Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 269, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

VALUE DISTRIBUTION OF DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS

YONG LIU, XIAOGUANG QI, HONGXUN YI

Abstract. In this article, we study the value distribution of difference polynomials of meromorphic functions, and obtain some results which can be viewed as discrete analogues of the results given by Yi and Yang [11]. We also consider the value distribution of

$$\varphi(z) = f(z)(f(z) - 1)\prod_{j=1}^{n} f(z + c_j).$$

1. Introduction and main results

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna theory (see, e.g., [7, 10]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of finite logarithmic measure. Then the meromorphic function α is called a small function of $f(z)$, if $T(r, \alpha) = S(r, f)$. If $f(z) - \alpha$ and $g(z) - \alpha$ have same zeros, counting multiplicity (ignoring multiplicity), then we say $f(z)$ and $g(z)$ share the small function α CM (IM). Denote

$$\delta(\alpha, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-\alpha})}{T(r, f)},$$

where α is a small function related to $f(z)$.

In the following sections, we denote by E a set of finite logarithmic measure, it is not necessarily the same at each appearance. In 1991, Yi and Yang [11] obtained the following theorem.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be meromorphic functions satisfying $\delta(\infty, f) = \delta(\infty, g) = 1$. If f' and g' share 1 CM and $\delta(0, f) + \delta(0, g) > 1$, then either $f \equiv g$ or $f'g' \equiv 1$.*

Lately, there has been an increasing interest in studying difference equations in the complex plane. For example, Halburd and Korhonen [3, 4] established a version of Nevanlinna theory based on difference operators. Ishizaki and Yanagihara [9] developed a version of Wiman-Valiron theory for difference equations of entire

---

2000 *Mathematics Subject Classification.* 30D35, 39B12.
Key words and phrases. Meromorphic function; difference polynomial; uniqueness; finite order.

Submitted June 18, 2013. Published December 5, 2013.

functions of small growth. Also Chiang and Feng [1] has a difference version of Wiman-Valiron. The main purpose of this article is to establish partial difference counterparts of Theorem 1.1. Our results can be stated as follows.

Theorem 1.2. *Let $c_j, a_j, b_j (j = 1, 2, \dots, k)$ be complex constants, and let $f(z)$ and $g(z)$ be two nonconstant finite order meromorphic functions satisfying $\delta(\infty, f) = \delta(\infty, g) = 1$. Let $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$ and $L(g) = \sum_{i=1}^{k} b_i g(z + c_i)$. Suppose that $L(f) \cdot L(g) \not\equiv 0$. If $L(f)$ and $L(g)$ share 1 CM and $\delta(0, f) + \delta(0, g) > 1$, then $L(f)L(g) \equiv 1$ or $L(f) \equiv L(g)$.*

Theorem 1.3. *Suppose that $f(z)$ is a nonconstant meromorphic function. Let $\delta_f = \sum_{a\in\mathbb{C}} \delta(a, f)$. If $\Delta_c f(z) = f(z + c) - f(z) \not\equiv 0$ $(c \in \mathbb{C} \setminus \{0\})$, then*
$$N\Big(r, \frac{1}{\Delta_c f(z)}\Big) \le \Big((1 - \frac{\delta_f}{2} + \varepsilon)T(r, \Delta_c f(z)\Big) \quad (r \notin E),$$
where ε is any fixed positive number.

Recently, Zhang [13] considered the value distribution of difference polynomial of entire functions, and obtain the following result.

Theorem 1.4. *Let $f(z)$ be a transcendental entire function of finite order, and $\alpha(z)$ be a small function with respect to $f(z)$. Suppose that c is a non-zero complex constant and n is an integer. If $n \ge 2$, then $f^n(z)(f(z) - 1)f(z + c) - \alpha(z)$ has infinitely many zeros.*

A natural question arises: If $n = 1$, whether we can get a similar conclusion? The following theorems give a partial answer to this question.

Theorem 1.5. *Let $f(z)$ be a finite order transcendental entire function with one Borel exceptional value d according to the condition or its proof described. Let $c_j (j = 1, \dots, n), b$ be complex constants. If $d^{n+2} - d^{n+1} - b \ne 0$, then $\varphi(z) = f(z)(f(z) - 1) \prod_{j=1}^{n} f(z + c_j) - b$ has infinitely many zeros and $\lambda(\varphi(z) - b) = \rho(f)$.*

Theorem 1.6. *Let $f(z)$ be a finite order transcendental entire function, and let c_j $(j = 1, \dots, n)$, b_j $(j = 1, \dots, n), b$ be complex constants. If $f(z)$ or $f(z) - 1$ has infinitely many multi-order zeros, then $f(z)(f(z) - 1) \prod_{i=1}^{n} (f(z + c_i) - b_i) - b$ has infinitely many zeros.*

2. Proof of Theorem 1.2

We need the following lemmas. The first lemma is a difference analogue of the logarithmic derivative lemma.

Lemma 2.1 ([3])**.** *Let $f(z)$ be a meromorphic function of finite order and let c be a non-zero complex number. Then for any small periodic function $a(z)$ with period c,*
$$m\Big(r, \frac{f(z + c) - f(z)}{f(z) - a(z)}\Big) = S(r, f).$$

The following lemma is essential for our proof and is due to Heittokangas et al., see [8, Theorems 6 and 7].

Lemma 2.2 ([5])**.** *Let $f(z)$ be a meromorphic function of finite order, $c \ne 0$ be fixed. Then*
$$\overline{N}(r, f(z + c)) \le \overline{N}(r, f(z)) + S(r, f),$$

$$N(r, f(z+c)) \leq N(r, f(z)) + S(r, f).$$

Lemma 2.3. *Let f be a nonconstant meromorphic function of finite order such that $\delta(\infty, f) = 1$ and $\delta(0, f) > 0$. Let $L(f)$ be as in Theorem 1.2. Then*

$$T(r, f) \leq \Big(\frac{1}{\delta(0, f)} + \varepsilon\Big) T(r, L(f)), \quad r \notin E,$$

and

$$N\Big(r, \frac{1}{L(f)}\Big) < (1 - \delta(0, f) + \varepsilon + o(1)) T(r, L(f)), \quad r \notin E,$$

where $\varepsilon > 0$ can be fixed arbitrarily.

Proof. From $\delta(\infty, f) = 1$, we have

$$N(r, f) = o(T(r, f)).$$

Then from Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} T(r, L(f)) &= m(r, L(f)) + N(r, L(f)) \\ &\leq m(r, f) + m\Big(r, \frac{L(f)}{f}\Big) + kN(r, f) + o(T(r, f)) \\ &\leq (1 + o(1)) T(r, f)), \quad r \notin E. \end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned} m\Big(r, \frac{1}{f}\Big) &\leq m\Big(r, \frac{1}{L(f)}\Big) + m\Big(r, \frac{L(f)}{f}\Big) \\ &= m\Big(r, \frac{1}{L(f)}\Big) + o(T(r, f)) \\ &= T(r, L(f)) - N\Big(r, \frac{1}{L(f)}\Big) + o(T(r, f)), \quad r \notin E. \end{aligned} \tag{2.2}$$

By the definition of $\delta(0, f)$, we obtain

$$m\Big(r, \frac{1}{f}\Big) \geq (\delta(0, f) - \varepsilon) T(r, f), \tag{2.3}$$

where $\varepsilon > 0$ can be fixed arbitrarily. Combining (2.2) and (2.3) yields

$$(\delta(0, f) - \varepsilon) T(r, f) < T(r, L(f)), \quad r \notin E;$$

that is,

$$T(r, f) < \Big(\frac{1}{\delta(0, f)} + \varepsilon\Big) T(r, L(f)), \quad r \notin E.$$

By (2.1), (2.2) and (2.3), we have

$$(\delta(0, f) - \varepsilon + o(1)) T(r, L(f)) < T(r, L(f)) - N\Big(r, \frac{1}{L(f)}\Big), \quad r \notin E;$$

that is,

$$N\Big(r, \frac{1}{L(f)}\Big) < (1 - \delta(0, f) + \varepsilon + o(1)) T(r, L(f)), \quad r \notin E.$$

□

Lemma 2.4 ([11]). *Let f_1, f_2 and f_3 be three meromorphic functions satisfying*

$$\sum_{i=1}^{3} f_i \equiv 1.$$

Assume that f_1 is not constant, and

$$\sum_{i=1}^{3} N_2\big(r, \frac{1}{f_i}\big) + \sum_{i=1}^{3} N(r, f_j) \leq (\lambda + o(1))T(r) \quad (r \in I),$$

where $\lambda < 1$, $T(r) = \max\{T(r, f_i) | i = 1, 2, 3)\}$, $N_2(r, 1/f_j)$ is the counting function of zeros of $f_j (j = 1, 2, 3)$, where a multiple zero is counted two times, and a simple zero is counted once. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Proof of Theorem 1.2. Set $I_1 = \{r : T(r, L(f)) \geq T(r, L(g))\} \subseteq (0, \infty)$ and $I_2 = (0, \infty) \backslash I_1$. Then there is at least one I_i $(i = 1, 2)$ such that I_i has infinite logarithmic measure. Without loss of generality, we may suppose that I_1 has infinite logarithmic measure.

Because $\delta(0, f) + \delta(0, g) > 1$, it follows that $\delta(0, f) > 0$ and $\delta(0, g) > 0$. Lemma 2.3 yields

$$\begin{gathered} T(r, f) = O(T(r, L(f))), \quad r \in I_1 \backslash E, \\ T(r, g) = O(T(r, L(g))) = O(T(r, L(f))), \quad r \in I_1 \backslash E. \end{gathered}$$

Thus,

$$\begin{gathered} N(r, f) = o(T(r, L(f))), \quad r \in I_1 \backslash E, \\ N(r, g) = o(T(r, L(f))), \quad r \in I_1 \backslash E. \end{gathered}$$

Since $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$ and $L(g) = \sum_{i=1}^{k} b_i g(z + c_i)$ share 1 CM, we have

$$\frac{L(f) - 1}{L(g) - 1} = h(z), \tag{2.4}$$

where

$$N(r, h) + N(r, \frac{1}{h}) \leq kN(r, f) + kN(r, g) + o(T(r, f)) + o(T(r, g)) = o(T(r, L(f))),$$

for $r \in I_1 \backslash E$. Let $f_1 = L(f)$, $f_2 = h(z)$, $f_3 = -L(g)h(z)$. Then we obtain $f_1 + f_2 + f_3 \equiv 1$, and

$$\sum_{i=1}^{3} N(r, f_i) \leq kN(r, f) + kN(r, g) + 2N(r, h) = o(T(r)), \quad r \in I_1 \backslash E,$$

where $T(r) = \max_{1 \leq i \leq 3}\{T(r, f_j)\}$. For any ε satisfying $0 < \varepsilon < \frac{\delta(0,f)+\delta(0,g)-1}{4}$, by Lemma 2.3, we obtain

$$\begin{aligned} \sum_{i=1}^{3} N_2\Big(r, \frac{1}{f_j}\Big) &\leq N\Big(r, \frac{1}{L(f)}\Big) + N\Big(r, \frac{1}{L(g)}\Big) + 2N(r, h) \\ &\leq (2 - \delta(0, f) - \delta(0, g) + o(1) + 2\varepsilon)T(r) \\ &= (\lambda + o(1))T(r), \quad r \in I_1 \backslash E, \end{aligned}$$

where $\lambda = 2 - \delta(0, f) - \delta(0, g) + 2\varepsilon < 1$. If $f_1(z) = L(f)$ is a constant, by Lemma 2.3, we see that $f(z)$ is also a constant, a contradiction. Hence, $f_1(z)$ is not constant.

By Lemma 2.4, we obtain $f_2 \equiv 1$ or $f_3 \equiv 1$. If $f_2 \equiv 1$, then we obtain $L(f) \equiv L(g)$. If $f_3 \equiv 1$, we have $L(f) \equiv -\frac{1}{h}$, $L(g) \equiv -h$, and so $L(f)L(g) \equiv 1$. □

3. Proof of Theorem 1.3

We need the following lemmas.

Lemma 3.1 ([5, 6]). *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then*

$$T(r, f(z + |c|)) = T(r, f(z)) + S(r, f).$$

Remark 3.2. It is shown in [2, p. 66], that for $c \in \mathbb{C} \setminus \{0\}$, we have

$$(1 + o(1))T(r - |c|, f(z)) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f(z))$$

hold as $r \to \infty$, for a general meromorphic function. By this and Lemma 3.1, we obtain

$$T(r, f(z + c)) = T(r, f(z)) + S(r, f)$$

Proof of Theorem 1.3. Without loss of generality, we assume that there exist infinitely many values a such that $\delta(a, f) > 0$. Then there is an sequence $\{a_i\}_{i=1}^{\infty}$ satisfying $a_i \neq a_j (i \neq j)$ and $\sum_{i=1}^{\infty} \delta(a_i, f) = \delta_f$. Hence for any fixed positive number ε, there exists an integer q such that

$$\delta = \sum_{i=1}^{q} \delta(a_i, f) > \delta_f - \frac{\varepsilon}{3}. \tag{3.1}$$

Set

$$F(z) = \sum_{i=1}^{q} \frac{1}{f(z) - a_i}.$$

Then

$$\sum_{i=1}^{q} m\Big(r, \frac{1}{f - a_i}\Big) = m(r, F) + O(1). \tag{3.2}$$

Hence by Lemma 2.1, we obtain

$$\begin{aligned} m(r, F) &\leq m\Big(r, \frac{1}{\Delta_c f(z)}\Big) + \sum_{i=1}^{q} m\Big(r, \frac{\Delta_c f(z)}{f - a_i}\Big) + O(1) \\ &= T(r, \Delta_c f(z)) - N\Big(r, \frac{1}{\Delta_c f(z)}\Big) + S(r, f). \end{aligned} \tag{3.3}$$

and

$$\sum_{i=1}^{q} m\Big(r, \frac{1}{f - a_i}\Big) \geq \Big(\delta_f - \frac{\varepsilon}{3}\Big)T(r, f), \quad r \notin E. \tag{3.4}$$

From (3.1)-(3.4) and Remark 1, we have

$$\begin{aligned} N(r, \frac{1}{\Delta_c f(z)}) &\leq T(r, \Delta_c f(z)) - (\delta_f - \frac{2}{3}\varepsilon)T(r, f) \\ &\leq T(r, \Delta_c f(z)) - (\frac{\delta_f}{2} - \varepsilon)T(r, \Delta_c f(z)) \\ &= (1 - \frac{\delta_f}{2} + \varepsilon)T(r, \Delta_c f(z)), \quad r \notin E. \end{aligned}$$

□

4. Proof of Theorem 1.5

The following lemma is a generalization of Borel's Theorem on linear combinations of entire functions.

Lemma 4.1 ([10, pp. 79-80]). *Let $f_j(z)$ $(j=1,2,\dots,n; n\geq 2)$ be meromorphic functions, $g_j(z)$ $(j=1,2,\dots,n)$ be entire functions, and assume they satisfy*

(i) $f_1(z)e^{g_1(z)}+\dots+f_k(z)e^{g_k(z)}\equiv 0$;
(ii) *when $1\leq j<k\leq n$, then $g_j(z)-g_k(z)$ is not a constant.*
(iii) *when $1\leq j\leq n$, $1\leq h<k\leq n$, then*

$$T(r,f_j)=o\{T(r,e^{g_h-g_k})\}\quad (r\to\infty,\ r\notin E_1),$$

where $E\subset(1,\infty)$ is of finite logarithmic measure.

Then $f_j\equiv 0$ $(j=1,\dots,n)$.

Proof of Theorem 1.5. Set $\varphi(z)=f(z)(f(z)-1)\prod_{j=1}^n f(z+c_j)$. Next we prove that $\rho(\varphi)=\rho(f)$. We write $\varphi(z)$ as

$$\varphi(z)=f^{n+1}(z)(f(z)-1)\prod_{j=1}^n\Big(\frac{f(z+c_j)}{f(z)}\Big) \tag{4.1}$$

By Lemma 2.1, we obtain

$$\begin{aligned}T(r,\varphi)=m(r,\varphi)&\leq(n+1)m(r,f)+m(r,f-1)+\sum_{i=1}^n m\Big(r,\frac{f(z+c_j)}{f(z)}\Big)+S(r,f)\\ &=(n+2)T(r,f)+S(r,f).\end{aligned} \tag{4.2}$$

On the other hand,

$$\begin{aligned}(n+2)T(r,f)&=(n+2)m(r,f)\\ &=m(r,f^{n+1}(z)(f(z)-1))+S(r,f)\\ &\leq m(r,\varphi)+\sum_{i=1}^n m\Big(r,\frac{f(z)}{f(z+c_i)}\Big)+S(r,f)\\ &=T(r,\varphi)+S(r,f).\end{aligned} \tag{4.3}$$

By (4.2) and (4.3), we have

$$\rho(\varphi)=\sigma(f). \tag{4.4}$$

Suppose that d is the Borel exceptional value of $f(z)$. Then we can write $f(z)$ as

$$f(z)=d+g(z)\exp\{\alpha z^k\}, \tag{4.5}$$

where α is a nonzero constant, $k\geq 1$ is an integer, and $g(z)$ is an entire function such that $g(z)(\not\equiv 0),\rho(g)<k$. By (4.5), we have

$$f(z+c_j)=d+g(z+c_j)g_j(z)\exp\{\alpha z^k\},(j=1,2,\dots,n), \tag{4.6}$$

where $g_j(z)=\exp\{\alpha\binom{k}{1}z^{k-1}c_j+\alpha\binom{k}{2}z^{k-2}c_j^2+\dots+\alpha c_j^k\}$, $\rho(g_j)=k-1$. Equality (4.4) implies that $\rho(\varphi-b)=\rho(f)$. Next, we prove that $\lambda(\varphi(z)-b)=\rho(f)$. Suppose, contrary to the assertion, that $\lambda(\varphi(z)-b)<\rho(f)$. Then

$$\varphi(z)-b=u(z)\exp\{\beta z^k\}, \tag{4.7}$$

where $u(z)$ is an entire function with $\rho(u) \le \max\{\lambda(\varphi(z)-b), k-1\} < k$, and β is a nonzero constant. From (4.5)-(4.7), we have

$$\begin{aligned} &g^2(z)\prod_{i=1}^{n} g(z+c_i)g_i(z)\exp\{(n+2)\alpha z^k\} + G_{n+1}(z)\exp\{(n+1)\alpha z^k\} \\ &+\dots + G_1(z)\exp\{\alpha z^k\} + d^{n+2} - d^{n+1} - b = u(z)\exp\{\beta z^k\}, \end{aligned} \tag{4.8}$$

where $G_i(z)$ $(i = 1, \dots, n+1)$ are difference polynomials in $g(z)$, $g_1(z)$, $g_2(z)$, $\dots$, $g_n(z)$, $g_1(z+c_1)$, $g_2(z+c_2)$, $\dots$, $g_n(z+c_n)$. Since $g(z) \not\equiv 0$, by comparing the growth of both side of (4.8), we have $\beta = (n+2)\alpha$. Hence we can rewritten (4.8) as

$$\begin{aligned} &(g^2(z)\prod_{i=1}^{n} g(z+c_i)g_i(z) - u(z))\exp\{(n+2)\alpha z^k\} + G_{n+1}(z)\exp\{(n+1)\alpha z^k\} \\ &+\dots + G_1(z)\exp\{\alpha z^k\} + d^{n+2} - d^{n+1} - b = 0. \end{aligned} \tag{4.9}$$

By Lemma 4.1 and (4.9), we obtain that $d^{n+2} - d^{n+1} - b = 0$, a contradiction. Hence, we obtain $\lambda(\varphi(z)-b) = \rho(f)$. $\square$

Proof of Theorem 1.6. Suppose that $f(z)$ or $f(z)-1$ has infinitely many multi-order zeros. If $b = 0$, then $H(z)$ has infinitely many zeros. Next we suppose that $b \neq 0$. If $H(z) - b$ has only finitely many zeros, then $H(z)$ can be rewritten as

$$H(z) = f(z)(f(z)-1)\prod_{i=1}^{n}(f(z+c_i)-b_i) - b = p(z)e^{q(z)}, \tag{4.10}$$

where $p(z), q(z)$ are polynomials. Suppose that $H(z)$ is a polynomial. Then we have

$$H(z) = f(z)(f(z)-1)\prod_{i=1}^{n}(f(z+c_i)-b_i) - b = P(z), \tag{4.11}$$

where $P(z)$ is a polynomial. From (4.11), we have

$$\begin{aligned} (n+2)T(r,f) &= T(r,H(z)) + S(r,f) \\ &= T(r, f(z)(f(z)-1)\prod_{i=1}^{n}(f(z+c_i)-b_i) - b) + S(r,f) \\ &= T(r,P(z)) + S(r,f) = O(\log r) + S(r,f). \end{aligned}$$

This is impossible, since $f(z)$ is transcendental. Hence $H(z)$ is transcendental, so we get $p(z) \not\equiv 0$, $\deg q(z) \ge 1$, by this, we obtain $p'(z) + p(z)q(z) \not\equiv 0$. Differential (4.10) and eliminating $e^{q(z)}$, we get

$$\begin{aligned} &\frac{(f(z)(f(z)-1)\prod_{i=1}^{n}(f(z+c_i)-b_j))'}{f(z)(f(z)-1)\prod_{i=1}^{n}(f(z+c_i)-b_j)} \\ &= \frac{p'(z)+p(z)q'(z)}{p(z)} - b\frac{p'(z)+p(z)q'(z)}{p(z)f(z)(f(z)-1)\prod_{i=1}^{n}(f(z+c_i)-b_j)} \end{aligned}$$

Since $f(z)$ or $f(z)-1$ has infinitely many multi-order zeros, there exists a sufficiently large point z_0 such that the multiplicity of the zero of $f(z)(f(z)-1)$ at z_0 is k

$(k \geq 2)$, and $p'(z_0) + p(z_0)q(z_0) \neq 0$, $p(z_0) \neq 0$. We can easily obtain that the multiplicity of

$$\frac{(f(z_0)(f(z_0)-1)\prod_{i=1}^{n}(f(z_0+c_i)-b_i))'}{f(z_0)(f(z_0)-1)\prod_{i=1}^{n}(f(z_0+c_i)-b_i)} = \infty$$

is 1, and the multiplicity of

$$\frac{p'(z_0)+p(z)q'(z_0)}{p(z_0)} - b\frac{p'(z_0)+p(z_0)q'(z_0)}{p(z_0)f(z_0)(f(z_0)-1)\prod_{i=1}^{n}(f(z_0+c_i)-b_i)} = \infty$$

is k $(l \geq k \geq 2)$. From the above equation, we get a contradiction. Hence $H(z)$ takes every value b infinitely often. □

Acknowledgements. The first author thanks Prof Risto Korhonen for his hospitality during the study period in Department of Physics and Mathematics, University of Eastern Finland, and for many valuable suggestions to the present paper. The authors are also grateful to the anonymous referee for providing many comments and suggestions for helping us to improve the paper.

This work was supported by the NNSF of China (No.10771121), the NSFC Tianyuan Mathematics Youth Fund(No. 11226094), the NSF of Shandong Province, China (No. ZR2012AQ020 and No. ZR2010AM030) and the Fund of Doctoral Program Research of University of Jinan (XBS1211).

References

[1] Y. M. Chiang, S. J. Feng; *On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions*, Trans. Amer. Math. Soc. 361 (2009), 3767-3791.

[2] A. A. Goldberg, I. V. Ostrovskii; *Distribution of Values of Meromorphic Functions*. Nauka, Mosow, 1970.

[3] R. G. Halburd, R. Korhonen; *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. 94 (2006), 463-478.

[4] R. G. Halburd, R. Korhonen; *Difference analogue of the lemma on the logarithmic derivative with applications to difference equatons*, J. Math. Anal. Appl. 314 (2006), 477-487.

[5] R. G. Halburd, R. Korhonen; *Finite order solutions and the discrete Painlevé equations*, Proc. London Math. Soc. 94 (2007), 443-474.

[6] R. G. Halburd, R. Korhonen; *Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations*. J- Phys. A. 40 (2007), 1-38.

[7] W. Hayman; *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[8] J. Heittokangas, R. Korhonen, I. Laine, Rieppo, J. Zhang; *Value sharing results for shift of meromorphic functions, and sufficent conditions for periodicity*. J. Math. Anal. Appl. 355 (2009), 352-363.

[9] K. Ishizaki, N. Yanagihara; *Wiman-Valiron method for difference equations*, Nagoya Math. J. 175 (2004), 75-102.

[10] C. C. Yang, H. X. Yi; *Uniqueness of Meromorphic Functions*, Kluwer, Dordrecht, 2003.

[11] H. X. Yi, C. C. Yang; *Unicity theorems for two meromorphic functions with their first derivatives have the same 1 points*. Acta Math. Sinica 34(5) (1991), 675-680.

[12] H. X. Yi; *Meromorphic functions that share two or three values*, Kodai Math. J. 13 (1990), 363-372.

[13] J. Zhang; *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. Appl. 367(2) (2010), 401-408.

Yong Liu
Department of Mathematics, Shaoxing College of Arts and Sciences, Shaoxing, Zhejiang 312000, China.
Department of Physics and Mathematics, Joensuu Campus, University of Eastern Finland, P.O. Box 111, Joensuu FI-80101, Finland
E-mail address: liuyongsdu@aliyun.com

Xiaoguang Qi
Department of Mathematics, Jinan University, 250022 Jinan Shandong, China
E-mail address: xiaoguangqi@yahoo.cn

Hongxun Yi
School of Mathematics, Shandong University, 250100 Jinan Shandong, China
E-mail address: hxyi@sdu.edu.cn