Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 265, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

IMPULSIVE FRACTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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Abstract. In this article, we apply Bohnenblust-Karlin's fixed point theorem to prove the existence of mild solutions for a class of impulsive fractional equations inclusions with infinite delay. An example is given to illustrate the theory.

1. Introduction

Recently, the subject of fractional differential equations has emerged as an important area of investigation. Indeed, we can find numerous applications of fractional-order derivatives in the mathematical modeling of physical and biological phenomena in various fields of science and engineering. For details, including some applications and recent results, see the monographs of Abbas et al. [1], Baleanu et al. [8], Diethelm [22], Hilfer [26], Kilbas et al. [27], Lakshmikantham et al. [29], Podlubny [33], and Tarasov [38].

On the other hand, the theory of impulsive differential equations appear frequently in applications because many evolutionary process from fields as physics, aeronautic, economics, engineering, population dynamics, etc. (see the monographs of Bainov and Simeonov [7], Benchohra et al. [13], Lakshmikantham et al. [28], and Samoilenko and Perestyuk [36] and the papers [15, 35]).

Fractional differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors, see [5, 11, 14, 19, 32] and the references therein. For some recent development on fractional differential inclusions, we refer the reader to the papers [3, 4, 12]. Recently, Benchohra et al. [10] studied the existence of solutions of differential inclusions with Riemann-Liouville fractional derivative. Cernea [17, 18] established some Filippov type existence theorems for solutions of fractional semilinear differential inclusions involving Caputo's fractional derivative in Banach spaces.

Motivated by the papers cited above, in this paper, we consider the existence of a class of impulsive fractional differential inclusions with infinite delay described by the form

$$ {}^{C}D_t^{\alpha} x(t) - Ax(t) \in F(t, x_t, x(t)), \quad t \in J = (t_k, t_{k+1}],\ k = 0, \dots, m, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 26A33, 34A08, 34A37, 34A60, 34G20, 34H05, 34K09.
Key words and phrases. Impulsive fractional differential inclusions; α-resolvent family; Caputo fractional derivative; mild solution; multivalued map; fixed point; Banach space.

Submitted September 10, 2013. Published November 30, 2013.

$$\Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \tag{1.2}$$

$$x(t) = \phi(t), \quad t \in (-\infty, 0], \tag{1.3}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, $T > 0$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an α-resolvent family $(S_\alpha(t))_{t\geq 0}$, $F : J \times \mathcal{B} \times E \to \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E). Here, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_k : E \to E$, $k = 1, 2, \dots, m$, are multivalued maps, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and the left limit of $x(t)$ at $t = t_k$, respectively. We denote by x_t the element of $\mathcal{B}$ defined by $x_t(\theta) = x(t + \theta), \theta \in (-\infty, 0]$. Here x_t represents the history of the state from $-\infty$ up to the present time t. We assume that the histories x_t belongs to some abstract phase space $\mathcal{B}$, to be specified later, and $\phi \in \mathcal{B}$.

2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

Let $(E, \|\cdot\|)$ be a complex Banach space. Let $C = C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_C = \sup\{\ |y(t)| : t \in J\ \}.$$

Let $L(E)$ be the Banach space of all linear and bounded operators on E. Let $L^1(J, E)$ be the space of E-valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

Denote

$$P_{cl}(E) = \{Y \in P(E) : Y \text{ closed}\}, \quad P_b(E) = \{Y \in P(E) : Y \text{ bounded }\},$$
$$P_{cp}(E) = \{Y \in P(E) : Y \text{ compact}\},$$
$$P_{cp,c}(E) = \{Y \in P(E) : Y \text{ compact and convex}\}.$$

A multivalued map $G : E \to P(E)$ is convex (closed) valued if $G(E)$ is convex (closed) for all $x \in E$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on E if for each $x_0 \in E$ the set $G(x_0)$ is a nonempty, closed subset of E, and if for each open set U of E containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq U$.

A map G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(E)$. If the multivalued map G is completely continuous with nonempty compact values, then G is upper semi continuous (u.s.c.) if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). For more details on multivalued maps see the books of Deimling [21], and Górniewicz [23].

Definition 2.1. The multivalued map $F : J \times \mathcal{B} \times E \to \mathcal{P}(E)$ is said to be Carathéodory if

(i) $t \mapsto F(t, v, w)$ is measurable for each $(v, w) \in \mathcal{B} \times E$;
(ii) $(v, w) \mapsto F(t, v, w)$ is upper semicontinuous for almost all $t \in J$.

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.2. Let $\alpha > 0$ and $f : \mathbb{R}_+ \to E$ be in $L^1(\mathbb{R}_+, E)$. Then the Riemann-Liouville integral is given by:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds.$$

Recall that the Laplace transform of a function $f \in L^1(\mathbb{R}_+, E)$ is defined by

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt, \quad \mathrm{Re}(\lambda) > \omega,$$

if the integral is absolutely convergent for $\mathrm{Re}(\lambda) > \omega$. For more details on the Riemann-Liouville fractional derivative, we refer the reader to [20].

Definition 2.3. [33] The Caputo derivative of order α for a function $f : [0, +\infty) \to \mathbb{R}$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)(s)}}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0,\ n-1 \le \alpha < n.$$

If $0 < \alpha \le 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds.$$

Obviously, The Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is

$$L\{D_t^\alpha f(t), \lambda\} = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^{(k)}(0), \quad n-1 \le \alpha < n, n \in \mathbb{N}.$$

To define the mild solution of the problems (1.1)–(1.3) we recall the following definition.

Definition 2.4. A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:

(1) $\rho(A) \subset \sum_{(\theta,\omega)} := \{\lambda \in C : \lambda \neq \omega,\ |arg(\lambda - \omega)| < \theta\}$.
(2) $\|R(\lambda, A)\|_{L(E)} \le \frac{M}{|\lambda-\omega|}$, $\lambda \in \sum_{(\theta,\omega)}$.

Sectorial operators are well studied in the literature. For details see [24].

Definition 2.5. [6] If A is a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$, then we say that A is the generator of an α-resolvent family if there exists $\omega \ge 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A))$ ($\rho(A)$ being the resolvent set of A) and

$$(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \mathrm{Re}\, \lambda > \omega,\ x \in E.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.6 ([2, Def. 2.1]). if A is a closed linear operator with domain $D(A)$ defined on a Banach space E and $\alpha > 0$, then we say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(E)$ such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \operatorname{Re}\lambda > \omega,\ x \in E,$$

in this case, $S_\alpha(t)$ is called the solution operator generated by A. For more details see [31, 34].

In this article, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [25]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1) If $x : (-\infty, T\,] \to E$ is such that $x_0 \in \mathcal{B}$, then for every $t \in J$, $x_t \in \mathcal{B}$ and
$$\|x(t)\| \leq C\|x_t\|_{\mathcal{B}}, \tag{2.1}$$
where $C > 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that
$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}, \tag{2.2}$$
for $t \in [0, T]$ and x as in (A1).

(A3) The space $\mathcal{B}$ is complete.

Now we state the following lemmas which are necessary to establish our main result.

Let $S_{F,x}$ be a set defined by

$$S_{F,x} = \{v \in L^1(J, E) : v(t) \in F(t, x_t, x(t)) \text{ a.e. } t \in J\}.$$

Lemma 2.7 ([30])**.** *Let E be a Banach space. Let $F : J \times \mathcal{B} \times E \to P_{cp,c}(E)$ be an L^1-Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^1(J, E)$ to $C(J, E)$, then the operator*

$$\begin{aligned} \Psi \circ S_F : C(J, E) &\to P_{cp,c}(C(J, E)), \\ x \quad &\mapsto (\Psi \circ S_F)(x) := \Psi(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

The next result is known as the Bohnenblust-Karlin's fixed point theorem.

Lemma 2.8 ([16])**.** *Let E be a Banach space and $D \in P_{cl,c}(E)$. Suppose that the operator $G : D \to P_{cl,c}(D)$ is upper semicontinuous and the set $G(D)$ is relatively compact in E. Then G has a fixed point in D.*

3. Main results

In this section we shall present and prove our main result. Before going further we need the following lemma [37].

Lemma 3.1. *Consider the Cauchy problem*

$$\begin{gathered} D_t^\alpha x(t) = Ax(t) + F(t), \quad 0 < \alpha < 1, \\ x(0) = x_0, \end{gathered} \tag{3.1}$$

if f satisfies the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem (3.1) *is*

$$x(t) = T_\alpha(t)x_0 + \int_0^t S_\alpha(t-s)F(s)ds,$$

where

$$T_\alpha(t) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda, \quad S_\alpha(t) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda,$$

$\hat{B}_r$ denotes the Bromwich path. $S_\alpha(t)$ is called the α-resolvent family and $T_\alpha(t)$ is the solution operator, generated by A.

Theorem 3.2 ([9, 37])**.** *If $\alpha \in (0,1)$ and $A \in \mathbb{A}^\alpha(\theta_0, \omega_0)$, then for any $x \in E$ and $t > 0$, we have*

$$\|T_\alpha(t)\|_{L(E)} \le Me^{\omega t}, \quad \|S_\alpha(t)\|_{L(E)} \le Ce^{\omega t}(1 + t^{\alpha-1}), \quad t > 0,\ \omega > \omega_0.$$

Let

$$\widetilde{M}_T = \sup_{0\le t\le T}\|T_\alpha(t)\|_{L(E)}, \quad \widetilde{M}_s = \sup_{0\le t\le T} Ce^{\omega t}(1 + t^{\alpha-1}),$$

so we have

$$\|T_\alpha(t)\|_{L(E)} \le \widetilde{M}_T, \quad \|S_\alpha(t)\|_{L(E)} \le t^{\alpha-1}\widetilde{M}_s.$$

Let us consider the set of functions

$$\begin{aligned} \mathcal{B}_1 = \Big\{ & x : (-\infty, T] \to E \text{ such that } x|_{J_k} \in C(J_k, E) \text{ and there exist} \\ & x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-),\ x_0 = \phi, k = 1, 2, \dots, m\Big\}. \end{aligned}$$

Endowed with the seminorm

$$\|x\|_{\mathcal{B}_1} = \sup\{|x(s)| : s \in [0,T]\} + \|\phi\|_{\mathcal{B}},\ x \in \mathcal{B}_1,$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$.

From Lemma 3.1, we can define the mild solution of system (1.1) as follows.

Definition 3.3. A function $x : (-\infty, T] \to E$ is called a mild solution of (1.1)-(1.3) if the following holds: $x_0 = \phi \in \mathcal{B}$ on $(-\infty, 0]$, $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \dots, m$, the restriction of $x(\cdot)$ to the interval J_k, $(k = 0, 1, \dots, m)$ is continuous and there exists $v(\cdot) \in L^1(J_k, E)$, such that $v(t) \in F(t, x_t, x(t))$ a.e. $t \in [0,T]$, and x satisfies the integral equation

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ \int_0^t S_\alpha(t-s)v(s)ds, & t \in [0, t_1]; \\ T_\alpha(t - t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)v(s)ds, & t \in (t_1, t_2]; \\ \dots, \\ T_\alpha(t - t_m)(x(t_m^-) + I_m(x(t_m^-))) + \int_{t_m}^t S_\alpha(t-s)v(s)ds, & t \in (t_m, T]. \end{cases} \tag{3.2}$$

We shall introduce the following hypotheses:

(H1) The semigroup $S_\alpha(t)$ is compact for $t > 0$.

(H2) The multivalued map $F : J \times \mathcal{B} \times E \to E$ is Carathéodory, with compact convex values.

(H3) There exists a function $\mu \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to (0,+\infty)$ such that

$$\|F(t,v,w)\| \le \mu(t)\psi\left(\|v\|_{\mathcal{B}} + \|w\|_E\right), \quad (t,v,w) \in J\times\mathcal{B}\times E.$$

(H4) $I_k : E \to E$ is continuous, and there exists $\Omega > 0$ such that

$$\Omega = \max_{1\le k\le m}\{\|I_k(x)\|,\ x \in D_r\}.$$

Theorem 3.4. *Assume that* (H1)–(H4) *hold. Then problem* (1.1)-(1.3) *has a mild solution on* $(-\infty,T]$.

Proof. We transform problem (1.1) into a fixed-point problem. Consider the multivalued operator $N : \mathcal{B}_1 \to \mathcal{P}(\mathcal{B}_1)$ defined by $N(h) = \{h \in \mathcal{B}_1\}$ with

$$h(t) = \begin{cases} \phi(t), & t \in (-\infty,0];\\ \int_0^t S_\alpha(t-s)v(s)ds, & t\in[0,t_1];\\ T_\alpha(t-t_1)(x(t_1^-)+I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)v(s)ds, & t\in(t_1,t_2];\\ \dots, & \\ T_\alpha(t-t_m)(x(t_m^-)+I_m(x(t_m^-))) & \\ +\int_{t_m}^t S_\alpha(t-s)v(s)ds, \quad v\in S_{F,x}, & t\in(t_m,T]. \end{cases}$$

It is clear that the fixed points of the operator N are mild solutions of problem (1.1). Let us define $y(.) : (-\infty,T] \to E$ as

$$y(t) = \begin{cases} \phi(t), & t\in(-\infty,0];\\ 0, & t\in J. \end{cases}$$

Then $y_0 = \phi$. For each $z \in C(J,E)$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & t\in(-\infty,0];\\ z(t), & t\in J. \end{cases}$$

Let $x_t = y_t + \overline{z}_t, t \in (-\infty,T]$. It is easy to see that $x(.)$ satisfies (3.2) if and only if $z_0 = 0$ and for $t \in J$, we have

$$z(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds, & t\in[0,t_1];\\ T_\alpha(t-t_1)[y(t_1^-)+\overline{z}(t_1^-)+I_1(y(t_1^-)+\overline{z}(t_1^-))] & \\ +\int_{t_1}^t S_\alpha(t-s)v(s)ds, & t\in(t_1,t_2];\\ \dots, & \\ T_\alpha(t-t_m)[y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))] & \\ +\int_{t_m}^t S_\alpha(t-s)v(s)ds, & t\in(t_m,T], \end{cases}$$

where $v(s) \in S_{F,y+\overline{z}}$. Let

$$\mathcal{B}_2 = \{z \in \mathcal{B}_1 : z_0 = 0\}.$$

For any $z \in \mathcal{B}_2$, we have

$$\|z\|_{\mathcal{B}_2} = \sup_{t\in J}\|z(t)\| + \|z_0\|_{\mathcal{B}} = \sup_{t\in J}\|z(t)\|.$$

Thus $(\mathcal{B}_2, \|.\|_{\mathcal{B}_2})$ is a Banach space. We define the operator $P : \mathcal{B}_2 \to \mathcal{P}(\mathcal{B}_2)$ by $P(z) = \{h \in \mathcal{B}_2\}$ with

$$h(t) = \begin{cases} \int_0^t S_\alpha(t-s)v(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1)[y(t_1^-) + \overline{z}(t_1^-) + I_1(y(t_1^-) + \overline{z}(t_1^-))] \\ + \int_{t_1}^t S_\alpha(t-s)v(s)ds, & t \in (t_1, t_2]; \\ \dots, \\ T_\alpha(t-t_m)[y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-) + \overline{z}(t_m^-))] \\ + \int_{t_m}^t S_\alpha(t-s)v(s)ds, & t \in (t_m, T], \end{cases}$$

where $v(s) \in S_{F, y+\overline{z}}$. It is clear that the operator N has a fixed point if and only if P has a fixed point. So let us prove that P has a fixed point. Let

$$D_r = \{z \in \mathcal{B}_2 : z(0) = 0, \|z\|_{\mathcal{B}_2} \le r\},$$

where r is any fixed finite real number which satisfies the inequality

$$r > \widetilde{M}_T(r + \Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha} \psi(C_2^* \|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \int_0^T \mu(s)ds.$$

It is clear that D_r is a closed, convex, bounded set in $\mathcal{B}_2$. We need the following lemma.

Lemma 3.5. *Set*

$$C_1^* = \sup_{t \in J} C_1(t), \quad C_2^* = \sup_{\eta \in J} C_2(\eta). \tag{3.3}$$

Then for any $z \in D_r$ we have

$$\|y_t + \overline{z}_t\|_{\mathcal{B}} \le C_2^* \|\phi\|_{\mathcal{B}} + C_1^* r,$$

Proof. Using (2.2) and (3.3), we obtain

$$\begin{aligned} \|y_t + \overline{z}_t\|_{\mathcal{B}} &\le \|y_t\|_{\mathcal{B}} + \|\overline{z}_t\|_{\mathcal{B}} \\ &\le C_1(t) \sup_{0 \le \tau \le t} \|y(\tau)\| + C_2(t)\|y_0\|_{\mathcal{B}} + C_1(t) \sup_{0 \le \tau \le t} \|z(\tau)\| + C_2(t)\|z_0\|_{\mathcal{B}} \\ &\le C_2(t)\|\phi\|_{\mathcal{B}} + C_1(t) \sup_{0 \le \tau \le t} \|z(\tau)\| \\ &\le C_2^* \|\phi\|_{\mathcal{B}} + C_1^* r. \end{aligned}$$

The proof is complete. □

Now we shall show that P satisfies all the assumptions of Lemma 2.8. The proof will be given in several steps.

Step 1: $P(z)$ is convex for each $z \in \mathcal{B}_2$. Indeed, if h_1 and h_2 belong to $P(z)$, then there exist $v_1, v_2 \in S_{F, y+\overline{z}}$ such that, for $t \in J$ and $i = 1, 2$, we have

$$h_i(t) = \begin{cases} \int_0^t S_\alpha(t-s)v_i(s)ds, & t \in [0, t_1]; \\ T_\alpha(t-t_1)[y(t_1^-) + \overline{z}(t_1^-) + I_1(y(t_1^-) + \overline{z}(t_1^-))] \\ + \int_{t_1}^t S_\alpha(t-s)v_i(s)ds, & t \in (t_1, t_2]; \\ \dots, \\ T_\alpha(t-t_m)[y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-) + \overline{z}(t_m^-))] \\ + \int_{t_m}^t S_\alpha(t-s)v_i(s)ds, & t \in (t_m, T]. \end{cases}$$

Let $d \in [0,1]$. Then for each $t \in [0,t_1]$, we get

$$dh_1(t) + (1-d)h_2(t) = \int_0^t S_\alpha(t-s)[dv_1(s) + (1-d)v_2(s)]ds.$$

Similarly, for any $t \in (t_i, t_{i+1}]$, $i = 1,\dots,m$, we have

$$\begin{aligned} dh_1(t) + (1-d)h_2(t) &= T_\alpha(t-t_i)[y(t_i^-) + \overline{z}(t_i^-) + I_i(y(t_i^-) + \overline{z}(t_i^-))] \\ &\quad + \int_{t_i}^t S_\alpha(t-s)[dv_1(s) + (1-d)v_2(s)]ds \end{aligned}$$

Since F has convex values, $S_{F,y+\overline{z}}$ is convex, we see that

$$dh_1 + (1-d)h_2 \in P(z).$$

Step 2: $P(D_r) \subset D_r$. Let $h \in P(z)$ and $z \in D_r$, for $t \in [0,t_1]$, then by Lemma 3.5, we have

$$\begin{aligned} \|h(t)\| &\le \int_0^t \|S_\alpha(t-s)\|\|v(s)\|ds \\ &\le \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(\tau)\psi(\|y_s + \overline{z}_s\| + \|y(s) + \overline{z}(s)\|)ds \\ &\le \widetilde{M}_S \frac{T^\alpha}{\alpha}\psi(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\int_0^t \mu(s)ds < r. \end{aligned}$$

Moreover, when $t \in (t_i, t_{i+1}], i = 1,\dots,m$, we have the estimate

$$\begin{aligned} \|h(t)\| &\le \|T_\alpha(t-t_i)[z(t_i^-) + I_i(z(t_i^-))]\| + \int_{t_i}^t \|S_\alpha(t-s)\|\|v(s)\|ds \\ &\le \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \int_0^t (t-s)^{\alpha-1}\mu(\tau)\psi(\|y_s + \overline{z}_s\| + \|y(s) + \overline{z}(s)\|)ds \\ &\le \widetilde{M}_T(r+\Omega) + \widetilde{M}_S \frac{T^\alpha}{\alpha}\psi(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\int_{t_i}^T \mu(s)ds < r, \end{aligned}$$

which proves that $P(D_r) \subset D_r$.

Step 3: We will prove that $P(D_r)$ is equicontinuous. Let $\tau_1, \tau_2 \in [0,t_1]$, with $\tau_1 < \tau_2$, we have

$$\begin{aligned} \|h(\tau_2) - h(\tau_1)\| &\le \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|\|v(s)\|ds \\ &\quad + \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\|\|v(s)\|ds \\ &\le Q_1 + Q_2, \end{aligned}$$

where

$$\begin{aligned} Q_1 &= \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|\|v(s)\|ds \\ &\le \psi(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r)\int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|\mu(s)ds. \end{aligned}$$

Since $\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|_{L(E)} \le 2\widetilde{M}_s(t_1 - s)^{\alpha-1}$ which belongs to $L^1(J, \mathbb{R}_+)$ for $s \in [0,t_1]$, and $S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s) \to 0$ as $\tau_1 \to \tau_2, S_\alpha$ is strongly continuous.

This implies that

$$\lim_{\tau_1 \to \tau_2} Q_1 = 0.$$

Where

$$\begin{aligned} Q_2 &= \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\| \|v(s)\| ds \\ &\leq \frac{\widetilde{M}_s(\tau_2 - \tau_1)^\alpha}{\alpha} \psi(C_2^* \|\phi\|_{\mathcal{B}} + (C_1^* + 1)r) \int_{\tau_1}^{\tau_2} \mu(s) ds. \end{aligned}$$

Hence, we deduce that

$$\lim_{\tau_1 \to \tau_2} Q_2 = 0.$$

Similarly, for $\tau_1, \tau_2 \in (t_i, t_{i+1}], i = 1, \dots, m$, we have

$$\begin{aligned} &\|h(\tau_2) - h(\tau_1)\| \\ &\leq \|T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i)\| \left[\|z(t_i^-)\| + \|I_i(z(t_i^-))\| \right] + Q_1 + Q_2 \\ &\leq \|T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i)\| (r + \Omega) + Q_1 + Q_2. \end{aligned}$$

Since T_α is also strongly continuous, so $T_\alpha(\tau_2 - t_i) - T_\alpha(\tau_1 - t_i) \to 0$ as $\tau_1 \to \tau_2$. Thus, from the above inequalities, we have

$$\lim_{\tau_1 \to \tau_2} \|h(\tau_2) - h(\tau_1)\| = 0.$$

So, $P(D_r)$ is equicontinuous.

As a consequence of Steps 1, 2 and 3 with the Arzelá-Ascoli theorem we conclude that $P : \mathcal{B}_2 \to \mathcal{P}(\mathcal{B}_2)$ is completely continuous.

Step 4: P has a closed graph. Suppose that $z_n \to z_*, h_n \in P(z_n)$ with $h_n \to h_*$. We claim that $h_* \in P(z_*)$. In fact, the assumption $h_n \in P(z_n)$ implies that there exists $v_n \in S_{F, y_n + \overline{z}_n}$ such that, for each $t \in [0, t_1]$,

$$h_n(t) = \int_0^t S_\alpha(t - s) v_n(s) ds.$$

We will show that there exists $v_* \in S_{F, z_*}$ such that, for each $t \in [0, t_1]$,

$$h_*(t) = \int_0^t S_\alpha(t - s) v_*(s) ds.$$

Consider the linear continuous operator $\Upsilon : L^1([0, t_1], E) \to C([0, t_1], E)$,

$$v \mapsto (\Upsilon v)(t) = \int_0^t S_\alpha(t - s) v(s) ds.$$

By Lemma 2.7, we know that $\Upsilon o S_F$ is a closed graph operator. Moreover, for every $t \in [0, t_1]$, we obtain

$$h_n(t) \in \Upsilon(S_{F, y_n + \overline{z}_n}).$$

Since $z_n \to z_*$ and $h_n \to h_*$, it follows, that for every $t \in [0, t_1]$,

$$h_*(t) = \int_0^t S_\alpha(t - s) v_*(s) ds,$$

for some $v_* \in S_{F, y_* + \overline{z}_*}$.

Similarly, for any $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$h_n(t) = T_\alpha(t - t_i) \left[y_n(t_i^-) + \overline{z}_n(t_i^-) + I_i(y_n(t_i^-) + \overline{z}_n(t_i^-)) \right]$$

$$+\int_{t_i}^{t} S_\alpha(t-s)v_n(s)ds.$$

We must prove that there exists $v_* \in S_{F,y_*+\overline{z}_*}$ such that, for each $t \in (t_i, t_{i+1}]$,

$$\begin{aligned}h_*(t) &= T_\alpha(t-t_i)\left[y_*(t_i^-) + \overline{z}_*(t_i^-) + I_i(y_*(t_i^-) + \overline{z}_*(t_i^-))\right] \\ &\quad + \int_{t_i}^{t} S_\alpha(t-s)v_*(s)ds.\end{aligned}$$

Now, for every $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$\begin{aligned}&\Big\|\Big(h_n(t) - T_\alpha(t-t_i)\left[y_n(t_i^-) + \overline{z}_n(t_i^-) + I_i(y_n(t_i^-) + \overline{z}_n(t_i^-))\right]\Big) \\ &-\Big(h_*(t) - T_\alpha(t-t_i)\left[y_*(t_i^-) + \overline{z}_*(t_i^-) + I_i(y_*(t_i^-) + \overline{z}_*(t_i^-))\right]\Big)\Big\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Consider the linear continuous operator $\Upsilon : L^1((t_i, t_{i+1}], E) \to C((t_i, t_{i+1}], E)$,

$$v \mapsto (\Upsilon v)(t) = \int_{t_i}^{t} S_\alpha(t-s)v(s)ds.$$

From Lemma 2.7, it follows that $\Upsilon o S_F$ is a closed graph operator. Also, from the definition of Υ, we have that, for every $t \in (t_i, t_{i+1}], i = 1, \dots, m$,

$$\Big(h_n(t) - T_\alpha(t-t_i)\left[y_n(t_i^-) + \overline{z}_n(t_i^-) + I_i(y_n(t_i^-) + \overline{z}_n(t_i^-))\right]\Big) \in \Upsilon(S_{F,y_n+\overline{z}_n}).$$

Since $z_n \to z_*$, for some $v_* \in S_{F,y_*+\overline{z}_*}$ it follows that, for every $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned}h_*(t) &= T_\alpha(t-t_i)\left[y_*(t_i^-) + \overline{z}_*(t_i^-) + I_i(y_*(t_i^-) + \overline{z}_*(t_i^-))\right] \\ &\quad + \int_{t_i}^{t} S_\alpha(t-s)v_*(s)ds.\end{aligned}$$

Hence the multivalued operator P is upper semi-continuous.

It follows from Lemma 2.8 that P has a fixed point $z \in \mathcal{B}_2$. Then the operator N has a fixed point which gives rise to a mild solution to problem (1.1)-(1.3). This completes the proof. □

4. An example

To apply our abstract results, we consider the impulsive fractional integro-differential inclusion

$$\begin{gathered}\frac{\partial_t^q}{\partial t^q}v(t,\zeta) - \frac{\partial^2}{\partial \zeta^2}v(t,\zeta) \in \int_{-\infty}^{0} H(t, v(\theta,\zeta))\eta(t,\theta,\zeta)d\theta \\ v(t,0) = 0, \quad v(t,\pi) = 0 \\ v(t,\zeta) = v_0(\theta,\zeta), \quad -\infty < \theta \le 0 \\ \Delta v(t_k)(\zeta) = \int_{-\infty}^{t_k} p_k(t_k - y)dy\cos(v(t_k)(\zeta))\end{gathered} \tag{4.1}$$

where $0 < q < 1$, $t \in [0,T]$, $\zeta \in [0,\pi]$, $\gamma : (-\infty, 0] \to \mathbb{R}$, $p_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \dots, m$, and $H : [0,T] \times \mathbb{R} \to P(\mathbb{R})$ is an u.s.c. multivalued map with compact convex values.

Set $E = L^2([0,\pi])$, $D(A) \subset E \to E$ is the map defined by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E : \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \quad \forall \omega \in E, \ \textit{and every } t > 0.$$

From these expressions, it follows that $\{T(t)\}_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$; that is, $A \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$. For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ defined by

$$\mathcal{B}_\gamma := \big\{\phi \in C((-\infty, 0], E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\big\}$$

endowed with the norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \ \theta \leq 0\}.$$

Clearly, we can see that $\mathcal{B}_\gamma$ is an admissible phase space which satisfies (A1)–(A3). Set

$$\begin{gathered}
x(t)(\zeta) = v(t, \zeta), \quad t \in [0, T], \zeta \in [0, \pi]; \\
\phi(\theta)(\zeta) = v_0(\theta, \zeta), \quad \theta \in (-\infty, 0], \zeta \in [0, \pi]; \\
F(t, \varphi, x(t))(\zeta) = \int_{-\infty}^{0} H(t, \varphi(\theta)(\zeta))\eta(t, \theta, \zeta)d\theta, \quad t \in [0, T], \zeta \in [0, \pi]; \\
I_k(x(t_k^-))(\zeta) = \int_{-\infty}^{0} p_k(t_k - y)dy \cos(x(t_k)(\zeta)), \quad k = 1, 2, \dots, m.
\end{gathered}$$

Then problem (4.1) can be rewritten in the abstract form (1.1). If conditions (H1)–(H4) are fulfilled, then from Lemma 2.8, system (4.1) has a mild solution on $(-\infty, T]$.

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