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DETERMINATION OF A SOURCE TERM FOR A TIME FRACTIONAL DIFFUSION EQUATION WITH AN INTEGRAL TYPE OVER-DETERMINING CONDITION

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Abstract. We consider a linear heat equation involving a fractional derivative in time, with a nonlocal boundary condition. We determine a source term independent of the space variable, and the temperature distribution for a problem with an over-determining condition of integral type. We prove the existence and uniqueness of the solution, and its continuous dependence on the data.

1. Introduction

In this article, we are concerned with the linear heat equation

$$D^{\alpha}_{0_+}(u(x,t)-u(x,0))-\varrho u_{xx}(x,t)=F(x,t), \quad (x,t)\in Q_T, \tag{1.1}$$

with initial and nonlocal boundary conditions

$$u(x,0)=\varphi(x), \quad x\in(0,1), \tag{1.2}$$

$$u(0,t)=u(1,t), \quad u_x(1,t)=0, \quad t\in(0,T], \tag{1.3}$$

where $Q_T=(0,1)\times(0,T]$, ϱ is a positive constant, $D^{\alpha}_{0_+}$ stands for the Riemann-Liouville fractional derivative of order $0<\alpha<1$ in the time variable (see formula (2.4)) and $\varphi(x)$ is the initial temperature.

Our choice of the term $D^{\alpha}_{0_+}(u(x,\cdot)-u(x,0))(t)$ rather than the usual term $D^{\alpha}_{0_+}u(x,\cdot)(t)$ is not only to avoid the singularity at zero, but also to include certain initial conditions.

For (1.1)-(1.3) the direct problem is the determination of $u(x,t)$ in $\bar{Q}_T$ such that $u(\cdot,t)\in C^2([0,1],\mathbb{R})$ and $D^{\alpha}_{0_+}(u(x,.)-u(x,0))\in C((0,T],\mathbb{R})$, when the initial temperature $\varphi(x)$ and the source term $F(x,t)$ are given and continuous.

Letting the source term have the form $F(x,t)=a(t)f(x,t)$, then the inverse problem consists of determining a source term $a(t)$ and the temperature distribution $u(x,t)$, from the initial temperature $\varphi(x)$ and boundary conditions (1.3). This problem is not uniquely solvable. The inverse problem of determining $a(t)$ was already considered in the literature for parabolic equations, see for example [5] and [2].

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To have the inverse problem uniquely solvable, we propose the over-determination condition

$$\int_0^1 u(x,t)dx = g(t), \quad t \in [0,T], \tag{1.4}$$

where $g \in AC([0,T],\mathbb{R})$ (the space of absolutely continuous functions). The solvability of inverse problems with such type of integral over-determination has been considered in the literature [5, 7].

It is well known (see [13] and references therein) that standard reaction diffusion equations and transport equations, commonly used to explain physical phenomena, show in some situations a disagreement with experimental data, due to non Gaussian diffusion. Among the several descriptions of this anomalous diffusion, one is by using fractional derivatives in time or space, or both, in reaction diffusion equations and transport equations. There are several publications on this topic, see [13, 11, 17] and references therein. Nonlocal boundary conditions arise from many important application in heat conduction and thermo-elasticity, see [2, 1].

When we want to solve (1.1)-(1.3) by using the Fourier method, i.e. by using separation of variables, we have to consider the spectral problem

$$X'' = -\lambda X, \quad x \in (0,1), \tag{1.5}$$

$$X(0) = X(1), \quad X'(1) = 0. \tag{1.6}$$

This boundary-value problem is non-self-adjoint, and the set of eigenvectors of the spectral problem (1.5)-(1.6) is not complete in the space $L^2(0,1)$. Following [4], we supplement the set of eigenvectors with the associated eigenvectors making the set complete on $L^2(0,1)$. Another complete set of eigenvectors and associated eigenvectors of the adjoint problem of problem (1.5)-(1.6) is obtained in the Appendix.

A solution of the inverse problem is a pair of functions $\{u(x,t), a(t)\}$ satisfying $u(.,t) \in C^2[(0,1),\mathbb{R}]$, $D^\alpha_{0_+}(u(x,.) - u(x,0)) \in C([0,T],\mathbb{R})$ such that $a \in C([0,T],\mathbb{R}^+ \cup \{0\})$, and for a given initial data the over determination condition (1.4) is satisfied.

Our approach for the solvability of the inverse problem is based on the expansion of the solution $u(x,t)$ by using the bi-orthogonal system of functions obtained from the boundary-value problem (1.5)-(1.6) and its adjoint problem.

Let us mention that in [3], the authors considered the inverse problem of determination of the order of the fractional derivative and the diffusion coefficient for the one dimensional diffusion equation (they considered the fractional time derivative defined in the sense of Caputo). They proved the unique determination of the order of the the fractional derivative and the diffusion coefficient (independent of time); their proof is based on the eigenfunctions expansion of the weak solution to the problem along the Gelfand-Levitan theory.

In [9] the inverse problem of finding the temperature distribution and a source term independent of the time variable for the one dimensional fractional diffusion equation with the nonlocal boundary condition

$$u(1,t) = 0, \quad u_x(0,t) = u_x(1,t), \quad t \in [0,T],$$

has been considered. The authors used two sets of Riesz basis (which form a bi-orthogonal system) for the space $L^2(0,1)$ in order to prove the existence and uniqueness for the solution of the inverse problem. In [19], the inverse problem of the determination of the source term (which is independent of the time variable) for the fractional diffusion equation

$$ {}^{C}D_{0_+}^{\alpha} u(x,t) - u_{xx}(x,t) = f(x), \quad (x,t) \in Q_T, \tag{1.7} $$

$$ u(x,0) = \varphi(x), \quad x \in [0,1], \tag{1.8} $$

$$ u_x(1,t) = 0, \quad u_x(0,t) = 0, \quad t \in [0,T], \tag{1.9} $$

where ${}^{C}D_{0_+}^{\alpha}$, for $0 < \alpha < 1$ stands for the Caputo fractional derivative in the time variable has been addressed. They proved the unique determination of the source term by using analytic continuation along with Duhamel's principle.

Some authors also consider the regularizing techniques for the solution of the inverse problem of the one dimensional linear time fractional heat equation. Murio [14] proposed a space marching regularizing scheme using mollification techniques for the solution of the inverse time fractional heat equation. The fractional derivative is considered in the sense of Caputo's definition. In [12] the author considers the problem of identification of the diffusion coefficient and the order of the fractional derivative for the one dimensional time fractional diffusion equation. The author presents the results by considering Riemann-Liouville's and Caputo's definition of the fractional derivative.

Recently, Kirane et al [10] considered two dimensional inverse source problem for time fractional diffusion equation and prove the well posedness of the inverse source problem using Fourier method. Jin and Rundell [6] consider the problem of recovering a spatially varying potential for a one dimensional time fractional diffusion equation from the flux measurements at a particular time. They proved the result of uniqueness of the potential using Green's function theory and propose a reconstruction method by a quasi Newton type iterative scheme. Li et al [18] propose algorithms for simultaneous inversion of order of fractional derivative and a space dependent diffusion coefficient for a one dimensional time fractional diffusion equation. They use the inverse eigenvalue problem for proving the uniqueness results of the inverse problem. An optimal perturbation algorithm for regularization using sigmoid-type function is proposed for the numerical inversion of order of fractional derivative and diffusion coefficient.

The rest of the paper is organized as follows: in Section 2, for the sake of the reader we present some basic definitions and results needed in the sequel. In Section 3, we present our main results concerning the existence, uniqueness and continuous dependence of the solution of the inverse problem. Section 4 concludes the paper by describing the results obtained in this paper.

2. Preliminaries

In this section, we recall basic definitions and notations from fractional calculus (see [16]). For an integrable function $f : \mathbb{R}^+ \to \mathbb{R}$, the left sided Riemann-Liouville fractional integral of order $0 < \alpha < 1$ is defined by

$$ J_{0_+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t > 0, \tag{2.1} $$

where $\Gamma(\alpha)$ is the Euler Gamma function. The integral (2.1) can be written as a convolution

$$ J_{0_+}^{\alpha} f(t) = (\phi_\alpha \star f)(t), \tag{2.2} $$

where

$$\phi_\alpha := \begin{cases} t^{\alpha-1}/\Gamma(\alpha), & t > 0, \\ 0 & t \le 0. \end{cases} \tag{2.3}$$

The left sided Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of the continuous function $f$is defined by

$$D^{\alpha}_{0_+} f(t) := \frac{d}{dt} J^{1-\alpha}_{0_+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau. \tag{2.4}$$

Notice that the Riemann-Liouville fractional derivative of a constant is not equal to zero.

The Laplace transform of the Riemann-Liouville integral of order $0 < \alpha < 1$ of a function with at most an exponential growth is

$$\mathcal{L}\{J^{\alpha}_{0_+} f(t) : s\} = \mathcal{L}\{f(t) : s\}/s^\alpha.$$

For $0 < \alpha < 1$, we have

$$J^{\alpha}_{0_+} D^{\alpha}_{0_+} \Big(f(t) - f(0)\Big) = f(t) - f(0). \tag{2.5}$$

The Mittag-Leffler function plays an important role in the theory of fractional differential equations; for any $z \in \mathbb{C}$ the Mittag-Leffler function with parameter ξ is

$$E_\xi(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + 1)} \quad (\mathrm{Re}(\xi) > 0). \tag{2.6}$$

In particular, $E_1(z) = e^z$.

The Mittag-Leffler function of two parameters $E_{\xi,\beta}(z)$ which is a generalization of (2.6) is defined by

$$E_{\xi,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + \beta)} \quad (z, \beta \in \mathbb{C};\ \mathrm{Re}(\xi) > 0). \tag{2.7}$$

Let us set $e_\xi(t, \mu) := E_\xi(-\mu t^\xi)$ where $E_\xi(t)$ is the Mittag-Leffler function with one parameter ξ as defined in (2.6) and μ is a positive real number.

The Mittag-Leffler functions $e_\alpha(t; \mu)$, $e_{\alpha,\beta}(t; \mu) := t^{\beta-1} E_{\alpha,\beta}(-\mu t^\alpha)$ for $0 < \alpha \le 1$, $0 < \alpha \le \beta \le 1$ respectively, and $\mu > 0$ are *completely monotone* functions; i.e.,

$$(-1)^n [e_\alpha(t;\mu)]^n \ge 0, \quad \text{and} \quad (-1)^n [e_{\alpha,\beta}(t;\mu)]^n \ge 0, \quad n \in \mathbb{N} \cup \{0\}.$$

Furthermore, we have

$$E_{\alpha,\beta}(\mu t^\alpha) \le M, \quad t \in [b, c], \tag{2.8}$$

where $[b, c]$ is a finite interval with $b \ge 0$, and

$$\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\beta}(\mu t^\alpha) d\tau < \infty,$$

on $[b, c]$ (see [15] page 9).

Let $\mathcal{H}$ be a Hilbert space with the scalar product $(\cdot, \cdot)$. Two sets S_1 and S_2 of functions of $\mathcal{H}$ form a bi-orthogonal system of functions if a one-to-one correspondence can be established between them such that the scalar product of two corresponding functions is equal to unity and the scalar product of two non-corresponding functions is equal to zero, i.e.,

$$(f_i, g_j) = \delta_{ij},$$

where $f_i \in S_1$, $g_i \in S_2$ and δ_{ij} is the Kronecker symbol.

Lemma 2.1. [9] *Let $\mathcal{G} : \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function such that $\mathcal{G} \in L^1(\mathbb{R})$. The solution of the equation*

$$v(t) + \mu J_{0_+}^{\alpha} v(t) = \mathcal{G}(t)$$

for $\mu \in \mathbb{R}^+$ satisfies the following integral equation

$$v(t) = \int_0^t \mathcal{G}'(t-\tau)e_\alpha(\tau,\mu)d\tau + \mathcal{G}(0)e_\alpha(t,\mu).$$

For the proof see Lemma 3.1 in [9].

3. Main Results

Our approach to the solvability of the inverse problem is based on the expansion of the solution $u(x,t)$ in a Riesz basis of the space $L^2(0,1)$ obtained from the eigenfunctions and associated eigenfunctions of the spectral problem (1.5)-(1.6). The unique expansion of the function in terms of the Riesz basis is assured by a bi-orthogonal system of functions formed from the spectral problem (1.5)-(1.6) and its adjoint problem.

3.1. A bi-orthogonal system of functions.

The sets of functions

$$\{2,\ \{4\cos(2\pi nx)\}_{n=1}^\infty,\ \{4(1-x)\sin(2\pi nx)\}_{n=1}^\infty\} \tag{3.1}$$

and

$$\{x,\ \{x\cos(2\pi nx)\}_{n=1}^\infty,\ \{\sin(2\pi nx)\}_{n=1}^\infty\} \tag{3.2}$$

are obtained from the non-self-adjoint spectral problem (1.5)-(1.6) and its adjoint problem

$$Y'' = -\lambda Y, \quad x \in (0,1), \tag{3.3}$$

$$Y'(0) = Y'(1), \quad Y(0) = 0, \tag{3.4}$$

respectively (see Appendix).

The set of functions (3.1) and (3.2) is complete in $L^2(0,1)$ and forms a Riesz basis in $L^2(0,1)$. Furthermore, set of functions (3.1)-(3.2) constitutes a bi-orthogonal system with the one to one correspondence

$$\begin{array}{cccc} \{\underbrace{2}, & \underbrace{\{4\cos(2\pi nx)\}_{n=1}^\infty}, & \underbrace{\{4(1-x)\sin(2\pi nx)\}_{n=1}^\infty}\}, \\ \downarrow & \downarrow & \downarrow \\ \{\ x, & \{x\cos(2\pi nx)\}_{n=1}^\infty, & \{\sin(2\pi nx)\}_{n=1}^\infty\ \}. \end{array}$$

The set of bi-orthogonal functions formed from (3.1) and (3.2) plays an important role in proving

existence and uniqueness of the solution of the inverse problem.

3.2. Existence and uniqueness of the solution of the inverse problem.

For the proof of the main result, i.e., Theorem 3.1 we will use properties of the bi-orthogonal system of functions and application of the Banach fixed point theorem. We have the following theorem

Theorem 3.1. *Suppose the following conditions hold:*

(A1) $\varphi \in C^4([0,1])$, $\varphi(1) = \varphi(0)$, $\varphi'(1) = 0$, $\varphi''(0) = \varphi''(1)$, $\varphi'''(1) = 0$;

(A2) $f \in C^4([\overline{Q}_T,\mathbb{R}])$, $f(0,t) = f(1,t)$, $f_x(1,t) = 0$, $f_{xx}(0,t) = f_{xx}(1,t)$, $f_{xxx}(1,t) = 0$, $\int_0^1 f(x,t)\,dx \neq 0$ *and*

$$0 < \frac{1}{M_1} \leq \Big|\int_0^1 f(x,t)dx\Big|;$$

(A3) $g \in AC([0,T])$, *and* $g(t)$ *satisfies the consistency condition* $\int_0^1 \varphi(x)dx = g(0)$,

then the inverse problem has a unique solution.

Proof. We write the solution $u(x,t)$ of the inverse problem for the linear system (1.1)-(1.4) in the form

$$u(x,t) = 2u_0(t) + \sum_{n=1}^{\infty} u_{1n}(t)4\cos(2\pi nx) + \sum_{n=1}^{\infty} u_{2n}(t)4(1-x)\sin(2\pi nx) \quad (3.5)$$

where $u_0(t), u_{1n}(t), u_{2n}(t)$ for $n \in \mathbb{N}$ are to be determined.

Let $\{f_0(t),\ f_{1n}(t),\ f_{2n}(t)\}$ be the coefficients of the series expansion of $f(x,t)$ in the basis (3.1) which are given by

$$\begin{gathered} f_0(t) = \int_0^1 f(x,t)x\,dx, \quad f_{1n}(t) = \int_0^1 f(x,t)x\cos(2\pi nx)\,dx, \\ f_{2n}(t) = \int_0^1 f(x,t)\ \sin 2\pi nx\,dx. \end{gathered} \quad (3.6)$$

Using properties of the bi-orthogonal system we have

$$u_0(t) = (u(x,t), x), \quad (3.7)$$

where $(f,g) := \int_0^1 f(x)g(x)\,dx$ is the scalar product in $L^2(0,1)$. By virtue of (3.7), we have

$$D_{0_+}^{\alpha}(u_0(t) - u_0(0)) = (D_{0_+}^{\alpha}(u(x,t) - u(x,0)), x).$$

Using (1.1) we can write

$$D_{0_+}^{\alpha}(u_0(t) - u_0(0)) = ((\varrho u_{xx} + a(t)f(x,t)), x).$$

On computing we obtain the following linear fractional differential equation

$$D_{0_+}^{\alpha}(u_0(t) - u_0(0)) = a(t)f_0(t). \quad (3.8)$$

Alike, we obtain the linear fractional differential equations

$$D_{0_+}^{\alpha}(u_{2n}(t) - u_{2n}(0)) + 2\pi n\varrho u_{2n}(t) = a(t)f_{2n}(t), \quad (3.9)$$

$$D_{0_+}^{\alpha}(u_{1n}(t) - u_{1n}(0)) + 4\pi^2 n^2 \varrho u_{1n}(t) - 4\pi n\varrho u_{2n}(t) = a(t)f_{1n}(t). \quad (3.10)$$

The solution of the linear fractional differential equation (3.8) is

$$u_0(t) = \varphi_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}a(\tau)f_0(\tau)d\tau.$$

The solutions of the linear fractional differential equations (3.9)-(3.10) are

$$u_{2n}(t) = \int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n)a(\tau)f_{2n}(\tau)d\tau + \varphi_{2n}e_{\alpha}(t,\lambda_n),$$

$$u_{1n}(t) = 2\lambda_n \int_0^t h(t-\tau)a(\tau)f_{2n}(\tau)d\tau + \int_0^t e_{\alpha,\alpha}(t-\tau, \lambda_n^2/\varrho)a(\tau)f_{1n}(\tau)d\tau \\ + 2\lambda_n\varphi_{2n}\int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n)e_\alpha(\tau,\lambda_n^2/\varrho)d\tau + \varphi_{1n}e_\alpha(t,\lambda_n^2/\varrho),$$

where we have used Lemma 2.1, $\lambda_n := 2\pi n\varrho$,

$$\varphi_0 = \int_0^1 \varphi(x)x\,dx, \quad \varphi_{1n} = \int_0^1 \varphi(x)x\cos(2\pi nx)\,dx, \quad \varphi_{2n} = \int_0^1 \varphi(x)\ \sin 2\pi nx\,dx,$$

$$h(t) = \int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n)e_{\alpha,\alpha}(\tau,\lambda_n^2/\varrho)d\tau.$$

In the above calculations we have used the following relations

$$h\star(f\star g) = (h\star f)\star g, \quad D^\alpha_{0_+}(f\star g) = (D^\alpha_{0_+}f\star g), \\ D^{1-\alpha}_{0_+}e_\alpha(t,\lambda_n) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha) =: e_{\alpha,\alpha}(t,\lambda_n).$$

Taking fractional derivative $D^\alpha_{0_+}$ under the integral sign of the over-determination condition (1.4) and in view of the consistency relation we have

$$\int_0^1 D^\alpha_{0_+}(u(x,t)-u(x,0))\,dx = D^\alpha_{0_+}(g(t)-g(0)),$$

which by using (1.1) and integration by parts leads to

$$a(t) = \Big(\int_0^1 f(x,t)dx\Big)^{-1}\Big(D^\alpha_{0_+}(g(t)-g(0)) + \rho u_x(0,t)\Big). \tag{3.11}$$

Recall that $\int_0^1 f(x,t)dx \neq 0$ and we have

$$f(x,t) = 2f_0(t) + \sum_{n=1}^\infty f_{1n}(t)4\cos(2\pi nx) + \sum_{n=1}^\infty f_{2n}(t)4(1-x)\sin(2\pi nx),$$

where $f_0(t)$, $f_{1n}(t)$ and $f_{2n}(t)$ are given by (3.6), then

$$\int_0^1 f(x,t)dx = 2f_0(t) + \frac{2}{\pi}\sum_{n=1}^\infty \frac{f_{2n}(t)}{n}, \tag{3.12}$$

and

$$u_x(0,t) = \sum_{n=1}^\infty 8\pi n\varphi_{2n}e_\alpha(t,\lambda_n) + \sum_{n=1}^\infty 8\pi n\int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n)a(\tau)f_{2n}(\tau)d\tau. \tag{3.13}$$

Let $B(a(t)) := a(t)$, where the operator B is defined by

$$B(a(t)) = \Big(\int_0^1 f(x,t)dx\Big)^{-1}\Big(D^\alpha_{0_+}(g(t)-g(0)) + \rho u_x(0,t)\Big).$$

By (3.12) and (3.13) we have the Volterra integral equation

$$B(a(t)) = \mathcal{F}(t) + \Big(2f_0(t) + \frac{2}{\pi}\sum_{n=1}^\infty \frac{f_{2n}(t)}{n}\Big)^{-1}\int_0^t K(t,\tau)a(\tau)d\tau, \tag{3.14}$$

where

$$\mathcal{F}(t) = \Big(\int_0^1 f(x,t)\,dx\Big)^{-1}\Big(D^\alpha_{0_+}(g(t)-g(0)) + \sum_{n=1}^\infty 8\pi n\rho\varphi_{2n}e_\alpha(t,\lambda_n)\Big), \tag{3.15}$$

and

$$K(t,\tau)=\sum_{n=1}^{\infty}8\pi n\rho f_{2n}(\tau)e_{\alpha,\alpha}(t-\tau,\lambda_n)=\sum_{n=1}^{\infty}8\pi n\rho f_{2n}(t-\tau)e_{\alpha,\alpha}(\tau,\lambda_n). \quad (3.16)$$

Before we proceed further, notice that under the assumptions (A2), the series $\sum_{n=1}^{\infty}8\pi n f_{2n}$ is uniformly convergent by the Weierstrass M-test because the series is bounded from above by the uniformly convergent numerical series

$$\sum_{n=1}^{\infty}|f_{2n}^{(4)}|/(2\pi^3 n^3),$$

where $f_{2n}^{(4)}$ is the coefficient of the Fourier sine series of the function $f^{(4)}(x)$. Furthermore, $f_{2n}^{(4)}$ for $n\in\mathbb{N}$ are bounded by the Bessel's inequality, indeed we have

$$\sum_{n=1}^{\infty}[f_{2n}^{(4)}]^2\le \mathcal{C}\|f^{(4)}\|_{L^2(0,1)}^2,$$

where $\mathcal{C}$ is a constant independent of t and n. Thus, we have $\sum_{n=1}^{\infty}8\pi n f_{2n}\le C$, where $(C$ is a constant independent of t and n.

Setting $T<(M_1MC)^{-1}$, where M_1 is from assumption (A2) of Theorem 3.1, M is from the inequality (2.8). Consider the space $C([0,T])$, equipped with the Chebyshev norm

$$\|f\|:=\max_{0\le t\le T}|f(t)|.$$

We shall show that $B:C([0,T])\to C([0,T])$ and the mapping B is a contraction. For $a\in C([0,1])$, using (2.8) and assumptions (A2), we have $u_x(0,t)$ continuous function. Indeed, the series in the expression of $u_x(0,t)$ (see 3.13) is uniformly convergent on $[0,T]$ and represents a continuous function. The term $D_{0_+}^{\alpha}(g(t)-g(0))$ being the difference of two continuous functions; i.e., a and $u_x(0,t)$ are continuous. We have

$$\begin{aligned}|B(a)-B(c)|&\le M_1\int_0^t|a(\tau)-c(\tau)|\,|K(t,\tau)|d\tau\\ &\le MTCM_1\max_{0\le t\le T}|a(\tau)-c(\tau)| \qquad (3.17)\\ \|B(a)-B(c)\|=\max_{0\le t\le T}|B(a)-B(c)|&\le MTCM_1\|a-c\|,\end{aligned}$$

Thus the mapping B is a contraction for $t\in[0,T]$. This assures unique determination of $a\in C([0,T])$ by the Banach fixed point theorem.

3.3. **Uniqueness of the solution.** Let $\{u(x,t),a(t)\}$ and $\{v(x,t),b(t)\}$ be two solution sets of the inverse problem then

$$\begin{aligned} u(x,t)-v(x,t) &= 2\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(a(\tau)-b(\tau))f_0(\tau)d\tau\Big)\\ &\quad + 4(1-x)\sin(2\pi nx)\Big(\int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n)f_{2n}(\tau)(a(\tau)-b(\tau))d\tau\Big)\\ &\quad + 4\cos(2\pi nx)\Big(4\pi n\int_0^t h(t-\tau)f_{2n}(\tau)(a(\tau)-b(\tau))d\tau\\ &\quad + \int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n)f_{1n}(\tau)(a(\tau)-b(\tau))d\tau\Big), \end{aligned} \tag{3.18}$$

and

$$a(t)-b(t)=\int_0^t K(t,\tau)(a(\tau)-b(\tau))d\tau.$$

Due to the estimate (3.17), we have $a=b$ and by substituting $a=b$ in (3.18), we obtain $u=v$.

Let us mention that under assumptions (A1)–(A3) and following [9], we shall show that the series solution for $u(x,t)$ given by (3.5) and the series corresponding to $u_{xx}(x,t)$ are uniformly convergent and represent continuous function on Q_T. Also, we shall show that the series corresponding to $u(x,t)-u(x,0)$ is α differentiable.

Let

$$M^* = \max\{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3\}, \quad \text{and} \quad \max_{0<t\le T} a(t)=N,$$

where $e_\alpha(t,\lambda_n)\le \mathcal{M}_1$, $e_{\alpha,\alpha}(t,\lambda_n)\le\mathcal{M}_2$, $e_\alpha(t,\lambda_n^2/\varrho)\le\mathcal{M}_3$. Then the series (3.5) is bounded above by the uniformly convergent series

$$\begin{aligned} &|\varphi_0| + NT^{2-\alpha}|f_0| + \sum_{n=1}^\infty (16\pi^4 n^4)^{-1}\Big(M^*|\varphi_{2n}^{(4)}| + NM^*T|f_{2n}^{(4)}| + \lambda_n N|f_{2n}^{(4)}|\\ &+ NM^*T|f_{1n}^{(4)}| + \lambda_n T(M^*)^2|\varphi_{1n}^{(4)}| + M^*|\varphi_{1n}^{(4)}|\Big), \end{aligned}$$

where $\varphi_{1n}^{(4)}$, $\varphi_{2n}^{(4)}$ and $f_{1n}^{(4)}$, $f_{2n}^{(4)}$ are the coefficients of the Fourier cosine and the Fourier sine series of the functions $\varphi^{(4)}(x)$ and $f^{(4)}(x,t)$, respectively. These functions are bounded by virtue of Bessel's inequality. By the Weierstrass M-test the series (3.5) is uniformly convergent.

Let us show that the series corresponding to $u(x,t)-u(x,0)$, i.e.,

$$\begin{aligned} u(x,t)-u(x,0) &= 2(u_0(t)-u_0(0)) + \sum_{n=1}^\infty (u_{1n}(t)-u_{1n}(0))4\cos(2\pi nx)\\ &\quad + \sum_{n=1}^\infty (u_{2n}(t)-u_{2n}(0))4(1-x)\sin(2\pi nx), \end{aligned} \tag{3.19}$$

is α differentiable and for this we use the result from [16], which states:

> For a sequence of functions f_i, $i\in\mathbb{N}$ defined on the interval $(a,b]$. Suppose the following conditions are fulfilled:
>
> (1) for a given $\alpha>0$ the fractional derivatives $D_{0+}^\alpha f_i(t)$, for $i\in\mathbb{N}$, $t\in(a,b]$ exists,

(2) the series $\sum_{i=1}^{\infty} f_i(t)$ and the series $\sum_{i=1}^{\infty} D_{0_+}^{\alpha} f_i(t)$ are uniformly convergent on the interval $[a+\epsilon, b]$ for any $\epsilon > 0$.

Then the function defined by the series $\sum_{i=1}^{\infty} f_i(t)$ is α differentiable and satisfies

$$D_{0_+}^{\alpha} \sum_{i=1}^{\infty} f_i(t) = \sum_{i=1}^{\infty} D_{0_+}^{\alpha} f_i(t). \tag{3.20}$$

We need to show that the series

$$\begin{aligned}
&2D_{0_+}^{\alpha}(u_0(t) - u_0(0)) + \sum_{n=1}^{\infty} D_{0_+}^{\alpha}(u_{1n}(t) - u_{1n}(0)) 4\cos(2\pi n x) \\
&+ \sum_{n=1}^{\infty} D_{0_+}^{\alpha}(u_{2n}(t) - u_{2n}(0)) 4(1-x)\sin(2\pi n x),
\end{aligned} \tag{3.21}$$

is uniformly convergent. Since $D_{0_+}^{\alpha} e_{\alpha,\alpha}(t,\lambda_n) = -\lambda_n e_{\alpha,\alpha}(t,\lambda_n)$ and $D_{0_+}^{\alpha} h(t) = -\lambda_n e_{\alpha,\alpha}(t,\lambda_n) \star e_{\alpha,\alpha}(t,\lambda_n)$, we have

$$D_{0_+}^{\alpha}\Big(u_0(t) - u_0(0)\Big) = a(t) f_0(t), \tag{3.22}$$

$$\begin{aligned}
&D_{0_+}^{\alpha}\Big(u_{2n}(t) - u_{2n}(0)\Big) \\
&= -\lambda_n \varphi_{2n} e_{\alpha}(t,\lambda_n) - \lambda_n \int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n) a(\tau) f_{2n}(\tau) d\tau,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&D_{0_+}^{\alpha}\Big(u_{1n}(t) - u_{1n}(0)\Big) \\
&= -2\lambda_n^2 \Big(\int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n) e_{\alpha,\alpha}(\tau,\lambda_n^2/\varrho) d\tau\Big) \star a(t) f_{2n}(t) \\
&- \lambda_n^2/\varrho \int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n^2/\varrho) a(\tau) f_{1n}(\tau) d\tau \\
&- 2\lambda_n^2 \varphi_{2n} \int_0^t e_{\alpha,\alpha}(t-\tau,\lambda_n) e_{\alpha}(\tau,\lambda_n^2/\varrho) d\tau - \lambda_n^2/\varrho \varphi_{1n} e_{\alpha}(t,\lambda_n^2/\varrho).
\end{aligned} \tag{3.24}$$

From the expressions of fractional derivative (3.22)-(3.24), we have

$$\begin{gathered}
\Big|D_{0_+}^{\alpha}\Big(u_0(t) - u_0(0)\Big)\Big| \le 2N|f_0|, \\
\Big|D_{0_+}^{\alpha}\Big(u_{2n}(t) - u_{2n}(0)\Big)\Big| \le M^* \lambda_n |\varphi_{2n}| + N\lambda_n M^* T |f_{2n}|, \\
\Big|D_{0_+}^{\alpha}\Big(u_{1n}(t) - u_{1n}(0)\Big)\Big| \le 2\lambda_n M^* T N |f_{2n}| + NTM^* \rho \lambda_n^2 |f_{1n}| + 2M^* T \lambda_n^2 |\varphi_{2n}| \\
+ M^* |\varphi_{1n}| \lambda_n^2/\rho.
\end{gathered}$$

Due to the assumptions of the Theorem 3.1, we have

$$\begin{gathered}
\varphi_{2n} = \frac{1}{16\pi^4 n^4} \int_0^1 \varphi^{(4)}(x) \sin(2\pi n x)\, dx = \frac{1}{16\pi^4 n^4} \varphi_{2n}^{(4)}, \\
\varphi_{1n} = \frac{1}{16\pi^4 n^4} \varphi_{1n}^{(4)}, \quad f_{2n} = \frac{1}{16\pi^4 n^4} f_{2n}^{(4)}.
\end{gathered}$$

The series (3.21) is bounded from above by the uniformly convergent series

$$2N|f_0| + \varrho M^* \sum_{n=1}^{\infty} \Big(\frac{4\pi n T + 1}{8\pi^3 n^3} \, (|\varphi_{2n}^{(4)}| + NT|f_{2n}^{(4)}|) + \frac{NT|f_{1n}^{(4)}|}{4\pi^2 n^2} + \frac{|\varphi_{1n}^{(4)}|}{4\pi^2 n^2} \Big),$$

consequently, the series (3.21) is uniformly convergent by the Weierstrass M-test. Hence the series (3.19) is α-differentiable with respect to the time variable and the relation (3.20) holds true.

Similarly we can show that the series corresponding to $u_{xx}(x,t)$ is uniformly convergent and represents continuous function. □

3.4. **Continuous dependence of the solution on the data.** Let $\mathcal{T}$ be the set of triples $\{\varphi, f, g\}$ where the functions φ, f, g satisfy the assumptions of Theorem 3.1 and

$$\|\varphi\|_{C^3([0,1])} \le M_2, \quad \|f\|_{C^3([Q_T])} \le M_3, \quad \|g\|_{AC([0,1])} \le M_4.$$

For $\psi \in \mathcal{T}$, we define the norm

$$\|\psi\| = \|\varphi\|_{C^3([0,1])} + \|f\|_{C^3([Q_T])} + \|g\|_{AC([0,1])}.$$

Before presenting the result about the stability of the solution of the inverse problem let us mention that the series

$$\sum_{n=1}^{\infty} \frac{1}{2\pi^3 n^4} |f_{2n}^{(4)}| \le M_5,$$

is uniformly convergent, where $f_{2n}^{(4)}$ are the coefficients of the sine Fourier expansion of the function $f^{(4)}(.,t)$. The functions $\{f_{2n}^{(4)}\}_{n=1}^{\infty}$ are bounded by virtue of the Bessel's inequality.

Setting T such that

$$T < (M_1 \mathcal{N})^{-1} \tag{3.25}$$

where M_1 is from the assumption (A1) and $\mathcal{N} := M M_5$ (M is from (2.8)). Then we have the following theorem.

Theorem 3.2. *The solution $\{u(x,t), a(t)\}$ of the inverse problem* (1.1)-(1.4)*, under the assumptions of Theorem 3.1, depends continuously upon the data for T satisfying* (3.25)*.*

Proof. Let $\{u(x,t), a(t)\}$, $\{\tilde{u}(x,t), \tilde{a}(t)\}$ be solution sets of the inverse problem (1.1)-(1.4), corresponding to data $\psi = \{\varphi, f, g\}$, $\tilde{\psi} = \{\tilde{\varphi}, \tilde{f}, \tilde{g}\}$, respectively. From (3.16) we have

$$\|K\|_{C([0,T]) \times C([0,T])} \le M \sum_{n=1}^{\infty} \frac{1}{2\pi^3 n^4} |f_{2n}^{(4)}|.$$

Then

$$\|K\|_{C([0,T]) \times C([0,T])} \le \mathcal{N}.$$

For $g \in AC[0,T]$ the term $D_{0_+}^{\alpha}(g(t) - g(0))$ is continuous being the difference of continuous functions (see equation (3.11)). Furthermore, for any $\epsilon > 0$ the term $D_{0_+}^{\alpha}(g(t) - g(0))$ is bounded on the interval $(\epsilon, T]$. In the estimates below, we will use this fact frequently.

From (3.15) and (3.14) we have

$$\|\mathcal{F}\|_{C([0,T])} \le M_8, \quad \|a\|_{C([0,T])} \le \frac{M_8}{1 - T M_1 \mathcal{N}},$$

where $M_8 = M_1(M_6 + M M_7)$, M_6 is a bound of $D_{0_+}^{\alpha}(g(t) - g(0))$, $\sum_{n=1}^{\infty} 8\pi n \rho \varphi_{2n} \le M_7$ and, both M_6, M_7 are constants independent of t and n. Before we proceed to the next estimate notice that from the expansion of $f(x,t)$, the norm of f_0, f_{2n-1}

and f_{2n} for $n \in \mathbb{N}$ can be estimated by the norm of $f(x,t)$. Similarly, the norm of $\varphi_0, \varphi_{2n-1}$ and φ_{2n} can be estimated by the norm of $\varphi(x)$. From (3.15), We have

$$\begin{aligned}
\mathcal{F}(t) - \tilde{\mathcal{F}}(t) &= \Big(\int_0^1 f(x,t)dx\Big)^{-1}\Big[D^\alpha_{0_+}(g(t)-g(0)) + \sum_{n=1}^\infty 8\pi n\rho\varphi_{2n}e_\alpha(t,\lambda_n)\Big] \\
&\quad - \Big(\int_0^1 \tilde{f}(x,t)dx\Big)^{-1}\Big[D^\alpha_{0_+}(\tilde{g}(t)-\tilde{g}(0)) + \sum_{n=1}^\infty 8\pi n\rho\tilde{\varphi}_{2n}e_\alpha(t,\lambda_n)\Big] \\
\mathcal{F}(t) - \tilde{\mathcal{F}}(t) &= \Big(\int_0^1 f(x,t)dx \int_0^1 \tilde{f}(x,t)dx\Big)^{-1}\Big[\int_0^1 \tilde{f}(x,t)dx\,\Big(D^\alpha_{0_+}(g(t)-g(0)) \\
&\quad + \sum_{n=1}^\infty 8\pi n\rho\varphi_{2n}e_\alpha(t,\lambda_n)\Big) \\
&\quad - \int_0^1 f(x,t)dx\Big(D^\alpha_{0_+}(\tilde{g}(t)-\tilde{g}(0)) + \sum_{n=1}^\infty 8\pi n\rho\tilde{\varphi}_{2n}e_\alpha(t,\lambda_n)\Big)\Big] \\
\mathcal{F}(t) - \tilde{\mathcal{F}}(t) &= \Big(\int_0^1 f(x,t)dx \int_0^1 \tilde{f}(x,t)dx\Big)^{-1}\Big[\int_0^1 \tilde{f}(x,t)dx\Big(D^\alpha_{0_+}(g(t)-g(0)) \\
&\quad - D^\alpha_{0_+}(\tilde{g}(t)-\tilde{g}(0)) + \sum_{n=1}^\infty 8\pi n\rho e_\alpha(t,\lambda_n)\,(\varphi_{2n}-\tilde{\varphi}_{2n})\Big) \\
&\quad + D^\alpha_{0_+}(\tilde{g}(t)-\tilde{g}(0))\Big(\int_0^1 \tilde{f}(x,t)dx - \int_0^1 f(x,t)dx\Big) \\
&\quad + \sum_{n=1}^\infty 8\pi n\rho\tilde{\varphi}_{2n}e_\alpha(t,\lambda_n)\Big(\int_0^1 \tilde{f}(x,t)\,dx - \int_0^1 f(x,t)\,dx\Big)\Big].
\end{aligned}$$

Notice that we can consider $\varphi_{2n} - \tilde{\varphi}_{2n}$ as the Fourier coefficient of the function $\varphi - \tilde{\varphi}$; i.e.,

$$\varphi_{2n} - \tilde{\varphi}_{2n} = \int_0^1 (\varphi - \tilde{\varphi})(x)\ \sin(2\pi n x)\,dx.$$

Recall that $\lambda_n := 2\pi n\rho$, $e_\alpha(t,\lambda_n) := E_\alpha(-\lambda_n t^\alpha)$ and the following estimate for the Mittag-Leffler type function

$$|\lambda_n E_\alpha(-\lambda_n t^\alpha)| \le \frac{\lambda_n}{1+\lambda_n t^\alpha} \le C^*,$$

leads to the estimate

$$\|\mathcal{F} - \tilde{\mathcal{F}}\|_{C([0,T])} \le N_1\|\varphi - \tilde{\varphi}\|_{C([0,1])} + N_2\|f - \tilde{f}\|_{C^3([Q_T])} + N_3\|g - \tilde{g}\|_{AC([0,T])},$$

where $0 < 1/M_1 \le \left|\int_0^1 f(x,t)dx\right|$, $0 < 1/M_1 \le \left|\int_0^1 \tilde{f}(x,t)dx\right|$ and $N_1 = M_1^2M_3C^*$, $N_2 = M_1^2(M_6 + M_2C^*)$, $N_3 = M_1^2M_3C^*/\Gamma(1-\alpha)$ are constants independent of n.

From (3.16), we have

$$\|K - \tilde{K}\|_{C([0,T])\times C([0,T])} \le \sum_{n=1}^\infty 4\pi n\rho e_{\alpha,\alpha}(t-\tau,\lambda_n)\|f - \tilde{f}\|_{C^3([Q_T])}\,.$$

Recall that $e_{\alpha,\alpha}(t-\tau,\lambda_n) := (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)$. Then due to the estimate

$$|\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)| \le \frac{1}{t}\frac{(t-\tau)^\alpha\lambda_n}{1+\lambda_n(t-\tau)^\alpha} \le \mathcal{C}^*,$$

we have the estimate

$$\|K-\tilde{K}\|_{C([0,T])\times C([0,T])} \le 2C^*\|f-\tilde{f}\|_{C^3([Q_T])},$$

for $\mathcal{C}^*$ is a positive constant independent of n.

From (3.14), we obtain

$$\begin{aligned}
&a(t)-\tilde{a}(t)\\
&=\mathcal{F}(t)-\tilde{\mathcal{F}}(t)+\Big(\int_0^1 f(x,t)dx\Big)^{-1}\Big[\int_0^t K(t,\tau)a(\tau)d\tau-\int_0^t \tilde{K}(t,\tau)\tilde{a}(\tau)d\tau\Big]\\
&=\mathcal{F}(t)-\tilde{\mathcal{F}}(t)+\Big(\int_0^1 f(x,t)dx\Big)^{-1}\Big[\int_0^t a(\tau)\Big(K(t,\tau)-\tilde{K}(t,\tau)\Big)\,d\tau\\
&\quad-\int_0^t \tilde{K}(t,\tau)\Big(a(\tau)-\tilde{a}(\tau)\Big)\,d\tau\Big]
\end{aligned}$$

using the assumption $0<1/M_1\le\big|\int_0^1 f(x,t)dx\big|$ and due to the estimates of $\|\mathcal{F}-\tilde{\mathcal{F}}\|_{C([0,T])}$, $\|K-\tilde{K}\|_{C([0,T])\times C([0,T])}$, we have

$$\begin{aligned}
\|a-\tilde{a}\|_{C([0,T])} &\le \|\mathcal{F}-\tilde{\mathcal{F}}\|_{C([0,T])}+TM_1\mathcal{N}\|a-\tilde{a}\|_{C([0,T])}\\
&\quad+\frac{TM_1M_6}{1-TM_1\mathcal{N}}\|K-\tilde{K}\|_{C([0,T])\times C([0,T])}
\end{aligned}$$

or

$$(1-TM_1\mathcal{N})\|a-\tilde{a}\|_{C([0,T])}\le N_5\|\psi-\tilde{\psi}\|,$$

where

$$N_5=\max\big\{N_1,N_2+\frac{2C^*TM_1M_6}{1-TM_1\mathcal{N}},N_3\big\}.$$

For $t\in[0,T]$, we have

$$\|a-\tilde{a}\|_{C([0,T])}\le\frac{N_5}{1-TM_1\mathcal{N}}\|\psi-\tilde{\psi}\|.$$

From (3.5) a similar estimate can be obtained for $u-\tilde{u}$, which completes the proof. □

4. Appendix

The spectral problem (1.5)-(1.6) is a non-self-adjoint; it has the following conjugate (adjoint) problem:

$$Y''=-\lambda_nY,\quad x\in(0,1), \tag{4.1}$$

$$Y(0)=0,\quad Y'(0)=Y'(1). \tag{4.2}$$

In fact

$$\int_0^1 YX''=-X'(0)Y(0)+X(0)(Y'(0)-Y'(1))+\int_0^1 Y''X.$$

It is clear that the right side of this relation vanishes if $Y'(0)=Y'(1)$ and $Y(0)=0$.

The spectral problem (1.5)-(1.6) has the eigenvalues

$$\lambda_n=(2\pi n)^2\quad\text{for } n=0,1,2,\dots$$

and the eigenvectors

$$X_0=1,\ \text{for } \lambda_0=0,\quad X_n=\cos(2\pi nx),\ \text{for } \lambda_n=(2\pi n)^2\ n=1,2,\dots$$

The set of functions $\{X_0, X_n\}$ does not form a complete system and is not a basis for the space $L^2(0,1)$. To complete the basis (see [4]), we consider the associated eigenvectors $\tilde{X}$ for the λ_n corresponding to X_n defined as the solution of the problem

$$\tilde{X}'' = -\lambda_n \tilde{X} - X_n, \quad x \in (0,1), \tag{4.3}$$

$$\tilde{X}'(1) = 0, \quad \tilde{X}(0) = \tilde{X}(1). \tag{4.4}$$

If $\lambda_0 = 0$, problem (3.3)-(3.4) has no solution. For $\lambda_n = (2\pi n)^2$ for $n \in \mathbb{N}$, the problem (3.3)-(3.4) has the eigenvectors

$$\tilde{X}_n = \frac{(1-x)}{4\pi n} \sin(2\pi n x), \quad n \in \mathbb{N}.$$

Thus $S = \{X_0, X_n, \tilde{X}_n\}$ forms a complete system but not orthogonal.

We need another complete set of functions which together with the set S forms a bi-orthogonal system for the space $L^2(0,1)$. To obtain the other system, we shall consider the conjugate or adjoint problem (4.1)-(4.2).

Alike, solving (4.1)-(4.2), we obtain the eigenvectors $\{Y_0 = x, Y_n = x\cos(2\pi n)\}$, and associated eigenvectors are obtained from the boundary-value problem

$$\tilde{Y}'' = -\lambda_n \tilde{Y} - Y_n, \quad x \in (0,1), \tag{4.5}$$

$$\tilde{Y}(0) = 0, \quad \tilde{Y}'(0) = \tilde{Y}'(1). \tag{4.6}$$

The set $\tilde{S} = \{Y_0, Y_n, \tilde{Y}_n\}$, with

$$Y_0 = x, \quad Y_n = x\cos(2\pi n), \quad \tilde{Y}_n = \sin(2\pi n x)$$

is a complete system for the space $L^2(0,1)$.

The set of functions S and $\tilde{S}$ forms a bi-orthogonal system for the space $L^2(0,1)$. We can normalize the bi-orthogonal system and its final form is

$$\{X_0 = 2,\ X_n = \{4\cos(2\pi n x)\}_{n=1}^{\infty},\ \tilde{X}_n = \{4(1-x)\sin(2\pi n x)\}_{n=1}^{\infty}\}l \tag{4.7}$$

$$\{Y_0 = x,\ Y_n = \{x\cos(2\pi n x)\}_{n=1}^{\infty},\ \tilde{Y}_n = \{\sin(2\pi n x)\}_{n=1}^{\infty}\}. \tag{4.8}$$

Conclusion

The purpose of this paper is to determine the pair of functions $\{u(x,t), a(t)\}$, i.e., the temperature distribution and the source term for the fractional diffusion equation (1.1)-(1.4). The problem in solving the inverse problem is not only due to the nonlocal boundary conditions (1.3) but also due to the presence of the fractional derivative in time. The underlying spectral problem for (1.1)-(1.3) is non-self-adjoint.

For the solution of the inverse problem we use two basis for the space $L^2(0,1)$, which form the bi-orthogonal system (see Il'in [4] and Keldysh [8]). Due to this bi-orthogonal system we are able to expand the solution in terms of the functions of the bi-orthogonal system. We show the existence and uniqueness of the solution of the inverse problem using properties of the Mittag-Leffler function and using the over-determination condition of integral type (1.4). The result about the continuous dependence of the solution of the inverse problem on the data is proved.

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