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SYMMETRIC POSITIVE SOLUTIONS FOR ϕ-LAPLACIAN BOUNDARY-VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

WENGUI YANG

Abstract. In this article, we study the existence, multiplicity, and nonexistence of symmetric positive solutions for a class of four-order integral boundary value problems with ϕ-Laplacian operator. The arguments mainly rely on the Guo-Krasnosel'skii fixed point theorem of cone expansion and compression of norm type and Leggett-Williams fixed point theorem. Finally, some examples are presented to illustrate the main results.

1. Introduction

In this article, we consider the following ϕ-Laplacian boundary-value problems (BVP) with integral boundary conditions

$$
\begin{gathered}
\big[q(t)[\phi((p(t)u'(t))')]'\big]' = w(t)f(t,u(t)), \quad t\in[a,b],\\
\alpha u(a) - \beta p(a)u'(a) = \int_a^b g(s)u(s)ds,\\
\alpha u(b) + \beta p(b)u'(b) = \int_a^b g(s)u(s)ds,\\
\gamma\phi((p(a)u'(a))') - \delta q(a)[\phi((p(a)u'(a))')]' = \int_a^b h(s)\phi((p(s)u'(s))')ds,\\
\gamma\phi((p(b)u'(b))') + \delta q(b)[\phi((p(b)u'(b))')]' = \int_a^b h(s)\phi((p(s)u'(s))')ds.
\end{gathered}
\tag{1.1}
$$

We use the following assumptions throughout this article:

- (H0) ϕ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0,\infty)$ onto $(0,\infty)$ such that $\psi_1(u)\phi(v)\le\phi(uv)\le\psi_2(u)\phi(v)$ for all $u,v>0$. Moreover, $\phi,\phi^{-1}\in C^1(\mathbb{R})$, where ϕ^{-1} denotes the inverse of ϕ.
- (H1) $\alpha>0$, $\gamma>0$, and $\beta\ge 0$, $\delta\ge 0$.
- (H2) $p(t),q(t):[a,b]\to(0,\infty)$ are continuous functions, and $p(t),q(t)$ are symmetric on $[a,b]$.

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(H3) $w \in L^1[a,b]$ is nonnegative, symmetric on $[a,b]$ and $w(t) \not\equiv 0$ on any subinterval of $[a,b]$.
(H4) $f : [a,b] \times [0,\infty) \to [0,\infty)$ is continuous and $f(\cdot,u)$ is symmetric on $[a,b]$ for all $u \in [0,\infty)$.
(H5) $g, h \in C([a,b],[0,\infty))$ are symmetric functions on $[a,b]$, and $\mu \in [0,\alpha)$, $\nu \in [0,\gamma)$, where

$$\mu = \int_a^b g(s)ds, \quad \nu = \int_a^b h(s)ds.$$

Boundary-value problems with integral boundary conditions have gained considerable popularity and importance due to their application in different areas of applied mathematics and physics and so on. They include two, three, multi-point and nonlocal boundary-value problems as special cases. There have been some papers dealing with the existence and multiplicity of solutions or positive solutions for such problems by the use of some well-known fixed point theorems and upper and lower solutions method. For some recent developments on the subject, see [2, 3, 4, 12, 13, 22, 23, 25, 27] and the references therein.

Recently, many researchers have extensively studied the existence, multiplicity and nonexistence of symmetric positive solutions of boundary-value problems by using fixed point theorem; i.e., fixed point theorem of cone expansion and compression of norm type, fixed point theory in cones and Leggett-Williams fixed point theorem. To identify a few, we refer the reader to [1, 6, 8, 9, 11, 17, 21, 20] and the references therein. In particular, we would like to mention some results of Feng [5], Ma [16], Xu [19], Luo and Luo [15], and Zhang et al. [24], Zhang and Ge [26]. Feng [5] considered the following second-order nonlinear ordinary differential equation with integral boundary conditions

$$\begin{gathered}
(g(t)u'(t))' + w(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\
au(0) - b\lim_{t\to 0^+} g(t)u'(t) = \int_0^1 h(s)u(s)ds, \\
au(1) + b\lim_{t\to 1^-} g(t)u'(t) = \int_0^1 h(s)u(s)ds,
\end{gathered}$$

where $a, b > 0$, $g \in C^1([0,1],(0,\infty))$ is symmetric on $[0,1]$, $w \in L^p[0,1]$ for some $1 \le p \le +\infty$, and is symmetric on $[0,1]$, $f : [0,1] \times [0,+\infty)$ is continuous, $f(1-t,u) = f(t,u)$ for all $(t,u) \in [0,1] \times [0,+\infty)$, and $h \in L^p[0,1]$ is nonnegative and symmetric on $[0,1]$. The author investigated the existence of at least one symmetric positive solution by applying the theory of fixed point index in cones.

Ma [16] obtained the existence of at least one symmetric positive solution for fourth-order boundary-value problem with integral boundary conditions by using the fixed point index in cones.

$$\begin{gathered}
u''''(t) = w(t)f(u(t)), \quad 0 < t < 1, \\
u(0) = u(1) = \int_0^1 p(s)u(s)ds, \quad u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds,
\end{gathered} \tag{1.2}$$

where $p, q \in L^1[0,1]$, $w : (0,1) \to [0,+\infty)$ is continuous, symmetric on $[0,1]$ and may be singular at $t = 0$ and $t = 1$, $f : [0,1] \times [0,+\infty)$ is continuous, $f(1-t,u) = f(t,u)$ for all $(t,u) \in [0,1] \times [0,+\infty)$. Xu [19] studied the existence of three positive solutions for fourth-order singular nonlocal boundary-value problems

$u''''(t) = w(t)f(t,u(t)),\ 0 < t < 1$, with integral boundary conditions (1.2) by using Leggett-Williams fixed point theorem.

Luo and Luo [15] investigated the existence, multiplicity and nonexistence of symmetric positive solutions of the following fourth order boundary value problem with integral boundary conditions

$$\begin{gathered}\phi(u''(t))'' = w(t)f(t,u(t),u'(t)), \quad 0 < t < 1,\\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \quad \phi(u''(0)) = \phi(u''(1)) = \int_0^1 q(s)\phi(u''(s))ds,\end{gathered} \tag{1.3}$$

where ϕ is defined as in (H0), $p, q : [0,1] \to (0,\infty)$ are continuous functions, and p, q are symmetric on $[0,1]$, $f : [0,1]\times[0,+\infty)$ is continuous, $f(1-t,u) = f(t,u)$ for all $(t,u) \in [0,1]\times[0,+\infty)$. Zhang et al. [24] studied the existence and nonexistence of symmetric positive solutions for a special case of fourth order boundary value problem (1.3) when $\phi(t) = \phi_p(t) = |t|^{p-2}t$, $p > 1$, and $f(t,u(t),u'(t)) = f(t,u(t))$.

Zhang and Ge [26] considered the following second-order nonlinear ordinary differential equation with integral boundary conditions

$$\begin{gathered}(q(t)u'''(t))' = w(t)f(t,u(t)), \quad 0 < t < 1,\\ u(0) = u(1) = \int_0^1 g(s)u(s)ds\\ au''(0) - b\lim_{t\to 0^+} g(t)u'''(t) = \int_0^1 h(s)u''(s)ds,\\ au''(1) + b\lim_{t\to 1^-} g(t)u'''(t) = \int_0^1 h(s)u''(s)ds,\end{gathered}$$

where $a, b > 0$, $q \in C^1([0,1],(0,\infty))$ is symmetric on $[0,1]$, $w \in L^p[0,1]$ for some $1 \le p \le +\infty$, and is symmetric on $[0,1]$, $f : [0,1]\times[0,+\infty)$ is continuous, $f(1-t,u) = f(t,u)$ for all $(t,u) \in [0,1]\times[0,+\infty)$, and $g, h \in L^p[0,1]$ is nonnegative and symmetric on $[0,1]$. The authors obtained the existence of at least one symmetric positive solution based upon a specially constructed cone and the fixed point theory in a cone.

Motivated greatly by the above mentioned works, we establish sufficient conditions for the existence, multiplicity and nonexistence of symmetric positive solutions of BVP (1.1) by applying the Guo-Krasnosel'skii fixed point theorem of cone expansion and compression of norm type and Leggett-Williams fixed point theorem. Our paper improves and generalizes the results of mentioned results to some degree. At the end of this paper, some examples are presented to illustrate the main results.

2. Preliminaries

We recall that the function u is said to be concave on $[a,b]$, if $u(\lambda t_2 + (1-\lambda)t_1) \ge \lambda u(t_1) + (1-\lambda)u(t_1)$, $t_1, t_2 \in [a,b]$, $\lambda \in [0,1]$ and the function u is said to be symmetric on $[a,b]$, if $u(b-t+a) = u(t)$, $t \in [a,b]$. A function u is called a symmetric positive solution of (1.1) provided u is symmetric and positive on $[a,b]$, and satisfies the differential equation and the boundary value conditions in (1.1).

To prove the main results in this article, we will use the following lemmas.

Lemma 2.1. *Assume* (H0)–(H2) *hold and* $\mu \neq \alpha$. *Then for any* $v \in C[a,b]$, *the BVP*

$$
\begin{gathered}
(p(t)u'(t))' = \phi^{-1}(v(t)), \quad t \in [a,b], \\
\alpha u(a) - \beta p(a)u'(a) = \int_a^b g(s)u(s)ds, \\
\alpha u(b) + \beta p(b)u'(b) = \int_a^b g(s)u(s)ds,
\end{gathered} \tag{2.1}
$$

has a unique solution u *and* u *can be expressed in the form*

$$
u(t) = -\int_a^b H_1(t,s)\phi^{-1}(v(s))ds, \tag{2.2}
$$

where

$$
H_1(t,s) = G_1(t,s) + \frac{1}{\alpha - \mu}\int_a^b G_1(s,\tau)g(\tau)d\tau, \tag{2.3}
$$

$$
G_1(t,s) = \frac{1}{\Delta_1}\begin{cases} \big(\beta + \alpha\int_a^s \frac{d\tau}{p(\tau)}\big)\big(\beta + \alpha\int_t^b \frac{d\tau}{p(\tau)}\big), & a \le s \le t \le b, \\ \big(\beta + \alpha\int_a^t \frac{d\tau}{p(\tau)}\big)\big(\beta + \alpha\int_s^b \frac{d\tau}{p(\tau)}\big), & a \le t \le s \le b. \end{cases} \tag{2.4}
$$

Here

$$
\Delta_1 = \alpha\Big(2\beta + \alpha\int_a^b \frac{ds}{p(s)}\Big), \quad \mu = \int_a^b g(s)ds.
$$

Proof. First suppose that u is a solution of problem (2.1). It is easy to see by integration of both sides of (2.1) on $[a,t]$ that

$$
p(t)u'(t) = p(a)u'(a) + \int_a^t \phi^{-1}(v(s))ds.
$$

Letting $A = p(a)u'(a)$, then we have

$$
u'(t) = \frac{A}{p(t)} + \frac{1}{p(t)}\int_a^t \phi^{-1}(v(s))ds.
$$

Integrating again, we can get

$$
u(t) = u(a) + A\int_a^t \frac{1}{p(s)}ds + \int_a^t \frac{1}{p(s)}\int_a^s \phi^{-1}(v(\tau))\,d\tau\,ds.
$$

By the boundary condition, we obtain

$$
\begin{aligned}
\alpha u(a) - \beta A &= \int_a^b g(s)u(s)ds, \\
\alpha u(a) + \Big(\alpha\int_a^b \frac{ds}{p(s)} + \beta\Big)A &= \int_a^b g(s)u(s)ds - \beta\int_a^b g(s)\phi^{-1}(v(s))ds \\
&\quad - \alpha\int_a^b \frac{1}{p(s)}\int_a^s \phi^{-1}(v(\tau))\,d\tau\,ds.
\end{aligned}
$$

Then

$$
A = -\frac{1}{2\beta + \alpha\int_a^b \frac{ds}{p(s)}}\Big(\alpha\int_a^b \frac{1}{p(s)}\int_a^s \phi^{-1}(v(\tau))\,d\tau\,ds + \beta\int_a^b g(s)\phi^{-1}(v(s))ds\Big),
$$

and

$$\begin{aligned} u(a) = &-\frac{1}{\Delta_1}\Big(\alpha\int_a^b \frac{1}{p(s)}\int_a^s \phi^{-1}(v(\tau))\,d\tau\,ds + \beta\int_a^b g(s)\phi^{-1}(v(s))ds\Big) \\ &+ \frac{1}{\alpha}\int_a^b g(s)u(s)ds. \end{aligned}$$

Thus

$$\begin{aligned} u(t) &= \frac{1}{\alpha}\int_a^b g(s)u(s)ds + \int_a^t \frac{1}{p(s)}\int_a^s \phi^{-1}(v(\tau))\,d\tau\,ds \\ &\quad - \frac{1}{\Delta_1}\Big(\alpha\int_a^b \frac{1}{p(s)}\int_a^s \phi^{-1}(v(\tau))\,d\tau\,ds + \beta\int_a^b g(s)\phi^{-1}(v(s))ds\Big) \\ &\quad - \frac{\int_a^t \frac{1}{p(s)}ds}{2\beta + \alpha\int_a^b \frac{ds}{p(s)}}\Big(\alpha\int_a^b \frac{1}{p(s)}\int_a^s \phi^{-1}(v(\tau))\,d\tau\,ds + \beta\int_a^b g(s)\phi^{-1}(v(s))ds\Big) \\ &= \frac{1}{\alpha}\int_a^b g(s)u(s)ds - \int_a^b G_1(t,s)\phi^{-1}(v(s))ds, \end{aligned}$$

where $G_1(t,s)$ is defined by (2.4). Multiplying the above equation with $g(t)$ and integrating it again, we obtain

$$\int_a^b g(s)u(s)ds = -\frac{\alpha}{\alpha - \int_a^b g(s)ds}\int_a^b G_1(t,s)\phi^{-1}(v(s))ds.$$

It follows that

$$\begin{aligned} u(t) &= -\int_a^b G_1(t,s)\phi^{-1}(v(s))ds - \frac{1}{\alpha-\mu}\int_a^b G_1(t,s)\phi^{-1}(v(s))ds \\ &= -\int_a^b H_1(t,s)\phi^{-1}(v(s))ds, \end{aligned}$$

where $H_1(t,s)$ is defined in (2.3). The proof is complete. □

Lemma 2.2. *Assume (H1)-(H4) hold and $\nu \neq \gamma$. Then for any $u \in C[a,b]$, the BVP*

$$(q(t)v'(t))' = w(t)f(t,u(t)), \quad t \in [a,b],$$
$$\gamma v(a) - \delta q(a)v'(a) = \int_a^b h(s)v(s)ds, \quad \gamma v(b) + \delta q(b)v'(b) = \int_a^b h(s)v(s)ds,$$

has a unique solution v and v can be expressed in the form

$$v(t) = -\int_a^b H_2(t,s)w(s)f(s,u(s))ds,$$

where

$$H_2(t,s) = G_2(t,s) + \frac{1}{\gamma-\nu}\int_a^b G_2(s,\tau)h(\tau)d\tau, \tag{2.5}$$

$$G_2(t,s) = \frac{1}{\Delta_2}\begin{cases} \big(\delta + \gamma\int_a^s \frac{d\tau}{q(\tau)}\big)\big(\delta + \gamma\int_t^b \frac{d\tau}{q(\tau)}\big), & a \le s \le t \le b, \\ \big(\delta + \gamma\int_a^t \frac{d\tau}{q(\tau)}\big)\big(\delta + \gamma\int_s^b \frac{d\tau}{q(\tau)}\big), & a \le t \le s \le b. \end{cases} \tag{2.6}$$

Here

$$\Delta_2=\gamma\Big(2\delta+\gamma\int_a^b\frac{ds}{q(s)}\Big),\quad \nu=\int_a^b h(s)ds.$$

The above lemma can be proved in a way similar to Lemma 2.1.

Lemma 2.3. *Assume* (H5) *holds. Then for any $t,s\in[a,b]$, the following results are true.*

(i) $G_1(t,s)\ge 0$, $H_1(t,s)\ge 0$, $G_2(t,s)\ge 0$, $H_2(t,s)\ge 0$;

(ii) $\frac{\beta^2}{\Delta_1}\le G_1(t,s)\le G_1(s,s)\le\frac{\Lambda_1}{\Delta_1}$, $\frac{\beta^2\sigma_1}{\Delta_1}\le H_1(t,s)\le H_1(s,s)\le\frac{\sigma_1\Lambda_1}{\Delta_1}$, $\frac{\delta^2}{\Delta_2}\le G_2(t,s)\le G_2(s,s)\le\frac{\Lambda_2}{\Delta_2}$, $\frac{\delta^2\sigma_2}{\Delta_2}\le H_2(t,s)\le H_2(s,s)\le\frac{\sigma_2\Lambda_2}{\Delta_2}$ *with*

$$\Lambda_1=\Big(\beta+\alpha\int_a^b\frac{d\tau}{p(\tau)}\Big)^2,\quad \sigma_1=\frac{\alpha}{\alpha-\mu},$$
$$\Lambda_2=\Big(\delta+\gamma\int_a^b\frac{d\tau}{q(\tau)}\Big)^2,\quad \sigma_2=\frac{\gamma}{\gamma-\nu};$$

(iii) $G_1(b-t+a,b-s+a)=G_1(t,s)$, $H_1(b-t+a,b-s+a)=H_1(t,s)$, $G_2(b-t+a,b-s+a)=G_2(t,s)$, $H_2(b-t+a,b-s+a)=H_2(t,s)$;

(iv) $\rho_1G_1(s,s)\le H_1(t,s)\le\sigma_1G_1(s,s)$, $\rho_2G_2(s,s)\le H_2(t,s)\le\sigma_2G_2(s,s)$ *with*

$$\rho_1=\frac{\Delta_1}{(\alpha-\mu)\Lambda_1}\int_a^b G_1(s,s)g(s)ds,\quad \rho_2=\frac{\Delta_2}{(\gamma-\nu)\Lambda_2}\int_a^b G_1(s,s)h(s)ds;$$

where $H_1(t,s)$, $G_1(t,s)$, $H_2(t,s)$ and $G_2(t,s)$ are defined by (2.3)–(2.6)*, respectively.*

Proof. By simple computations, we have (i) and (ii). Firstly, we prove that (iii) holds. If $a\le t\le s\le b$, then $b-t+a\ge b-s+a$. In view of (2.4) and the assumption (H2), we get

$$\begin{aligned}
&G_1(b-t+a,b-s+a)\\
&=\frac{1}{\Delta_1}\Big(\beta+\alpha\int_a^{b-s+a}\frac{d\tau}{p(\tau)}\Big)\Big(\beta+\alpha\int_{b-t+a}^b\frac{d\tau}{p(\tau)}\Big)\\
&=\frac{1}{\Delta_1}\Big(\beta+\alpha\int_b^s\frac{d(b-\tau+a)}{p(b-\tau+a)}\Big)\Big(\beta+\alpha\int_t^a\frac{d(b-\tau+a)}{p(b-\tau+a)}\Big)\\
&=\frac{1}{\Delta_1}\Big(\beta+\alpha\int_s^b\frac{d\tau}{p(\tau)}\Big)\Big(\beta+\alpha\int_a^t\frac{d\tau}{p(\tau)}\Big)\\
&=G_1(t,s),\quad a\le t\le s\le b.
\end{aligned}$$

Similarly, we can prove that $G_1(b-t+a,b-s+a)=G_1(t,s)$, $a\le s\le t\le b$. So, we have $G_1(b-t+a,b-s+a)=G_1(t,s)$, for any $t,s\in[a,b]$. By (2.3) and (H5), we have

$$\begin{aligned}
&H_1(b-t+a,b-s+a)\\
&=G_1(b-t+a,b-s+a)+\frac{1}{\alpha-\mu}\int_a^b G_1(b-s+a,\tau)g(\tau)d\tau\\
&=G_1(t,s)+\frac{1}{\alpha-\mu}\int_b^a G_1(b-s+a,b-\tau+a)g(b-\tau+a)d(b-a\tau+a)\\
&=G_1(t,s)+\frac{1}{\alpha-\mu}\int_a^b G_1(s,\tau)g(\tau)d\tau=H_1(t,s).
\end{aligned}$$

Similarly, we can prove that $G_2(b-t+a, b-s+a) = G_2(t,s)$, $H_2(b-t+a, b-s+a) = H_2(t,s)$.

Next, we prove that (iv). First, we prove that $\frac{\Delta_1}{\Lambda_1} G_1(t,t)G_1(s,s) \le G_1(t,s)$ and $\frac{\Delta_2}{\Lambda_2} G_2(t,t)G_2(s,s) \le G_2(t,s)$. If $a \le t \le s \le b$, then

$$\frac{G_1(t,s)}{G_1(t,t)G_1(s,s)} = \frac{\Delta_1}{\big(\beta + \alpha \int_a^s \frac{d\tau}{p(\tau)}\big)\big(\beta + \alpha \int_t^b \frac{d\tau}{p(\tau)}\big)} \ge \frac{\Delta_1}{\Lambda_1}.$$

Similarly, we can prove that

$$\frac{G_1(t,s)}{G_1(t,t)G_1(s,s)} \ge \frac{\Delta_1}{\Lambda_1},$$

for $a \le s \le t \le b$. So, we have $\frac{\Delta_1}{\Lambda_1} G_1(t,t)G_1(s,s) \le G_1(t,s)$, for any $t,s \in [a,b]$. From (2.3), then we obtain

$$\begin{aligned} H_1(t,s) &= G_1(t,s) + \frac{1}{\alpha-\mu}\int_a^b G_1(s,\tau)g(\tau)d\tau \\ &\le G_1(s,s) + \frac{1}{\alpha-\mu}\int_a^b g(s)ds G_1(s,s) \\ &= \frac{\alpha}{\alpha-\mu} G_1(s,s) = \sigma_1 G_1(s,s). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} H_1(t,s) &= G_1(t,s) + \frac{1}{\alpha-\mu}\int_a^b G_1(s,\tau)g(\tau)d\tau \\ &\ge \frac{1}{\alpha-\mu}\int_a^b G_1(s,\tau)g(\tau)d\tau \\ &\ge \frac{\Delta_1}{(\alpha-\mu)\Lambda_1}\int_a^b G_1(\tau,\tau)g(\tau)d\tau G_1(s,s) = \rho_1 G_1(s,s). \end{aligned}$$

Hence we get $\rho_1 G_1(s,s) \le H_1(t,s) \le \sigma_1 G_1(s,s)$. Similarly, we can prove that $\rho_2 G_2(s,s) \le H_2(t,s) \le \sigma_2 G_2(s,s)$. $\square$

From Lemmas 2.1 and 2.2, we have the following result.

Lemma 2.4. *Assume* (H0)–(H5) *hold. If u is a solution of* (1.1)*, then*

$$u(t) = \int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds.$$

Now, we let $\mathbb{C} = C[a,b]$, then $\mathbb{C}$ is a Banach space with $\|u\| = \max_{t\in[a,b]} |u(t)|$, and define a cone $\mathbb{P}$ by $\mathbb{P} = \{x \in \mathbb{C} : u(t) \ge 0,\ u''(t) \le 0,\ u(t)$ is a symmetric and concave function on $[a,b]$ and $u(t) \ge \omega\|u\|\}$, where ω is defined as in Lemma 2.6. Also, define, for $0 < r < R$ two positive numbers, Ω_r and $\overline{\Omega}_{r,R}$ by $\Omega_r = \{u \in \mathbb{C} : \|u\| < r\}$ and $\overline{\Omega}_{r,R} = \{u \in \mathbb{C} : r \le u \le R\}$. Note that $\partial\Omega_r = \{u \in \mathbb{C} : \|u\| = r\}$.

Lemma 2.5 ([18])**.** *Assume* (H0) *holds. Then for any $u, v \in (0,\infty)$, we have $\psi_2^{-1}(u)v \le \phi^{-1}(u\phi(v)) \le \psi_1^{-1}(u)v$, where ψ_1^{-1} and ψ_2^{-1} denote the inverse of ψ_1 and ψ_2, respectively.*

Lemma 2.6. *Assume* (H0)–(H5) *hold. Then the solution $u(t)$ of* (1.1) *is positive and symmetric on $[a,b]$ and $\min_{t\in[a,b]} u(t) \ge \omega\|u\|$, where $\omega = \frac{\rho_1\psi_2^{-1}(\rho_2)}{\sigma_1\psi_1^{-1}(\sigma_2)}$.*

Proof. From (H0)–(H5) and Lemma 2.3, it is easy to prove that the solution $u(t)$ of (1.1) is positive on $[a,b]$. We need to prove only that $u(t)$ is symmetric on $[a,b]$. Combining (H3) and (H4) and Lemma 2.3 and 2.4, we obtain

$$
\begin{aligned}
& u(b-t+a) \\
&= \int_a^b H_1(b-t+a,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&= \int_b^a H_1(b-t+a,b-s+a) \\
&\quad \times \phi^{-1}\Big(\int_a^b H_2(b-s+a,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)d(b-s+a) \\
&= \int_b^a H_1(b-t+a,b-s+a)\phi^{-1}\Big(\int_b^a H_2(b-s+a,b-\tau+a) \\
&\times w(b-\tau+a)f(b-\tau+a,u(b-\tau+a))d(b-\tau+a)\Big)d(b-s+a) \\
&= \int_b^a H_1(b-t+a,b-s+a)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)d(b-s+a) \\
&= \int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds = u(t).
\end{aligned}
$$

Therefore, $u(t)$ is symmetric on $[a,b]$. From (H0) and Lemma 2.3—2.5, we obtain

$$
\begin{aligned}
u(t) &= \int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\le \sigma_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\sigma_2\int_a^b G_2(\tau,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\le \sigma_1\psi_1^{-1}(\sigma_2)\int_a^b G_1(s,s)ds\phi^{-1}\Big(\int_a^b G_2(\tau,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big).
\end{aligned}
$$

So we have

$$
\|u(t)\| \le \sigma_1\psi_1^{-1}(\sigma_2)\int_a^b G_1(s,s)ds\phi^{-1}\Big(\int_a^b G_2(\tau,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big). \tag{2.7}
$$

On the other hand, by Lemma 2.3—2.5, we have

$$
\begin{aligned}
u(t) &= \int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds && (2.8)\\
&\ge \rho_1\int_a^b G_1(s,s)\phi^{-1}\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds && (2.9)\\
&\ge \rho_1\psi_2^{-1}(\rho_2)\int_a^b G_1(s,s)ds\phi^{-1}\Big(\int_a^b G_2(\tau,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big) && (2.10)\\
&\ge \frac{\rho_1\psi_2^{-1}(\rho_2)}{\sigma_1\psi_1^{-1}(\sigma_2)}\|u\| = \omega\|u\|. && (2.11)
\end{aligned}
$$

Combined (2.7) with (2.8), we deduce inequality $\min_{t\in[a,b]} u(t) \ge \omega\|u\|$. □

We define the integral operator $T : \mathbb{C} \to \mathbb{C}$ by

$$(Tu)(t) = \int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds. \tag{2.12}$$

Lemma 2.7. *Assume* (H0)–(H5) *hold. Then $u \in \mathbb{C}$ is a solution of* (1.1) *if and only if u is a fixed point of the operator T.*

Lemma 2.8. *Assume* (H0)-(H5) *hold. Then $T : \mathbb{P} \to \mathbb{P}$ is a completely continuous operator.*

The next lemma is the Fixed point theorem of cone expansion and compression of norm type, see [7].

Lemma 2.9. *Let P be a cone of real Bananch space E, Ω_1 and Ω_2 be two bounded open sets in E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let operator $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the two conditions is satisfied.*

(i) $\|Tu\| \le \|u\|$, $\forall u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $\forall u \in P \cap \partial\Omega_2$.
(ii) $\|Tu\| \ge \|u\|$, $\forall u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $\forall u \in P \cap \partial\Omega_2$.

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let Q be a cone in a real Banach space E, a functional $\vartheta : Q \to Q$ is said to be increasing on Q provided $\vartheta(u) \le \vartheta(v)$, for all $u, v \in Q$ with $u \le v$. Let χ be a nonnegative continuous functional on Q in a real Banach space E, we define for each $d > 0$ the following set $Q(\chi, d) = \{u \in Q | \chi(u) < d\}$. Let Q be a cone in a real Banach space E, κ is said to be nonnegative continuous concave on Q provided $\kappa(\lambda u + (1-\lambda)v) \ge \lambda\kappa(u) + (1-\lambda)\kappa(v)$, $u, v \in [a,b]$ with $\lambda \in [0,1]$.

Let $a, b, r > 0$ be constants with Q and κ as defined above, we note

$$Q_r = \{u \in Q : \|u\| < r\}, \quad Q(\kappa, a, b) = \{u \in Q : \kappa(u) \ge a,\ \|u\| \le b\}.$$

Next, we have the Leggett-Williams fixed-point theorem, see [10, 14].

Lemma 2.10. *Assume E be a real Banach space and $Q \subset E$ be a cone. Let $T : \overline{Q}_c \to \overline{Q}_c$ be completely continuous and κ be a nonnegative continuous concave functional on Q such that $\kappa(u) \le \|u\|$, for $u \in \overline{Q}_c$. Suppose that there exist $0 < a < b < d \le c$ such that*

(i) $\{u \in Q(\kappa, b, d) \mid \kappa(u) > b\} \ne \emptyset$ *and* $\kappa(Tu) > b$, *for all* $u \in Q(\kappa, b, d)$.
(ii) $\|Tu\| \le a$, *for all* $\|u\| \le a$.
(iii) $\kappa(Tu) \ge b$, *for all* $u \in Q(\kappa, b, c)$ *with* $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2 and u_3 satisfying

$$\|u_1\| < a, \quad b < \kappa(u_2), \quad a < \|u_3\|, \quad \kappa(u_3) < b.$$

3. Main results

To state the following results, we need to introduce the symbols.

$$f^0 = \limsup_{u\to 0} \max_{t\in[a,b]} \frac{f(t,u)}{\phi(u)}, \quad f^\infty = \limsup_{u\to\infty} \max_{t\in[a,b]} \frac{f(t,u)}{\phi(u)},$$
$$f_0 = \liminf_{u\to 0} \min_{t\in[a,b]} \frac{f(t,u)}{\phi(u)}, \quad f_\infty = \liminf_{u\to\infty} \min_{t\in[a,b]} \frac{f(t,u)}{\phi(u)},$$
$$k_1 = \rho_1 \int_a^b G_1(s,s)ds\psi_2^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big),$$

$$k_2 = \sigma_1 \int_a^b G_1(s,s)ds\psi_1^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big).$$

Theorem 3.1. *Assume* (H0)-(H5) *hold. Furthermore, suppose one of the following conditions are satisfied.*

- (A1) *There exist two constants r and R with $0 < r \leq \frac{k_1}{k_2}R$ such that $f(t,u) \geq \phi\big(\frac{r}{k_1}\big)$ for $(t,u) \in [a,b]\times[0,r]$, and $f(t,u) \leq \phi\left(\frac{r}{k_2}\right)$ for $(t,u) \in [a,b]\times[0,R]$.*
- (A2) $f_0 > \psi_2\Big((\rho_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and* $f^\infty < \psi_1\Big((\sigma_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *(particularly, $f_0 = \infty$ and $f^\infty = 0$).*
- (A3) $f^0 < \psi_1\Big((\sigma_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and* $f_\infty > \psi_2\Big((\rho_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *(particularly, $f^0 = 0$ and $f_\infty = \infty$).*

Then the BVP (1.1) *has at least one symmetric positive solution.*

Proof. Let the operator T be defined by (2.12).

(A1) For $u \in \mathbb{P} \cap \partial\Omega_r$, we have $u \in [0,r]$, which implies $f(t,u) \geq \phi\big(\frac{r}{k_1}\big)$. Hence for $t \in [a,b]$, by Lemma 2.3, we obtain

$$\begin{aligned} u(t) &\geq \rho_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\phi\big(\frac{r}{k_1}\big)\Big)ds \\ &\geq \rho_1 \int_a^b G_1(s,s)ds\psi_2^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)\frac{r}{k_1} = r = \|u\|, \end{aligned}$$

which implies that for $u \in \mathbb{P} \cap \partial\Omega_r$,

$$\|Tu\| \geq \|u\|. \tag{3.1}$$

Next, for $u \in \mathbb{P} \cap \partial\Omega_R$, we have $u \in [0,R]$, which implies $f(t,u) \leq \phi\big(\frac{R}{k_2}\big)$. Hence for $t \in [a,b]$, by Lemma 2.3, we obtain

$$\begin{aligned} u(t) &\leq \sigma_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\phi\big(\frac{R}{k_2}\big)\Big)ds \\ &\leq \sigma_1 \int_a^b G_1(s,s)ds\psi_1^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)\frac{R}{k_2} = R = \|u\|, \end{aligned}$$

which implies that for $u \in \mathbb{P} \cap \partial\Omega_R$

$$\|Tu\| \leq \|u\|. \tag{3.2}$$

(A2) At first, in view of

$$f_0 > \psi_2\Big((\rho_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1},$$

there exists $r > 0$ such that $f(t,u) \geq (f_0 - \varepsilon_1)\phi(\|u\|)$, for $t \in [a,b]$, $\|u\| \in [0,r]$, where $\varepsilon \geq 0$ satisfies

$$\rho_1 \int_a^b G_1(s,s)ds\psi_2^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau(f_0 - \varepsilon_1)\Big) \geq 1.$$

Then, for $t \in [a,b]$, $u \in \mathbb{P} \cap \partial\Omega_r$, which implies $\|u\| \le r$, we have

$$
\begin{aligned}
(Tu)(t) &\ge \int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)(f_0-\varepsilon_1)\phi(\|u\|)d\tau\Big)ds \\
&\ge \rho_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau(f_0-\varepsilon_1)\phi(\|u\|)\Big)ds \\
&\ge \rho_1 \int_a^b G_1(s,s)ds\psi_2^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau(f_0-\varepsilon_1)\Big)\|u\| \ge \|u\|,
\end{aligned}
$$

which implies that for $u \in \mathbb{P} \cap \partial\Omega_r$,

$$
\|Tu\| \ge \|u\|. \tag{3.3}
$$

Next, turning to

$$
f^\infty < \psi_1\Big((\sigma_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1},
$$

there exists $\overline{R} > 0$ large enough such that $f(t,u) \le (f^\infty + \varepsilon_2)\phi(\|u\|)$, for $t \in [a,b]$, $\|u\| \in (\overline{R}, \infty)$, where $\varepsilon_2 > 0$ satisfies

$$
\psi_1\Big((\sigma_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1} - f^\infty - \varepsilon_2 > 0.
$$

Set $M = \max_{\|u\| \le \overline{R}, t\in[a,b]} f(t,u)$. Then $f(t,u) \le M + (f^\infty + \varepsilon_2)\phi(\|u\|)$. Choose

$$
\begin{aligned}
R > \max\Big\{ r, \overline{R}, \phi^{-1}\Big(M\Big[\psi_1\Big((\sigma_1 \int_a^b G_1(s,s)ds)^{-1}\Big) \\
\times \Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1} - f^\infty - \varepsilon_2\Big]\Big)\Big\}.
\end{aligned}
$$

Hence for $u \in \mathbb{P} \cap \partial\Omega_R$, we have

$$
\begin{aligned}
(Tu)(t) &\le \sigma_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)(M + (f^\infty+\varepsilon_2)\phi(\|u\|))d\tau\Big)ds \\
&\le \sigma_1 \int_a^b G_1(s,s)ds\phi^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big(\frac{M}{\phi(R)} + f^\infty + \varepsilon_2\Big)\phi(R)\Big) \\
&\le \sigma_1 \int_a^b G_1(s,s)ds\psi_1^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big(\frac{M}{\phi(R)} + f^\infty + \varepsilon_2\Big)\Big)R \\
&\le R = \|u\|,
\end{aligned}
$$

which implies that for $u \in \mathbb{P} \cap \partial\Omega_R$,

$$
\|Tu\| \le \|u\|. \tag{3.4}
$$

(A3) Considering

$$
f^0 < \psi_1\Big((\sigma_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1},
$$

there exists $r > 0$ such that $f(t,u) \le \eta\phi(u)$, for any $u \in [0,r]$, $t \in [a,b]$, where

$$
\eta \le \psi_1\Big((\sigma_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}.
$$

Then if Ω_r is the ball in $\mathbb{C}$ centered at the origin with radius r and if $u \in \mathbb{P} \cap \partial\Omega_r$, then we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[a,b]} \int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\le \sigma_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)\eta\phi(u(\tau))d\tau\Big)ds \\
&\le \sigma_1 \int_a^b G_1(s,s)ds\phi^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)\eta\phi(r)d\tau\Big) \\
&\le r\sigma_1 \int_a^b G_1(s,s)ds\psi_1^{-1}\Big(\eta\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big) \le r = \|u\|,
\end{aligned}$$

which implies that for $u \in \mathbb{P} \cap \partial\Omega_r$

$$\|Tu\| \le \|u\|. \tag{3.5}$$

On the other hand, we use the assumption

$$f_\infty > \psi_2\Big((\rho_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}.$$

Then there exists $R > 0$ large enough such that $f(t,u) \ge \varrho\phi(u)$ for any $u \in [R,\infty)$, $t \in [a,b]$, where

$$\varrho \ge \psi_2\Big((\rho_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}.$$

If we define $\Omega_R = \{u \in \mathbb{C} : \|u\| < R\}$, for $t \in [a,b]$ and $u \in \mathbb{P} \cap \partial\Omega_R$, we get

$$\begin{aligned}
(Tu)(t) &\ge \rho_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)\varrho\phi(u(\tau))d\tau\Big)ds \\
&\ge \rho_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)\varrho\phi(R)d\tau\Big)ds \\
&\ge R\rho_1 \int_a^b G_1(s,s)ds\psi_2^{-1}\Big(\varrho\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big) \ge r = \|u\|,
\end{aligned}$$

which implies that for $u \in \mathbb{P} \cap \partial\Omega_R$

$$\|Tu\| \ge \|u\|. \tag{3.6}$$

Applying Lemma 2.9 to (3.1) and (3.2), (3.3) and (3.4) or (3.5) and (3.6) yields that T has a fixed point $u \in \mathbb{P} \cap \overline{\Omega}_{r,R}$ with $0 \le r \le \|u\| \le R$. It follows from Lemma 2.9 that problem (1.1) has at least one symmetric positive solution u. ☐

Theorem 3.2. *Assume* (H0)-(H5) *hold. Furthermore, suppose one of the following conditions is satisfied.*

(A4) (i) $f_0 > \psi_2\Big((\rho_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and* $f_\infty > \psi_2\Big((\rho_1 \int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2 \int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *(particularly,* $f_0 = f_\infty = \infty$*).*

(ii) *There exists* $c > 0$ *satisfying* $f(t,u) < \phi\big(\frac{c}{k_2}\big)$, $(t,b) \in [a,b] \times [0,c]$.

(A5) (i) $f^0 < \psi_1\Big((\sigma_1\int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and* $f^\infty < \psi_1\Big((\sigma_1\int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\sigma_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *(particularly,* $f^0 = f^\infty = 0$*).*

(ii) *There exists* $c > 0$ *satisfying* $f(t,u) > \phi\big(\frac{c}{k_1}\big)$, $(t,b) \in [a,b]\times[0,c]$.

Then (1.1) *has at least two symmetric positive solutions* $u_1(t)$ *and* $u_2(t)$*, which satisfy* $0 < \|u_1\| < c < \|u_2\|$.

Proof. (A4) Consider (i). If

$$f_0 > \psi_2\Big((\rho_1\int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1},$$

it follows from the proof of (3.3) that we can choose r with $0 < r < c$ such that

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathbb{P}\cap\partial\Omega_r. \tag{3.7}$$

If

$$f_\infty > \psi_2\Big((\rho_1\int_a^b G_1(s,s)ds)^{-1}\Big)\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1},$$

then as in the proof of (3.6), we can choose R with $c < R$ such that

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathbb{P}\cap\partial\Omega_R. \tag{3.8}$$

Next, for $u \in \mathbb{P}\cap\partial\Omega_c$, we have $u \in [0,c]$, then from (ii), we obtain $f(t,u) < \phi\big(\frac{c}{k_2}\big)$. Thus for $t \in [a,b]$, like in the proof of (3.2), we get

$$\|Tu\| \leq \|u\|, \quad \text{for } u \in \mathbb{P}\cap\partial\Omega_c. \tag{3.9}$$

Applying Lemma 2.9 to (3.7) and (3.9), or (3.2) and (3.9) yields that T has a fixed point $u_1 \in \mathbb{P}\cap\overline{\Omega}_{r,c}$, and a fixed point $u_2 \in \mathbb{P}\cap\overline{\Omega}_{c,R}$. It follows from Lemma 2.7 that problem (1.1) has at least two symmetric positive solutions $u_1(t)$ and $u_2(t)$, which satisfy $0 < \|u_1\| < c < \|u_2\|$.

(A5) It can be proved in a way similar to (A2) and (A3) of Theorem 3.1 and (A4) of Theorem 3.2. The proof is complete. □

Now we define the nonnegative, continuous concave functional $\varphi : \mathbb{P} \to [0,\infty)$ by $\varphi(u) = \min_{t\in[a,b]} u(t)$. Obviously, for every $u \in \mathbb{P}$, we have $\varphi(u) \leq \|u\|$.

Theorem 3.3. *Assume* (H0)–(H5) *hold. In addition, there exist three positive constants* x, y *and* z *with* $0 < x < y < \omega z$ *such that*

(A6) $f(t,u) < \phi\big(\frac{z}{k_2}\big)$*, for all* $t \in [a,b]$*, and* $0 \leq u \leq z$.

(A7) $f(t,u) \geq \phi\big(\frac{y}{k_1}\big)$*, for all* $t \in [a,b]$*, and* $y \leq u \leq \frac{y}{\omega}$.

(A8) $f(t,u) \leq \phi\big(\frac{x}{k_2}\big)$*, for all* $t \in [a,b]$*, and* $0 \leq u \leq x$.

Then (1.1) *has at least three symmetric positive solutions* $u_1(t), u_2(t)$ *and* $u_3(t)$ *such that*

$$\|u_1\| < x, \quad y < \varphi(u_2), \quad x < \|u_3\|, \quad \varphi(u_3) < y.$$

Proof. We show that all the conditions of Lemma 2.10 are satisfied. We first assert that there exists a positive number z such that $T(\overline{Q}_z) \subset \overline{Q}_z$. By (A6), we obtain

$$\begin{aligned}\|Tu\| &= \max_{t\in[a,b]}(Tu)(t)\\ &= \max_{t\in[a,b]}\int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma_1 \int_a^b G_1(s,s)\phi^{-1}\Big(\sigma_2 \int_a^b G_2(\tau,\tau)w(\tau)\phi\big(\frac{z}{k_2}\big)d\tau\Big)ds \\
&\leq \frac{z\sigma_1}{k_2}\int_a^b G_1(s,s)ds\psi_1^{-1}\Big(\sigma_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big) = z.
\end{aligned}
$$

Therefore, we have $T(\overline{Q}_z) \subset \overline{Q}_z$. Especially, if $u \in \overline{Q}_x$, then assumption (A8) yields $T : \overline{Q}_x \to \overline{Q}_x$.

We now show that condition (i) of Lemma 2.10 is satisfied. Clearly, $\{u \in Q(\varphi, y, \frac{y}{\omega}) : \varphi(u) > y\} \neq \emptyset$. Moreover, if $u \in Q(\varphi, y, \frac{y}{\omega})$, then $\varphi(u) \geq y$, so $y \leq \|u\| \leq \frac{y}{\omega}$. By the definition of φ and (A7), we obtain

$$
\begin{aligned}
\|Tu\| &= \min_{t\in[a,b]}(Tu)(t) \\
&= \min_{t\in[a,b]}\int_a^b H_1(t,s)\phi^{-1}\Big(\int_a^b H_2(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\geq \rho_1\int_a^b G_1(s,s)\phi^{-1}\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)\phi\big(\frac{y}{k_1}\big)d\tau\Big)ds \\
&\geq \frac{y\rho_1}{k_1}\int_a^b G_1(s,s)ds\psi_2^{-1}\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big) = y.
\end{aligned}
$$

Therefore, condition (i) of Lemma 2.10 is satisfied.

Finally, we address condition (iii) of Lemma 2.10. For this we choose $u \in Q(\varphi, y, z)$ with $\|Tu\| > \frac{y}{\omega}$. Then from Lemma 2.6, we deduce

$$
\varphi(Tu) = \min_{t\in[a,b]}(Tu)(t) \geq \omega\|Tu\| > y.
$$

Hence, (iii) of Lemma 2.10 holds. By Lemma 2.10, then we obtain that (1.1) has at least three symmetric positive solutions $u_1(t), u_2(t)$ and $u_3(t)$ such that

$$
\|u_1\| < x, \quad y < \varphi(u_2), \quad x < \|u_3\|, \quad \varphi(u_3) < y.
$$

The proof is complete. □

Theorem 3.4. *Assume* (H0)–(H5) *hold. Furthermore, suppose one of the following conditions are satisfied:*

- (A9) $f(t,u) > \phi\big(\frac{\|u\|}{k_1}\big)$ *for all* $t \in [a,b], u \in [0,\infty)$.
- (A10) $f(t,u) < \phi\big(\frac{\|u\|}{k_2}\big)$ *for all* $t \in [a,b], u \in [0,\infty)$.

Then (1.1) *has no positive solution.*

Proof. Assume $u(t)$ is a positive solution of (1.1), we have

$$
\begin{aligned}
\|u\| &= \|Tu\| \geq (Tu)(t) \\
&> \rho_1\int_a^b G_1(s,s)\phi^{-1}\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)\phi\big(\frac{\|u\|}{k_1}\big)d\tau\Big)ds \\
&\geq \frac{\|u\|\rho_1}{k_1}\int_a^b G_1(s,s)ds\psi_2^{-1}\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big) = \|u\|.
\end{aligned}
$$

which is a contradiction. So, due to (A9), equation (1.1) has no positive solution. Similarly, due to (A9), we obtain that (1.1) has no positive solution. □

4. Further remarks

Consider the fourth-order integral BVP with with ϕ-Laplacian operator

$$
\begin{gathered}
\big[q(t)\phi(p(t)u''(t))\big]'' = w(t)f(t,u(t)), \quad t\in[a,b],\\
\alpha u(a)-\beta u'(a)=\int_a^b g(s)u(s)ds, \quad \alpha u(b)+\beta u'(b)=\int_a^b g(s)u(s)ds,\\
\gamma q(a)\phi(p(a)u''(a))-\delta[q(a)\phi(p(a)u''(a))]'=\int_a^b h(s)q(s)\phi(p(s)u''(s))ds,\\
\gamma q(b)\phi(p(b)u''(b))+\delta[q(b)\phi(p(b)u''(b))]'=\int_a^b h(s)q(s)\phi(p(s)u''(s))ds.
\end{gathered}
\tag{4.1}
$$

By methods analogous to the ones abov, we have the following results.

Lemma 4.1. *Assume* (H0)–(H2) *hold and $\mu\neq\alpha$. Then for any $v\in C[a,b]$, the BVP*

$$
\begin{gathered}
q(t)\phi(p(t)u''(t)) = v(t), \quad t\in[a,b],\\
\alpha u(a)-\beta u'(a)=\int_a^b g(s)u(s)ds, \quad \alpha u(b)+\beta u'(b)=\int_a^b g(s)u(s)ds,
\end{gathered}
\tag{4.2}
$$

has a unique solution u and u can be expressed in the form

$$
u(t)=-\int_a^b H_1^*(t,s)\frac{1}{p(s)}\phi^{-1}\Big(\frac{v(s)}{q(s)}\Big)ds, \tag{4.3}
$$

where

$$
H_1^*(t,s)=G_1^*(t,s)+\frac{1}{\alpha-\mu}\int_a^b G_1^*(s,\tau)g(\tau)d\tau, \tag{4.4}
$$

$$
G_1^*(t,s)=\frac{1}{\Delta_1^*}\begin{cases}[\beta+\alpha(s-a)][\beta+\alpha(b-t)], & a\le s\le t\le b,\\ [\beta+\alpha(t-a)][\beta+\alpha(b-s)], & a\le t\le s\le b.\end{cases} \tag{4.5}
$$

Here

$$
\Delta_1^*=\alpha[2\beta+\alpha(b-a)], \quad \mu=\int_a^b g(s)ds.
$$

Lemma 4.2. *Assume* (H1)–(H4) *hold and $\nu\neq\gamma$. Then for any $u\in C[a,b]$, the BVP*

$$
\begin{gathered}
v''(t)=w(t)f(t,u(t)), \quad t\in[a,b],\\
\gamma v(a)-\delta v'(a)=\int_a^b h(s)v(s)ds, \quad \gamma v(b)+\delta v'(b)=\int_a^b h(s)v(s)ds,
\end{gathered}
$$

has a unique solution v that can be expressed as

$$
v(t)=-\int_a^b H_2^*(t,s)w(s)f(s,u(s))ds,
$$

where

$$
H_2^*(t,s)=G_2^*(t,s)+\frac{1}{\gamma-\nu}\int_a^b G_2^*(s,\tau)h(\tau)d\tau,
$$

$$
G_2^*(t,s)=\frac{1}{\Delta_2^*}\begin{cases}[\delta+\gamma(s-a)][\delta+\gamma(b-t)], & a\le s\le t\le b,\\ [\delta+\gamma(t-a)][\delta+\gamma(b-s)], & a\le t\le s\le b.\end{cases}
$$

Here

$$\Delta_2^* = \gamma[2\delta + \gamma(b-a)], \quad \nu = \int_a^b h(s)ds.$$

Lemma 4.3. *Assume* (H5) *holds. Then for any $t, s \in [a, b]$, the following results are true.*

(i) $G_1^*(t,s) \ge 0$, $H_1^*(t,s) \ge 0$, $G_2^*(t,s) \ge 0$, $H_2^*(t,s) \ge 0$;

(ii) $\frac{\beta^2}{\Delta_1^*} \le G_1^*(t,s) \le G_1^*(s,s) \le \frac{\Lambda_1^*}{\Delta_1^*}$, $\frac{\beta^2\sigma_1}{\Delta_1^*} \le H_1^*(t,s) \le H_1^*(s,s) \le \frac{\sigma_1\Lambda_1^*}{\Delta_1^*}$, $\frac{\delta^2}{\Delta_2^*} \le G_2^*(t,s) \le G_2^*(s,s) \le \frac{\Lambda_2^*}{\Delta_2^*}$, $\frac{\delta^2\sigma_2}{\Delta_2^*} \le H_2^*(t,s) \le H_2^*(s,s) \le \frac{\sigma_2\Lambda_2^*}{\Delta_2^*}$ *with*

$$\Lambda_1^* = [\beta + \alpha(b-a)]^2, \quad \sigma_1 = \frac{\alpha}{\alpha - \mu}, \quad \Lambda_2^* = [\delta + \gamma(b-a)]^2, \quad \sigma_2 = \frac{\gamma}{\gamma - \nu};$$

(iii) $G_1^*(b-t+a, b-s+a) = G_1^*(t,s)$, $H_1^*(b-t+a, b-s+a) = H_1^*(t,s)$, $G_2^*(b-t+a, b-s+a) = G_2^*(t,s)$, $H_2^*(b-t+a, b-s+a) = H_2^*(t,s)$;

(iv) $\rho_1 G_1^*(s,s) \le H_1^*(t,s) \le \sigma_1 G_1^*(s,s)$, $\rho_2 G_2^*(s,s) \le H_2^*(t,s) \le \sigma_2 G_2^*(s,s)$ *with*

$$\rho_1^* = \frac{\Delta_1^*}{(\alpha-\mu)\Lambda_1^*}\int_a^b G_1^*(s,s)g(s)ds, \quad \rho_2^* = \frac{\Delta_2^*}{(\gamma-\nu)\Lambda_2^*}\int_a^b G_1^*(s,s)h(s)ds;$$

where $H_1^*(t,s)$, $G_1^*(t,s)$, $H_2^*(t,s)$ and $G_2^*(t,s)$ are defined by (4.4)–(4.2), *respectively.*

Lemma 4.4. *Assume* (H0)–(H5) *hold. If u is a solution of* (4.1), *then*

$$u(t) = \int_a^b H_1^*(t,s)\phi^{-1}\Big(\int_a^b H_2^*(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds.$$

Lemma 4.5. *Assume* (H0)-(H5) *hold. the solution $u(t)$ of* (4.1) *is positive and symmetric on $[a,b]$ and $\min_{t\in[a,b]} u(t) \ge \omega^*\|u\|$, where $\omega^* = \frac{\rho_1^*\psi_2^{-1}(\rho_2^*)}{\sigma_1\psi_1^{-1}(\sigma_2)}$.*

Let $\mathbb{P}^* = \{x \in \mathbb{C} : u(t) \ge 0,\ u''(t) \le 0,\ u(t)$ is a symmetric and concave function on $[a,b]$ and $u(t) \ge \omega^*\|u\|\}$. We introduce the integral operator $T^* : \mathbb{C} \to \mathbb{C}$ by

$$(T^*u)(t) = \int_a^b H_1^*(t,s)\phi^{-1}\Big(\int_a^b H_2^*(s,\tau)w(\tau)f(\tau,u(\tau))d\tau\Big)ds. \tag{4.6}$$

Lemma 4.6. *Assume* (H0)–(H5) *hold. Then $u \in \mathbb{C}$ is a solution of* (1.1) *if and only if u is a fixed point of the operator T^*.*

Lemma 4.7. *Assume* (H0)-(H5) *hold. Then $T^* : \mathbb{P} \to \mathbb{P}$ is a completely continuous operator.*

Now we need to introduce new notation.

$$k_1^* = \rho_1^*\int_a^b G_1^*(s,s)ds\psi_2^{-1}\Big(\rho_2^*\int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big),$$
$$k_2^* = \sigma_1\int_a^b G_1^*(s,s)ds\psi_1^{-1}\Big(\sigma_2\int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big).$$

Theorem 4.8. *Assume* (H0)–(H5) *hold. Furthermore, suppose one of the following conditions are satisfied.*

(A1*) *There exist two constants r and R with $0 < r \le \frac{k_1^*}{k_2^*}R$ such that $f(t,u) \ge \phi\big(\frac{r}{k_1^*}\big)$ for $(t,u) \in [a,b]\times[0,r]$, and $f(t,u) \le \phi\big(\frac{r}{k_2^*}\big)$ for $(t,u) \in [a,b]\times[0,R]$.*

(A2*) $f_0 > \psi_2\Big((\rho_1^* \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\rho_2^* \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and*
$f^\infty < \psi_1\Big((\sigma_1 \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$
(particularly, $f_0 = \infty$ and $f^\infty = 0$).

(A3*) $f^0 < \psi_1\Big((\sigma_1 \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and*
$f_\infty > \psi_2\Big((\rho_1^* \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\rho_2^* \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$
(particularly, $f^0 = 0$ and $f_\infty = \infty$).

Then (4.1) *has at least one symmetric positive solution.*

Theorem 4.9. *Assume* (H0)–(H5) *hold. Furthermore, suppose one of the following conditions are satisfied.*

(A4*) (i) $f_0 > \psi_2\Big((\rho_1^* \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\rho_2^* \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and*
$f_\infty > \psi_2\Big((\rho_1^* \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\rho_2^* \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$
(particularly, $f_0 = f_\infty = \infty$).
(ii) *There exists $c > 0$ satisfying $f(t,u) < \phi\big(\frac{c}{k_2^*}\big)$, $(t,b) \in [a,b] \times [0,c]$.*

(A5*) (i) $f^0 < \psi_1\Big((\sigma_1 \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$ *and*
$f^\infty < \psi_1\Big((\sigma_1 \int_a^b G_1^*(s,s)ds)^{-1}\Big)\Big(\sigma_2 \int_a^b G_2^*(\tau,\tau)w(\tau)d\tau\Big)^{-1}$
(particularly, $f^0 = f^\infty = 0$).
(ii) *There exists $c > 0$ satisfying $f(t,u) > \phi\Big(\frac{c}{k_1^*}\Big)$, $(t,b) \in [a,b] \times [0,c]$.*

Then (4.1) *has at least two symmetric positive solutions $u_1(t)$ and $u_2(t)$, which satisfy* $0 < \|u_1\| < c < \|u_2\|$.

Theorem 4.10. *Assume* (H0)–(H5) *hold. In addition, there exist three positive constants x, y and z with $0 < x < y < \omega^* z$ such that*

(A6*) $f(t,u) < \phi\big(\frac{z}{k_2^*}\big)$, *for all $t \in [a,b]$, and $0 \le u \le z$.*
(A7*) $f(t,u) \ge \phi\big(\frac{y}{k_1^*}\big)$, *for all $t \in [a,b]$, and $y \le u \le \frac{y}{\omega^*}$.*
(A8*) $f(t,u) \le \phi\big(\frac{x}{k_2^*}\big)$, *for all $t \in [a,b]$, and $0 \le u \le x$.*

Then (4.1) *has at least three symmetric positive solutions $u_1(t), u_2(t)$ and $u_3(t)$ such that*

$$\|u_1\| < x, \quad y < \varphi(u_2), \quad x < \|u_3\|, \quad \varphi(u_3) < y.$$

Theorem 4.11. *Assume* (H0)-(H5) *hold. Furthermore, suppose one of the following conditions are satisfied.*

(A9*) $f(t,u) > \phi\big(\frac{\|u\|}{k_1^*}\big)$ *for all $t \in [a,b], u \in [0,\infty)$.*
(A10*) $f(t,u) < \phi\big(\frac{\|u\|}{k_2^*}\big)$ *for all $t \in [a,b], u \in [0,\infty)$.*

Then (4.1) *has no positive solution.*

Remark 4.12. ϕ defined as in (H0) generalizes the projection $\varphi : \mathbb{R} \to \mathbb{R}$ which is an increasing homeomorphism and homomorphism with $\varphi(0) = 0$. A projection $\varphi : \mathbb{R} \to \mathbb{R}$, which generates the p-Laplacian operator $\varphi_p(u) = |u|^{p-2}u$ for $p > 1$, is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:

(i) If $x \le y$, then $\varphi(x) \le \varphi(y)$, for all $x, y \in \mathbb{R}$;
(ii) φ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in \mathbb{R}$. If ϕ is replaced by φ, so all of our results also hold.

5. Examples

Example 5.1. Consider the following fourth-order integral BVP with ϕ-Laplacian operator

$$\begin{gathered}
\Big[\frac{1}{1+\sin(\pi t)}\Big[\phi\Big(\big(\frac{u'(t)}{1+\sin(\pi t)}\big)'\Big)\Big]'\Big]' = w(t)f(t,u(t)), \quad t \in [0,1], \\
u(0) - u'(0) = \frac{1}{2}\int_0^1 u(s)ds, \quad u(1) + u'(1) = \frac{1}{2}\int_0^1 u(s)ds, \\
\phi((u'(0))') - [\phi((u'(0))')]' = \frac{1}{2}\int_0^1 \phi\Big(\big(\frac{u'(s)}{1+\sin(\pi s)}\big)'\Big)ds, \\
\phi((u'(1))') + [\phi((u'(1))')]' = \frac{1}{2}\int_0^1 \phi\Big(\big(\frac{u'(s)}{1+\sin(\pi s)}\big)'\Big)ds,
\end{gathered} \tag{5.1}$$

where $\phi(u) = |u|u$, $w(t) = 1/100$ and $f(t,u) = (1+\sin(\pi t))(1+u)$ for $(t,u) \in [0,1]\times[0,\infty)$.

Let $\psi_1(u) = \psi_2(u) = u^2$, $u > 0$. Then, by calculations we obtain that $\mu = \nu = 1/2$, $\Delta = \Delta_1 = \Delta_2 = 3 + \frac{2}{\pi}$,

$$\begin{aligned}
&G_1(t,s) = G_2(t,s) \\
&= \frac{1}{\Delta}\begin{cases} \big(1+\frac{1}{\pi}+s-\frac{1}{\pi}\cos(\pi s)\big)\big(2+\frac{1}{\pi}-t+\frac{1}{\pi}\cos(\pi t)\big), & 0 \le s \le t \le 1, \\ \big(1+\frac{1}{\pi}+t-\frac{1}{\pi}\cos(\pi t)\big)\big(2+\frac{1}{\pi}-s+\frac{1}{\pi}\cos(\pi s)\big), & 0 \le t \le s \le 1, \end{cases}
\end{aligned}$$

$$\begin{gathered}
\rho_1 = \rho_2 = \frac{(2+3\pi)(-24+3\pi+18\pi^2+13\pi^3)}{24\pi^2(1+\pi)^2} \approx 1.59199, \quad \sigma_1 = \sigma_2 = 2, \\
\omega \approx 0.710176, \quad k_1 = \frac{(2+3\pi)^{\frac{3}{2}}(-24+3\pi+18\pi^2+13\pi^3)^3}{17280\pi^{\frac{15}{2}}(1+\pi)^3} \approx 1.06639, \\
k_2 = \frac{1}{10\pi^{\frac{3}{2}}}\Big(\frac{13}{3} - \frac{8}{\pi^3} + \frac{1}{\pi^2} + \frac{6}{\pi}\Big)\sqrt{\frac{1}{3}(-24+3\pi+18\pi^2+13\pi^3)} \approx 1.50159.
\end{gathered}$$

Clearly, the conditions (H0)–(H5) hold. Next, we prove that the condition (A1) of Theorem 3.1 is satisfied. In fact, choosing $r = 1$ and $R = 6$, we have $r < \frac{k_1}{k_2}R$, $\phi\big(\frac{r}{k_1}\big) \approx 0.879356$, $\phi\big(\frac{R}{k_2}\big) \approx 15.9661$. For $(t,u) \in [0,1]\times[0,1]$, then $f(t,u) = (1+\sin(\pi t))(1+u) \ge 1 > 0.879356 \approx \phi\big(\frac{r}{k_1}\big)$.

For $(t,u) \in [0,1]\times[0,10]$, then $f(t,u) = (1+\sin(\pi t))(1+u) \le 14 < 15.9661 \approx \phi\big(\frac{R}{k_2}\big)$. Hence, by (A1) of Theorem 3.1, then BVP (5.1) has at least one symmetric positive solution.

Example 5.2. Consider the following fourth-order integral BVP with ϕ-Laplacian operator

$$\begin{gathered}
\Big[\frac{1}{1+t(1-t)}\Big[\phi\Big(\big(\frac{u'(t)}{1+t(1-t)}\big)'\Big)\Big]'\Big]' = w(t)f(t,u(t)), \quad t\in[0,1],\\
u(0)-u'(0)=\int_0^1 su(s)ds, \quad u(1)+u'(1)=\int_0^1 su(s)ds,\\
\phi((u'(0))')-[\phi((u'(0))')]' = \int_0^1 s\phi\Big(\big(\frac{u'(s)}{1+\sin(\pi s)}\big)'\Big)ds,\\
\phi((u'(1))')+[\phi((u'(1))')]' = \int_0^1 s\phi\Big(\big(\frac{u'(s)}{1+\sin(\pi s)}\big)'\Big)ds,
\end{gathered} \tag{5.2}$$

where $\phi(u)=u$, $w(t)=t^2(1-t)^2$ and $f(t,u)=\big((t-\frac12)^2+1\big)\big(\frac12+u\big)\big(\frac14+u^2\big)$ for $(t,u)\in[0,1]\times[0,\infty)$.

Let $\psi_1(u)=\psi_2(u)=u$, $u>0$. Then, by calculations we obtain that $\mu=\nu=1/2$, $\Delta=\Delta_1=\Delta_2=19/6$,

$$\begin{aligned}
&G_1(t,s)=G_2(t,s)\\
&=\frac{1}{\Delta}\begin{cases}\big(1-\frac{s}{6}(2s^2-3s-6)\big)\big(\frac{13}{6}+\frac{t}{6}(2t^2-3t-6)\big), & 0\le s\le t\le 1,\\ \big(1-\frac{t}{6}(2t^2-3t-6)\big)\big(\frac{13}{6}+\frac{s}{6}(2s^2-3s-6)\big), & 0\le t\le s\le 1,\end{cases}
\end{aligned}$$

$$\begin{gathered}
\rho_1=\rho_2=\frac{1745663}{1171170}\approx 1.49053, \quad \sigma_1=\sigma_2=2, \quad \omega=\frac{3047339309569}{5486556675600}\approx 0.555419,\\
k_1=\frac{263200328806935685377871}{711426606363060588000000}\approx 0.369961, \quad k_2=\frac{86370535759}{129667230000}\approx 0.666094.
\end{gathered}$$

Clearly, the conditions (H0)-(H5) hold. Next, we prove that the condition (A4) of Theorem 3.2 is satisfied.

$$f_0=f_\infty=\infty>2.70299\approx\psi_2\Big(\big(\rho_1\int_a^b G_1(s,s)ds\big)^{-1}\Big)\Big(\rho_2\int_a^b G_2(\tau,\tau)w(\tau)d\tau\Big)^{-1}.$$

On the other hand, choosing $c=\frac12$, for $(t,u)\in[0,1]\times[0,c]$, we have $f(t,u)\le f(0,c)=0.625<0.750645\approx\phi\big(\frac{c}{k_2}\big)$. Hence, by (A4) of Theorem 3.2, BVP (5.2) has at least two symmetric positive solutions u_1 and u_2 satisfying $0<\|u_1\|<\frac12<\|u_2\|$.

Example 5.3. Consider the BVP (5.2) with

$$f(t,u)=\begin{cases}\frac{1}{t(1-t)+79u}+8u^3, & 0\le t\le 1,\ u\le 1,\\ \frac{u}{t(1-t)+79u}+u+7, & 0\le t\le 1,\ 1<u,\end{cases} \tag{5.3}$$

and other conditions also hold. Choosing $x=1/8$, $y=2$ and $z=20$, then $0<x<y<\omega z$. Now, we can verify the validity of conditions (A6)-(A8) in Theorem 3.3. Indeed, by direct computations, we have

$$f(t,u)\le\frac{1}{79}+20+7\approx 27.0127<30.0258\approx\phi\big(\frac{z}{k_2}\big),$$

for all $t\in[0,1]$, and $0\le u\le z$.

$$f(t,u)\ge\frac{1}{80}+2+7=9.0125>5.40598\approx\phi\big(\frac{y}{k_1}\big),$$

for all $t \in [0,1]$, and $y \le u \le \frac{y}{\omega}$.

$$f(t,u) \le \frac{1}{79} + \frac{8}{8^3} \approx 0.0282832 < 0.187661 \approx \phi(\frac{x}{k_2}),$$

for all $t \in [0,1]$, and $0 \le u \le x$. Thus, according to Theorem 3.3, BVP (5.2) with (5.3) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\| < \frac{1}{8}, \quad 2 < \varphi(u_2), \quad \frac{1}{8} < \|u_3\|, \quad \varphi(u_3) < 2.$$

References

[1] R. I. Avery, J. Henderson; *Three symmetric positive solutions for a second-order boundary value problem*, Appl. Math. Lett. 13 (2000) 1-7.

[2] A. Boucherif; *Second-order boundary value problems with integral boundary conditions*, Nonlinear Anal. 70 (2009) 364-371.

[3] Y. Ding; *Positive solutions for integral boundary value problem with ϕ-Laplacian operator*, Bound. Value Probl. 2011 (2011), Art. ID 827510, 15 pages.

[4] W. Fan, X. Hao, L. Liu, Y. Wu; *Nontrivial solutions of singular fourth-order Sturm-Liouville boundary value problems with a sign-changing nonlinear term*, Appl. Math. Comput. 217 (2011), no. 15. 6700-6708.

[5] M. Feng; *Existence of symmetric positive solutions for a boundary value problem with integral boundary conditions*, Appl. Math. Lett. 24 (2011) 1419-1427.

[6] H. Feng, H. Pang, W. Ge; *Multiplicity of symmetric positive solutions for a multipoint boundary value problem with a one-dimensional p-Laplacian*, Nonlinear Anal. 69 (2008) 3050-3059.

[7] D. Guo, V. Lashmikanthan; *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[8] J.R. Graef, L. Kong; *Necessary and sufficient conditions for the existence of symmetric positive solutions of singular boundary value problems*, J. Math. Anal. Appl. 331 (2007) 1467-1484.

[9] J.R. Graef, L. Kong; *Necessary and sufficient conditions for the existence of symmetric positive solutions of multi-point boundary value problems*, Nonlinear Anal. 68 (2008) 1529-1552.

[10] J. Henderson and H.B. Thompson; *Multiple symmetric positive solutions for second order boundary value problems*, Proc. Amer. Math. Soc. 128 (2000) 2373-2379.

[11] D. Ji, W. Ge, Y. Yang; *The existence of symmetric positive solutions for Sturm-Liouville-like four-point boundary value problem with a p-Laplacian operator*, Appl. Math. Comput. 189 (2007) 1087-1098.

[12] P. Kang, Z. Wei, J. Xu; *Positive solutions to fourth-order singular boundary value problems with integral boundary conditions in abstract spaces*, Appl. Math. Comput. 206 (2008) 245-256.

[13] L. Kong; *Second order singular boundary value problems with integral boundary conditions*, Nonlinear Anal. 72 (2010) 2628-2638.

[14] R.W. Legget, R.L. Williams; *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana. Univ. Math. J. 28 (1979) 673-688.

[15] Y. Luo, Z. Luo; *Symmetric positive solutions for nonlinear boundary value problems with ϕ-Laplacian operator*, Appl. Math. Lett. 23 (2010) 657-664.

[16] H. Ma; *Symmetric positive solutions for nonlocal boundary value problems of fourth order*, Nonlinear Analysis 68 (2008) 645-651.

[17] D. Ma, W. Ge; *Existence and iteration of positive pseudo-symmetric solutions for a three-point second-order p-Laplacian BVP*, Appl. Math. Lett. 20 (2007) 1244-1249.

[18] H. Wang; *On the number of positive solutions of nonlinear systems*, J. Math. Anal. Appl. 281 (2003) 287-306.

[19] F. Xu; *Three symmetric positive solutions of fourth-order nonlocal boundary value problems*, Electron J. Qual. Theory Differ. Equ. 2011 (2011), No. 96, 1-11.

[20] J. Yang, Z. Wei, K. Liu; *Existence of symmetric positive solutions for a class of Sturm-Liouville-like boundary value problems*, Appl. Math. Comput. 214 (2009) 424-432.

[21] C. Zhai, R. Song, Q. Han; *The existence and the uniqueness of symmetric positive solutions for a fourth-order boundary value problem*, Comput. Math. Appl. 62 (2011) 2639-2647.

[22] X. Zhang; *Positive solutions for singular fourth-order integral boundary-value problem with p-Laplacian operator*, J. Math. Res. Exp. 30 (2010) 495-506.

[23] X. Zhang, M. Feng, W. Ge; *Existence results for nonlinear boundary-value problems with integral boundary conditions in Banach spaces*, Nonl. Anal. 69 (2008) 3310-3321.

[24] X. Zhang, M. Feng, W. Ge; *Symmetric positive solutions for p-Laplacian fourth-order differential equations with integral boundary conditions*, J. Comput. Appl. Math. 222 (2008) 561-573.

[25] X. Zhang, W. Ge; *Positive solutions for a class of boundary-value problems with integral boundary conditions*, Comput. Math. Appl. 58 (2009) 203-215.

[26] X. Zhang, W. Ge; *Symmetric positive solutions of boundary value problems with integral boundary conditions*, Appl. Math. Comput. 219 (2012) 3553-3564.

[27] X. Zhang, L. Liu; *Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator*, J. Math. Anal. Appl. 336 (2007) 1414-1423.

Wengui Yang

Ministry of Public Education, Sanmenxia Polytechnic, Sanmenxia 472000, China

E-mail address: wgyang0617@yahoo.com