zero in finite or infinite time. However, there has no paper concerning this case at present.

In the case when $\lambda > 0$, $\mu > 0$, the properties of the solution will be more complicated than the ones when $\lambda\mu < 0$ and $\lambda \le 0$, $\mu \le 0$. For this case, both gradient blowup and L^∞ blowup may occur under suitable conditions. However, as the local existence of the solution is unknown so far, when gradient blowup occurs and when L^∞ blowup occurs are also open.

Besides the gradient blowup and L^∞ blowup, the global existence is also an important property one would have interest. For the global existence, one can see from part (i) and part (ii) in Theorem 1.11 and Theorem 1.13 that the solution of (1.1) can exist globally under the assumptions that $m \le p-1$, $q \le p-1$ or $m \ge 1$, $p-1 < q \le p$ and that Ω is small enough if $q = p-1$ or $m = p-1$. Also, for the case of $\lambda > 0$ and $\mu > 0$, the existence of global solutions has been proved in [40] when $q \le p-1$ and $m \le p-1$. While for $q > p-1$ or$m > p-1$, there has no related results. We leave it to the interested readers as an open question.

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