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APPROXIMATE CONTROLLABILITY OF ABSTRACT IMPULSIVE FRACTIONAL NEUTRAL EVOLUTION EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

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Abstract. In this article, we study the approximate controllability of impulsive abstract fractional neutral evolution equations in Banach spaces. The main results are obtained by using Krasnoselkii's fixed point theorem, fractional calculus and methods of controllability theory. An application is provided to illustrate the theory. Here we have provided new definition of phase space for the impulsive and infinite delay term. Our result is new for the approximate controllability with infinite delay in Hilbert space.

1. Introduction

Differential equations of fractional order have proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [5, 6, 13, 1, 14, 18, 22, 23, 25]). Now a days, controllability theory for linear systems has already been well established, for finite and infinite dimensional systems (see [11]). Several authors have extended these concepts to infinite dimensional systems represented by nonlinear evolution equations in infinite-dimensional spaces, (see [12, 21, 24, 29, 30, 33, 34]). On the other hand, approximate controllability problems for fractional evolution equations in Hilbert spaces is not yet sufficiently investigated and there are only few works on it (see [29, 30, 34]). On the other hand, it has been observed that the existence or the controllability results proved by different authors are through an axiomatic definition of the phase space given by Hale and Kato [15]. However, as remarked by Hino, Murakami, and Naito [19], it has come to our attention that these axioms for the phase space are not correct for the impulsive systems with infinite delay, refer the work in [7, 8]. Benchohra et al. [2] discussed the controllability of first and second order neutral functional differential and integro-differential inclusions in a Banach space with non-local conditions, without impulse effect. Chang and Li [9] obtained the controllability result for functional integro-differential inclusions on an unbounded domain without impulse term. Benchohra et al. [3] studied the

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existence result for damped differential inclusion with impulse effect. Hernandez et al. [16] proved the existence of solutions for impulsive partial neutral functional differential equations for first and second order systems with infinite delay. In last couple of years, JinRong Wang et al. [35, 36, 37, 38, 39, 40, 41, 42] discussed various development in the theory of boundary value problems, optimal feedback control, and new concept of solutions for impulsive fractional evolution/differential equations.

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has been significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Benchohra et al. [4], Lakshmikantham et al. [20], and Samoilenko and Perestyuk [32], and the references therein. A neutral generalization of impulsive differential equations is abstract impulsive differential equations in Banach spaces. For general aspects of impulsive differential equations, see monographs given in [27, 28].

The purpose of this paper is to study the approximate controllability of impulsive fractional neutral evolution differential system with infinite delay using the new definition of the phase space for impulsive term and infinite delay term.

In this article, Section 2 provides the definitions and some preliminary results to be used in the main theorems stated and proved. In Section 3, we focus on existence of the solutions of (2.1). We study the main results in Section 4. Finally an application is given in Section 5 to justify the theory.

2. Preliminaries

Recently, in [10] the existence of solutions for an impulsive neutral functional differential equations of the form

$$\begin{gathered}\frac{d}{dt}(u(t)+F(t,u_t)) = A(t)u(t)+G(t,u_t), \quad t\in I,\ t\neq t_i\\ \Delta u(t_i) = I_i(u_{t_i}),\\ u_0=\varphi\in\mathcal{B},\end{gathered}$$

was studied using Leray-Schauder's alternative theorem. We consider the following impulsive fractional neutral evolution differential system with infinite delay:

$$\begin{gathered}\frac{d^\alpha}{dt^\alpha}[x(t)-h(t,x_t)] = Ax(t)+Bu(t)+f(t,x_t), \quad t\in[0,T],\ t\neq t_i\\ x(t)=\phi(t)\in\mathcal{B}_h,\\ \Delta x(t_i)=I_i(x_{t_i})\end{gathered} \tag{2.1}$$

where the state x takes values in a Banach space X, the control function takes values in a Hilbert space U. The functions h, f will be specified in the sequel and I is an interval of the form $[0,T)$; $0<t_1<t_2<\cdots<t_i<\cdots<T$ are prefixed numbers. Let $x_t(\cdot)$ denote $x_t(\theta)=x(t+\theta)$, $\theta\in(-\infty,0]$. Assume that $l:(-\infty,0]\to(-\infty,0)$ is a continuous function satisfying $l=\int_{-\infty}^0 l(t)dt<\infty$.

We present the abstract phase space $\mathcal{B}_h$. Assume that $h:\,]-\infty,0]\to]0,\infty[$ be a continuous function with $l=\int_{-\infty}^0 h(s)ds<+\infty$. Define,

$$\mathcal{B}_h := \big\{\phi:]-\infty,0]\to X \text{ such that, for any } r>0,\ \phi(\theta) \text{ is bounded and}$$

measurable function on $[-r,0]$ and $\displaystyle\int_{-\infty}^{0} h(s)\sup_{s\le\theta\le 0}|\phi(\theta)|ds<+\infty\}$.

Here, $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h}=\int_{-\infty}^{0} h(s)\sup_{s\le\theta\le 0}|\phi(\theta)|ds,\quad \forall\phi\in\mathcal{B}_h.$$

Then it is easy to show that $(\mathcal{B}_h,\|.\|_{\mathcal{B}_h})$ is a Banach space.

Lemma 2.1. *Suppose $y\in\mathcal{B}_h$; then, for each $t\in J$, $y_t\in\mathcal{B}_h$. Moreover,*

$$l|y(t)|\le\|y_t\|_{\mathcal{B}_h}\le l\sup_{s\le\theta\le 0}(|y(s)|+\|y_0\|_{\mathcal{B}_h}),$$

where $l:=\int_{-\infty}^{0}h(s)ds<+\infty$.

Proof. For any $t\in[0,a]$, it is easy to see that y_t is bounded and measurable on $[-a,0]$ for $a>0$, and

$$\begin{aligned}
\|y_t\|_{\mathcal{B}_h}&=\int_{-\infty}^{0} h(s)\sup_{\theta\in[s,0]}|y_t(\theta)|ds\\
&=\int_{-\infty}^{-t} h(s)\sup_{\theta\in[s,0]}|y(t+\theta)|ds+\int_{-t}^{0} h(s)\sup_{\theta\in[s,0]}|y(t+\theta)|ds\\
&=\int_{-\infty}^{-t} h(s)\sup_{\theta_1\in[t+s,t]}|y(\theta_1)|ds+\int_{-t}^{0} h(s)\sup_{\theta_1\in[t+s,t]}|y(\theta_1)|ds\\
&\le\int_{-\infty}^{-t} h(s)\Big[\sup_{\theta_1\in[t+s,0]}|y(\theta_1)|+\sup_{\theta_1\in[0,t]}|y(\theta_1)|\Big]ds+\int_{-t}^{0} h(s)\sup_{\theta_1\in[0,t]}|y(\theta_1)|ds\\
&=\int_{-\infty}^{-t} h(s)\sup_{\theta_1\in[t+s,0]}|y(\theta_1)|ds+\int_{-\infty}^{0} h(s)ds.\sup_{s\in[0,t]}|y(s)|\\
&\le\int_{-\infty}^{-t} h(s)\sup_{\theta_1\in[s,0]}|y(\theta_1)|ds+l.\sup_{s\in[0,t]}|y(s)|\\
&\le\int_{-\infty}^{0} h(s)\sup_{\theta_1\in[s,0]}|y(\theta_1)|ds+l\sup_{s\in[0,t]}|y(s)|\\
&=\int_{-\infty}^{0} h(s)\sup_{\theta_1\in[s,0]}|y_0(\theta_1)|ds+l\sup_{s\in[0,t]}|y(s)|\\
&=l\sup_{s\in[0,t]}|y(s)|+\|y_0\|_{\mathcal{B}_h}
\end{aligned}$$

Since $\phi\in\mathcal{B}_h$, then $y_t\in\mathcal{B}_h$. Moreover,

$$\|y_t\|_{\mathcal{B}_h}=\int_{-\infty}^{0} h(s)\sup_{\theta\in[s,0]}|y_t(\theta)|ds\ge|y_t(\theta)|\int_{-\infty}^{0}h(s)ds=l|y(t)|$$

The proof is complete. ☐

The phase space $\mathcal{B}_h$ defined above also satisfies the following properties:

(B1) If $x:(-\infty,\sigma+a]\to X$, $a>0$, $\sigma\in\mathbb{R}$ such that $x_\sigma\in\mathcal{B}_h$, and $x[\sigma,\sigma+a]\in PC([\sigma,\sigma+a],X)$, then for every $t\in[\sigma,\sigma+a)$ the following conditions hold:
  (i) x_t is in $\mathcal{B}_h$;
  (ii) $\|x(t)\|_X\le H\|x_t\|_{\mathcal{B}_h}$;

(iii) $\|x_t\|_{\mathcal{B}_h} \le K(t-\sigma)\sup\{\|x(s)\|_X : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|\mathcal{B}_h$, where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of x.

(B2) The space $\mathcal{B}_h$ is complete.

Example 2.2. The $PC_r \times L^2(g,X)$ be the phase space. Let $r > 0$ and $g : (-\infty,-r] \to R$ be a non- negative, locally Lebesgue integrable function. Assume that there is a non-negative measurable, locally bounded function $\eta(\cdot)$ on $(-\infty,0]$ such that $g(\xi+\theta) \le \eta(\xi)g(\theta)$ for all $\xi \in (-\infty,0]$ and $\theta \in (-\infty,-r]\backslash N_\xi$, where $N_\xi \subset (-\infty,-r]$ is a set with Lebesgue measure zero. We denote by $PC_R \times L^2(g,X)$ the set of all functions $\varphi : (-\infty,0] \to X$ such that $\varphi|[-r,0] \in PC([-r,0],X)$ and $\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|_X^2 d\theta < +\infty$. In $PC_r \times L^2(g,X)$, we consider the seminorm defined by

$$\|\varphi\|_{\mathcal{B}_h} = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\|_X + \Big(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|_X^2 d\theta\Big)^{1/2}.$$

From the proceeding conditions, the space $PC_r \times L^2(g,X)$ satisfies (B1) and (B2). Moreover, when $r = 0$, we can take $H = 1$, $K(t) = \big(1 + \int_{-t}^0 g(\theta)\big)^{\frac{1}{2}}$ and $M(t) = \eta(-t)$ for $t \ge 0$.

Let $0 < t_1 < t_2 < \dots < t_n < T$ be pre-fixed numbers. We introduce the space $PC = PC([0,a];X)$ formed by functions $u : [0,T] \to X$ such that are continuous at $t \ne t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists, for $i = 1,\dots n$. In this paper, we assume that PC is endowed with the norm $\|u\|_{PC} = \sup_{s\in[0,a]} \|u(s)\|_X$. It is clear that $(PC, \|\cdot\|_{PC})$ is a Banach space; see[17] for more details.

In what follows, we put $t_0 = 0$, $t_{n+1} = T$, and for $u \in PC$, we denote by $\bar{u_i} \in C([t_i,t_{i+1}],X)$, $i = 0,1,2,\dots,n$, the functions

$$\tilde{u_i}(t) = \begin{cases} u(t_i), & \text{for } t \in (t, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases}$$

Moreover, for $\mathcal{B}_h \subset PC$, we employ the notation $\mathcal{B}_{h_i}$, $i = 1,2,\dots,n$, for the sets $\tilde{\mathcal{B}}_{h_i} = \{\tilde{u_i} : u \in \mathcal{B}_h\}$.

Lemma 2.3 ([2]). *A set $B \subset PC$ is relatively compact in PC if and only if the set $\tilde{\mathcal{B}}_{h_i}$, is relatively compact in the space $C([t_i,t_{i+1}];X)$, for every $i = 0,1,\dots,n$.*

We introduced symbols which will be useful throughout this article. Let $(X, \|\cdot\|)$ be a separable reflexive Banach space and let $(X^*, \|\cdot\|_*)$ stands for its dual space with respect to the continuous pairing $\langle\cdot,\cdot\rangle$. We may assume, without loss of generality, that X and X^* are smooth and strictly convex, by virtue of renorming theorem (for example, see [21]). In particular, this implies that the duality mapping J of X into X^* given by the following relations

$$\|J(z)\|_* = \|z\|, \quad \langle J(z), z\rangle = \|z\|^2, \quad \text{for all } z \in X$$

is bijective, homogeneous, demicontinuous, i.e., continuous from X with a strong topology into X^* with weak topology and strictly monotonic. Moreover, $J^{-1} : X^* \to X$ is also duality mapping. Note that our results are new even for the approximate controllbility of impulsive fractional neutral differential equations with infinite delay in Hilbert spaces.

In this article, we also assume that $-A : D(A) \subset X \to X$ is the infinitesimal generator of a compact analytic semigroup $S(t), t > 0$, of uniformly bounded linear operator in X, that is, there exists $M > 1$ such that $\|S(t)\|_{L(X)} \le M$ for all $t \ge 0$. Without loss of generality, let $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$. Then for any $\beta > 0$, we can define $A^{-\beta}$ by $A^{-\beta} := \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} S(t) dt$.

It follows that each $A^{-\beta}$ is an injective continuous and homorphism of X. Hence we can define $A^\beta := (A^{-\beta})^{-1}$, which is a closed bijective linear operator in X. It can be shown that each A^β has dense domain and that $D(A^\gamma) \subset D(A^\beta)$ for $0 \le \beta \le \gamma$. Moreover $A^{\beta+\gamma}x = A^\beta A^\gamma x = A^\gamma A^\beta x$ for every $\beta, \gamma \in \mathbb{R}$ and $x \in D(A^\mu)$ with $\mu := \max(\beta, \gamma, \beta + \gamma)$, where $A^0 = I$, I is the identity in X.

We denote by X_β the Banach space of $D(A^\beta)$ equipped with norm $\|x\|_\beta := \|A^\beta x\|$ for $x \in D(A^\beta)$, which is equivalent to the graph norm of A^β. Then we have $X_\gamma \hookrightarrow X_\beta$, for $0 \le \beta \le \gamma$ (with $X_0 = X$), and the embedding is continuous. Moreover, A^β has the following basic properties.

Lemma 2.4 ([26]). *A^β has the following properties*

(i) *$S(t) : X \to X_\beta$ for each $t > 0$ and $\beta \ge 0$.*
(ii) *$A^\beta S(t)x = S(t)A^\beta x$ for each $x \in D(A^\beta)$ and $t \ge 0$.*
(iii) *for every t, $A^\beta S(t)$ is bounded in X and there exists $M_\beta > 0$ such that*
$$\|A^\beta S(t)\| \le M_\beta t^{-\beta}.$$
(iv) *$A^{-\beta}$ is a bounded linear operator for $0 \le \beta \le 1$ in X, there exists a constant C_β such that $\|A^{-\beta}\| \le C_\beta$ for $0 \le \beta \le 1$.*

We recall the following known definitions from fractional calculus. For more details, see [5, 31].

Definition 2.5. The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function f is defined as
$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,\ \alpha > 0,$$
provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.6. The Riemann- Liouville derivative of order α with the lower limit 0 for a function $f : [0, \infty) \to R$ is written as
$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,\ n-1 < \alpha < n.$$

Definition 2.7. The Caputo derivative of order α for a function $f : [0, \infty) \to R$ is written as
$${}^c D^\alpha f(t) = {}^L D^\alpha \Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \Big), \quad t > 0,\ n-1 < \alpha < n.$$

For $x \in X$, we define two families $\mathfrak{T}_\alpha(t) : t \ge 0$ and $\mathfrak{A}_\alpha(t) : t \ge 0$ of operators by
$$\mathfrak{T}_\alpha(t) = \int_0^\infty \Psi_\alpha(\theta) S(t^\alpha \theta) d\theta, \quad \mathfrak{A}_\alpha(t) = \alpha \int_0^\infty \theta \Psi_\alpha(\theta) S(t^\alpha \theta) d\theta,$$
where
$$\Psi_\alpha(\theta) = \frac{1}{\pi\alpha} \sum_{n=1}^\infty (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty)$$

is the function defined on $(0,\infty)$ which satisfies

$$\Psi_\alpha(\theta) \geq 0, \quad \int_0^\infty \Psi_\alpha(\theta)d\theta = 1,$$
$$\int_0^\infty \theta^\zeta \Psi_\alpha(\theta)d\theta = \frac{\Gamma(1+\zeta)}{\Gamma(1+\alpha\zeta)}, \quad \zeta \in (-1,\infty).$$

The following lemma follows from the results in [43].

Lemma 2.8. *The operators $\mathfrak{T}_\alpha$ and $\mathfrak{A}_\alpha$ have the following properties:*

(i) *For any fixed $t \geq 0$, any $x \in X_\beta$, the operators $\mathfrak{T}_\alpha(t)$ and $\mathfrak{A}_\alpha(t)$ are linear and bounded, i.e. for any $x \in X_\beta$,*
$$\|\mathfrak{T}_\alpha(t)\| \leq M\|x\|_\beta, \quad \|\mathfrak{A}_\alpha(t)\| \leq \frac{M}{\Gamma(\alpha)}\|x\|_\beta;$$
(ii) *The operators $S_\alpha(t)$ and $\mathfrak{A}_\alpha(t)$ are strongly continuous for all $t \geq 0$;*
(iii) *$S_\alpha(t)$ and $\mathfrak{A}_\alpha(t)$ are norm-continuous in X for $t > 0$;*
(iv) *$S_\alpha(t)$ and $\mathfrak{A}_\alpha(t)$ are compact operators in X for $t > 0$;*
(v) *For any $t > 0$, the restriction of $S_\alpha(t)$ to X_β and the restriction of $\mathfrak{A}_\alpha(t)$ to X_β are norm-continuous;*
(vi) *For every $t > 0$ the restriction of $S_\alpha(t)$ to X_β and the restriction of $\mathfrak{A}_\alpha(t)$ to X_β are compact operators in X_β;*
(vii) *For all $x \in X$ and $t \in [0,T]$,*
$$\|A^\beta \mathfrak{A}_\alpha(t)x\| \leq C_\beta t^{-\alpha\beta}\|x\|, \quad C_\beta := \frac{M_\beta \alpha \Gamma(2-\beta)}{\Gamma(1+\alpha(1-\beta))}.$$

In the following definition, we introduce the concept of a mild solution for (2.1).

Definition 2.9. A function $x(\cdot, u) \in PC([0,T],X)$ is said to be mild solution of (2.1) if for any $u \in L_2([0,T],U)$, and $t \in I$ the integral equation

$$\begin{aligned} x(t) &= \mathfrak{T}_\alpha(t)[\phi(0)+g(0,\phi)] - g(t,x_t) - \int_0^t (t-s)^{\alpha-1} A\mathfrak{A}_\alpha(t-s)g(s,x_s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathfrak{A}_\alpha(t-s)[Bu(s)+f(s,x_s)]ds + \sum_{t_i<t} \mathfrak{T}_\alpha(t-t_i)I_i(x_{t_i}) \\ &\quad + \sum_{t_i<t}[g(t,x_t)_{|t_i^+} - g(t,x_t)_{|t_i^-}], \end{aligned} \tag{2.2}$$

is satisfied.

Let $x(T,u)$ be the state value of (2.1) at terminal time T corresponding to the control u. Introduce the set $\mathfrak{R}(T) = \{x(T,u) : u \in L_2([0,T],U\}$, which is called the reachable set of system (2.2) at terminal time T, its closure in X is denoted by $\overline{\mathfrak{R}(T)}$.

Definition 2.10. System (2.1) is said to be approximately controllable on $[0,T]$ if $\overline{\mathfrak{R}(T)} = X$; that is, given an arbitrary $\varepsilon > 0$ it is possible to steer the system from the initial point x_0 to within a distance ε from all points in the space X at time T.

To investigate the approximate controllability of system (2.1), we assume the following conditions:

(H1) $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$ in X, $0 \in \rho(-A)$, $S(t)$ is compact for $t > 0$, and there exists a positive constant M such that $\|S(t)\| \le M$;

(H2) The function $g : [0, T] \times \mathcal{B}_h \to X$ is continuous and there exists some constant $M_g > 0$, $0 < \beta < 1$, such that g is X_β-valued and

$$\|A^\beta g(t, x) - A^\beta g(t, y)\| \le M_g \|x - y\|_{\mathcal{B}_h}, \quad x, y \in \mathcal{B}_h,\ t \in [0, T],$$
$$\|A^\beta g(t, x)\| \le M_g(1 + \|x\|_{\mathcal{B}_h}).$$

(H3) The function $f : [0, T] \times \mathcal{B}_h \to X$ satisfies following properties:
  (a) $f(t_1, .) : \mathcal{B}_h \to X$ is continuous for each $t \in [0, T]$ and for each $x \in \mathcal{B}_h, f(., x) : [0, T] \to X$ is strongly measurable;
  (b) There is a positive integrable function $n \in L^\infty([0, T], [0, +\infty))$ and a continuous nondecreasing function $\Lambda_f : [0, \infty) \to (0, \infty)$ such that for every $(t, x) \in [0, T] \times \mathcal{B}_h$, we have

$$\|f(t, x)\| \le n(t)\Lambda_f(\|x\|_{\mathcal{B}_h}), \quad \lim\inf_{r\to\infty} \frac{\Lambda_f(r)}{r} = \sigma_f < \infty.$$

(H4) The following inequality holds

$$\Big(1 + \frac{1}{\varepsilon} M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha - 1}\Big)\Big(M_g\|A^{-\beta}\|l + K(\alpha, \beta)M_g \frac{T^{\alpha\beta}}{\alpha\beta} l$$
$$+ \frac{M}{\Gamma(\alpha)} \frac{T^\alpha}{\alpha} \sigma_f \sup_{s\in J} n(s)\Big) + M \sum_{i=1}^{N} \sigma_i < 1,$$

where $M_B := \|B\|$, $M_{\mathfrak{A}} := \|\mathfrak{A}_\alpha\|$, and $K(\alpha, \beta) = \frac{\alpha M_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}$;

(H5) The maps $I_i : \mathcal{B}_h \to X$ are completely continuous and uniformly bounded, $i \in F = \{1, 2, \dots, N\}$. In what follows, we denote $N_i = \sup\{\|I_i(\phi)\| : \phi \in \mathcal{B}_h\}$ and

$$\lim\inf_{r\to\infty} \frac{N_i}{r} = \sigma_i < \infty\,;$$

(H6) There are positive constants L_i such that

$$\|I_i(\psi_1) - I_i(\psi_2)\| \le L_i\|\psi_1 - \psi_2\|_{\mathcal{B}_h}, \quad \psi_1, \psi_2 \in \mathcal{B}_h,\ i \in F;$$

(H7) For every $h \in X$, $z_\alpha(h) = \varepsilon(\varepsilon I + \Gamma_0^T J)^{-1}(h)$ converges to zero as $\varepsilon \to 0^+$ in strong topology, where

$$\Gamma_0^T := \int_0^T (T - s)^{2(\alpha-1)} \mathfrak{A}_\alpha(T - s)BB^* \mathfrak{A}_\alpha^*(T - s)ds,$$

and $z_\varepsilon(h)$ is a solution of the equation $\varepsilon z_\varepsilon + \Gamma_0^T J(z_\varepsilon) = \varepsilon h$.

Let $PC_T = \{x : x \in PC((-\infty, T], X), \quad x_0 = \phi \in \mathcal{B}_h\}$. Let $\|\cdot\|$ be the seminorm

$$\|x\|_T = \|x_0\|_{\mathcal{B}_h} + \sup_{0\le s\le T} \|x(s)\|, \quad x \in PC_T.$$

3. Existence theorem

To formulate the controllability problem in a form suitable for applying a fixed point theorem, it is assumed that the corresponding linear system is approximately controllable. Then it will be shown that the system (2.1) is approximately controllable if for all $\varepsilon > 0$ there exists a continuous function $x(\cdot) \in PC([0,T],X)$ such that

$$\begin{aligned} u_\varepsilon(t,s) &= (T-t)^{\alpha-1}B^*\mathfrak{A}_\alpha^*(T-t)J((\varepsilon I+\Gamma_0^T J)^{-1}p(x)),\\ x(t) &= \mathfrak{T}_\alpha(t)[\phi(0)+g(0,\phi)] - g(t,x_t) - \int_0^t (t-s)^{\alpha-1}A\mathfrak{A}_\alpha(t-s)g(s,x_s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathfrak{A}_\alpha(t-s)[Bu_\varepsilon(s,x)+f(s,x_s)]ds + \sum_{t_i<t}\mathfrak{T}_\alpha(t-t_i)I_i(x_{t_i}) \\ &\quad + \sum_{t_i<t}[g(t,x_t)_{|t_i^+} - g(t,x_t)_{|t_i^-}], \quad t\in I \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} p(x) &= h - \mathfrak{T}_\alpha(T)[\phi(0)+g(0,\phi)] + g(T,x_T) + \int_0^T (T-s)^{\alpha-1}A\mathfrak{A}_\alpha(T-s)g(s,x_s)ds \\ &\quad - \int_0^T (T-s)^{\alpha-1}\mathfrak{A}_\alpha(T-s) + f(s,x_s)ds, \quad h\in X. \end{aligned}$$

The control in (3.1) steers the system (2.1) from $\phi(0)$ to $h-\varepsilon J\big(\varepsilon I+\Gamma_0^T J)^{-1}p(x)\big)$ provided that the system (3.1) has a solution.

Theorem 3.1. *Assume that Assumptions* (H1)–(H6) *hold and* $\frac{1}{2} < \alpha \le 1$. *Then there exists a solution to the equation* (3.1).

The proof of the above theorem follows from Lemmas 3.2–3.7 and infinite dimensional analogue of Arzela- Ascoli theorem.

For $\varepsilon > 0$ consider the operator $\Phi_\varepsilon : PC_T \to PC_T$ defined by

$$(\Phi_\varepsilon x)(t) := \begin{cases} \phi(t), \quad t\in(-\infty,0]; \\ \mathfrak{T}_\alpha(t)[\phi(0)+g(0,\phi)] - g(t,x_t) - \int_0^t (t-s)^{\alpha-1}A\mathfrak{A}_\alpha(t-s)g(s,x_s)ds \\ + \int_0^t (t-s)^{\alpha-1}\mathfrak{A}_\alpha(t-s)[Bu_\varepsilon(s,x)+f(s,x_s)]ds \\ + \sum_{t_i<t}\mathfrak{T}_\alpha(t-t_i)I_i(x_{t_i}) \\ + \sum_{t_i<t}[g(t,x_t)_{|t_i^+} - g(t,x_t)_{|t_i^-}], \quad t\in[0,T]. \end{cases}$$

where

$$\begin{aligned} u_\varepsilon(t,s) &= (T-t)^{\alpha-1}B^*\mathfrak{A}_\alpha^*(T-t)J((\varepsilon I+\Gamma_0^T J)^{-1}p(x)) \\ &:= (T-t)^{\alpha-1}v_\varepsilon(t,x). \end{aligned}$$

It will be shown that for all $\varepsilon > 0$ the operator Φ_ε: $PC_T \to PC_T$ has a fixed point.

Suppose that $x(t) = \tilde\phi(t) + z(t)$, $t\in(-\infty,T]$, where

$$\tilde\phi(t) = \begin{cases} \phi(t), & \text{for } t\in(-\infty,0], \\ \mathfrak{T}_\alpha(t)\phi(0), & \text{for} t\in[0,T]. \end{cases}$$

Set $PC_T^0 = \{z\in C_T : z_0 = 0 \in \mathcal{B}_h\}$. For any $z\in PC_T^0$, we have

$$\|z\|_T = \|z_0\|_{\mathcal{B}_h} + \sup_{0\le s\le T}\|z(s)\| = \sup_{0\le s\le T}\|z(s)\|.$$

Thus $(PC_T^0, \|\cdot\|_T)$ is a Banach space. For each positive number $r>0$, set

$$B_r := \{z\in PC_T^0 : \|z\|_T \le r\}.$$

It is clear that B_r is bounded closed convex set in PC_T^0. For any $z \in B_r$ we have

$$\begin{aligned}
\|\tilde{\phi}_t + z_t\|_{\mathcal{B}_h} &\le \|\tilde{\phi}_t\|_{\mathcal{B}_h} + \|z_t\|_{\mathcal{B}_h} \\
&\le l \sup_{0\le s\le T} \|\tilde{\phi}(s)\| + \|\tilde{\phi}_0\|_{\mathcal{B}_h} + l \sup_{0\le s\le T} \|z(s)\| + \|Z_0\|_{\mathcal{B}_h} \\
&\le l(M\|\phi(0)\| + r) + \|\phi\|_{\mathcal{B}_h} := R(r).
\end{aligned}$$

Consider the maps $\prod_\varepsilon, \Theta_\varepsilon, \Upsilon_\varepsilon : PC_T^0 \to PC_T^0$ defined by

$$\Big(\prod_\varepsilon z\Big)(t) := \begin{cases} 0, & t \in (-\infty, 0]; \\ \mathfrak{T}_\alpha(t) - g(0,\phi)] - g(t, \tilde{\phi}_t + z_t) \\ - \int_0^t (t-s)^{\alpha-1} A\mathfrak{A}_\alpha(t-s) g(s, \tilde{\phi}_s + z_s) ds, & t \in [0,T] \end{cases}$$

$$\big(\Theta_\varepsilon z\big)(t) := \begin{cases} 0, \quad t \in (-\infty, 0]; \\ \int_0^t (t-s)^{\alpha-1} \mathfrak{A}_\alpha(t-s) \big[Bu_\varepsilon(s, \tilde{\phi} + z) + f(s, \tilde{\phi}_s + z_s)\big] ds, & t \in [0,T] \end{cases}$$

$$\big(\Upsilon_\varepsilon z\big)(t) := \begin{cases} 0, & t \in (-\infty, 0]; \\ \sum_{0<t_i<t} \mathfrak{T}_\alpha(t-t_i) I_i(\tilde{\phi}_{t_i} + z_{t_i}), & t \in [0,T]. \end{cases}$$

Obviously, the operator φ_ε has a fixed point if and only if operator $\prod_\varepsilon + \Theta_\varepsilon + \Upsilon_\varepsilon$ has a fixed point. In order to prove that $\prod_\varepsilon + \Theta_\varepsilon + \Upsilon_\varepsilon$ has a fixed point. We will employ the Krasnoselkii fixed point theorem.

Lemma 3.2. *Under Assumptions (H1)–(H6), for any $\varepsilon > 0$ there exists a positive number $r := r(\varepsilon)$ such that $(\prod_\varepsilon + \Theta_\varepsilon + \Upsilon_\varepsilon)(B_r) \subset B_r$.*

Proof. Let $\varepsilon > 0$ be fixed. If the statement were not true, then for each $r > 0$, there exists a function $z_r \in B_r$, but $(\prod_\varepsilon + \Theta_\varepsilon + \Upsilon_\varepsilon)(z_r) \notin B_r$. So for some $t = t(r) \in [0,T]$ one can show that

$$\begin{aligned}
r &\le \big\|((\prod_\varepsilon + \Theta_\varepsilon + \Upsilon_\varepsilon) z_r)(t)\big\| \\
&\le \big\|\mathfrak{T}_\alpha(t) g(0,\phi)\big\| + \big\|g(t, \tilde{\phi}_t + z_t)\big\| + \Big\|\int_0^t (t-s)^{\alpha-1} A\mathfrak{A}_\alpha(t-s) g(s, \tilde{\phi}_s + z_s) ds\Big\| \\
&\quad + \Big\|\int_0^t (t-s)^{\alpha-1} \mathfrak{A}_\alpha(t-s) f(s, \tilde{\phi}_s + z_s) ds\Big\| \\
&\quad + \Big\|\int_0^t (t-s)^{\alpha-1} \mathfrak{A}_\alpha(t-s) Bu_\varepsilon(s, \phi + z) ds\Big\| \\
&+ \Big\|\sum_{0<t_i<t} \mathfrak{T}_\alpha(t-t_i) I_i(\tilde{\phi}_{t_i} + z_{t_i})\Big\| \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{3.2}$$

Let us estimate I_i, $i = 1, \dots, 6$. By Assumption (H2), we have

$$I_1 \le M\|A^{-\beta}\|\|A^\beta g(0,\phi)\| \le MM_g\|A^{-\beta}\|(1 + \|\phi\|_{\mathcal{B}_h}), \tag{3.3}$$

$$\begin{aligned}
I_2 &\le \|A^{-\beta}\|\|\|A^\beta g(t, \tilde{\phi}_t + z_t) \\
&\le M_g\|A^{-\beta}\|\big(1 + \|\tilde{\phi}_t + z_t\|_{\mathcal{B}_h}\big) \\
&\le M_g\|A^{-\beta}\|\big(1 + R(r)\big).
\end{aligned} \tag{3.4}$$

Using Lemma 2.4 and Hölder inequality, one can deduce that

$$
\begin{aligned}
I_3 &\leq \Big\| \int_0^t (t-s)^{\alpha-1} A^{1-\beta} \mathfrak{A}_\alpha(t-s) A^\beta g(s, \tilde{\phi}_s + z_s) ds \Big\| \\
&\leq \frac{M_{1-\beta}\alpha\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \int_0^t (t-s)^{\alpha\beta-1} \|A^\beta g(s, \tilde{\phi}_s + z_s)\| \, ds \\
&\leq K(\alpha,\beta) \int_0^t (t-s)^{\alpha\beta-1} M_g \big(1 + \|\tilde{\phi}_s + z_s)\|_{\mathcal{B}_h}\big) ds \\
&\leq K(\alpha,\beta) M_g \frac{T^{\alpha\beta}}{\alpha\beta}(1+R(r)).
\end{aligned}
\tag{3.5}
$$

Using Assumption (H3), we have

$$
\begin{aligned}
I_4 &\leq \int_0^t \|(t-s)^{\alpha-1} \mathfrak{A}_\alpha(t-s) f(s, \tilde{\phi}_s + z_s)\| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, \tilde{\phi}_s + z_s)\| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} n(s) \Lambda_f\big(\|\tilde{\phi}_s + z_s)\|\big) ds \\
&\leq \frac{M}{\Gamma(\alpha)} \frac{T^\alpha}{\alpha} \Lambda_f\big(R(r)\big) \sup_{s\in J} n(s).
\end{aligned}
\tag{3.6}
$$

Combining the estimates (3.2)-(3.6) yields

$$
\begin{aligned}
&I_1 + I_2 + I_3 + I_4 \\
&< M M_g \|A^{-\beta}\|(1 + \|\phi\|_{\mathcal{B}_h}) + M_g \|A^{-\beta}\|\big(1 + R(r)\big) \\
&\quad + K(\alpha,\beta) M_g \frac{T^{\alpha\beta}}{\alpha\beta}(1+R(r)) + \frac{M}{\Gamma(\alpha)}) \frac{T^\alpha}{\alpha} \Lambda_f\big(R(r)\big) \sup_{s\in J} n(s) := \Lambda.
\end{aligned}
\tag{3.7}
$$

On the other hand,

$$
\begin{aligned}
I_5 &\leq \int_0^t \|(t-s)^{\alpha-1} \mathfrak{A}_\alpha(t-s) B u_\varepsilon(s, \phi + z)\| ds \\
&= \int_0^t \|(t-s)^{\alpha-1} (T-s)^{\alpha-1} \mathfrak{A}_\alpha(t-s) B B^* \mathfrak{A}_\alpha^*(T-t) \\
&\quad \times J((\varepsilon I + \Gamma_0^T J)^{-1} p(\phi + z))\| ds \\
&\leq \int_0^t \|(t-s)^{\alpha-1} (T-s)^{\alpha-1} \mathfrak{A}_\alpha(t-s) B B^* \mathfrak{A}_\alpha^*(T-t)\| ds \\
&\quad \times \|J((\varepsilon I + \Gamma_0^T J)^{-1} p(\phi + z))\| \\
&\leq M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha-1} \|J((\varepsilon I + \Gamma_0^T J)^{-1} p(\phi + z))\| \\
&\leq M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha-1} \|(\varepsilon I + \Gamma_0^T J)^{-1} p(\phi + z)\| \\
&\leq \frac{1}{\varepsilon} M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha-1} \|p(\phi + z)\| \\
&\leq \frac{1}{\varepsilon} M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha-1} \Delta
\end{aligned}
$$

and

$$I_6 \le \|\sum_{0<t_i<t} \mathfrak{T}_\alpha(t-t_i) I_i(\tilde{\phi}_{t_i} + z_{t_i})\| \le M \sum_{i=1}^{N} N_i$$

Thus

$$\begin{aligned} r &\le \|((\prod_\varepsilon + \Theta_\varepsilon + \Upsilon_\varepsilon)(z_r)(t)\| \\ &\le \Delta + \frac{1}{\varepsilon} M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Delta + M\sum_{i=1}^{N} N_i \\ &= \Big(1 + \frac{1}{\varepsilon} M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big)\Delta + M\sum_{i=1}^{N} N_i \end{aligned}$$

Dividing both sides by r and taking $r \to \infty$, we obtain that

$$\begin{aligned} &\Big(1 + \frac{1}{\varepsilon} M_B^2 M_{\mathfrak{A}}^2 \frac{T^{2\alpha-1}}{2\alpha-1}\Big)\Big(M_g\|A^{-\beta} l + M_g K(\alpha,\beta) M_g \frac{T^{\alpha\beta}}{\alpha\beta} l \\ &+ \frac{M}{\Gamma(\alpha)}\frac{T^\alpha}{\alpha}\sigma_f \sup_{s\in J} n(s)\Big) + M\sum_{i=1}^{N}\sigma_i \ge 1, \end{aligned}$$

which is a contradiction to Assumption (H4). Thus $(\prod_\varepsilon + \Theta_\varepsilon + \Upsilon_\varepsilon)(B_r) \subset B_r$ for some $r > 0$. □

Lemma 3.3. *Let Assumptions* (H1)–(H4) *hold. Then Θ_1 is contractive.*

Proof. Let $x, y \in B_r$. Then

$$\begin{aligned} &\|(\prod_\varepsilon x)(t) - (\prod_\varepsilon y)(t)\| \\ &\le \|g\big(t, \tilde{\phi}_t + x_t\big) - g\big(t, \tilde{\phi}_t + y_t\big)\| \\ &\le \|\int_0^t (t-s)^{\alpha-1} A\mathfrak{A}_\alpha(t-s)\Big(g\big(s, \tilde{\phi}_s + x_s\big) - g\big(s, \tilde{\phi}_s + y_s\big)\Big)\|ds \\ &\le \|A^{-\beta}\| M_g \|x_t - y_t\|_{\mathcal{B}_h} \\ &\quad + K(\alpha,\beta)\int_0^t (t-s)^{\alpha\beta-1}\|A^\beta\Big(g\big(s, \tilde{\phi}_s + x_s\big) - g\big(s, \tilde{\phi}_s + y_s\big)\Big)\|ds \\ &\le \|A^{-\beta}\| M_g \|x_t - y_t\|_{\mathcal{B}_h} + K(\alpha,\beta) M_g \int_0^t (t-s)^{\alpha\beta-1}\|x_s - y_s\|_{\mathcal{B}_h} ds. \end{aligned}$$

Hence

$$\|(\prod_\varepsilon x)(t) - (\prod_\varepsilon y)(t)\| \le M_g l\Big(\|A^{-\beta}\| + K(\alpha,\beta)\frac{T^{\alpha\beta}}{\alpha\beta}\Big) \sup_{0\le s\le T}\|x(s) - y(s)\|,$$

where we have used the fact that $x_0 = y_0 = 0$. Thus

$$\sup_{0\le t\le T}\|(\prod_\varepsilon x)(t) - (\prod_\varepsilon y)(t)\| \le M_g l\Big(\|A^{-\beta}\| + K(\alpha,\beta)\frac{T^{\alpha\beta}}{\alpha\beta}\Big) \sup_{0\le s\le T}\|x(s) - y(s)\|,$$

so Θ_1 is a contraction by Assumption (H4). □

Lemma 3.4. *Let Assumptions* (H1)–(H4) *hold. Then θ_ε maps bounded sets to bounded sets in B_r.*

Proof. By a similar argument as Lemma 3.2, we obtain

$$\|(\Theta_\varepsilon z)(t)\| < \Big(1+\frac{1}{\varepsilon}M_B^2M_{\mathfrak{A}}^2\frac{T^{2\alpha-1}}{2\alpha-1}\Big)\frac{M}{\Gamma(\alpha)}\frac{T^\alpha}{\alpha}\Lambda_f(R(r))\sup_{s\in J}n(s) := r_1(\varepsilon)$$

which implies that $(\Theta_\varepsilon z)\in B_{r_1(\varepsilon)}$. □

Lemma 3.5. *Let Assumptions* (H1)–(H4) *hold. Then the set* $\{\Theta_\varepsilon z) : z\in B_r\}$ *is an equicontinuous family of functions on [0,T].*

Proof. Let $0<\eta<t<T$ and $\delta>0$ such that $\|\mathfrak{A}_\alpha(s_1)-\mathfrak{A}_\alpha(s_2)\|<\eta$ for every $s_1,s_2\in[0,T]$ with $|s_1-s_2|<\delta$. For $z\in B_r, 0<|h|<\delta, t+h\in[0,T]$, we have

$$\begin{aligned}
&\|(\Theta_\varepsilon)(t+h)-(\Theta_\varepsilon)(t)\|\\
&\le \|\int_0^t \Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)\mathfrak{A}_\alpha(t+h-s)\big[(T-s)^{\alpha-1}Bv_\varepsilon\big(s,\tilde\phi+z\big)\\
&\quad + f\big(s,\tilde{\phi_s}+z_s\big)\big]ds\|\\
&\le \|\int_0^{t+h}(t+h-s)^{\alpha-1}\mathfrak{A}_\alpha(t+h-s)\big[(T-s)^{\alpha-1}Bv_\varepsilon\big(s,\tilde\phi+z\big)\\
&\quad + f\big(s,\tilde{\phi_s}+z_s\big)\big]ds\|\\
&\le \|\int_0^{t}(t-s)^{\alpha-1}\big(\mathfrak{A}_\alpha(t+h-s)-\mathfrak{A}_\alpha(t-s)\big)\big[(T-s)^{\alpha-1}Bv_\varepsilon\big(s,\tilde\phi+z\big)\\
&\quad + f\big(s,\tilde{\phi_s}+z_s\big)\big]ds\|
\end{aligned}$$

Applying Lemma 2.4 and Hölder inequality, we have

$$\begin{aligned}
&\|(\Theta_\varepsilon z)(t+h)-(\Theta_\varepsilon z)(t)\|\\
&\le \frac{M}{\Gamma(\alpha)}\Lambda_f(R(r))\int_0^t\Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)n(s)ds\\
&+\frac{M}{\Gamma(\alpha)}\frac{1}{\varepsilon}M_BM_{\mathfrak{A}}\Delta\int_0^t\Big((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\Big)(T-s)^{\alpha-1}ds\\
&+\frac{M}{\Gamma(\alpha)}\Lambda_f(R(r))\int_0^t\Big((t-s)^{\alpha-1}n(s)ds\\
&+\frac{M}{\Gamma(\alpha)}\frac{1}{\varepsilon}M_BM_{\mathfrak{A}}\Delta\int_t^{t+h}((t+h-s)^{\alpha-1}(T-s)^{\alpha-1}ds\\
&+\frac{\eta T^\alpha}{\alpha}\Lambda_f(R(r))\int_0^t(t-s)^{\alpha-1}n(s)ds\\
&+\frac{\eta T^\alpha}{\alpha}\frac{1}{\varepsilon}M_BM_{\mathfrak{A}}\Delta\int_0^t(t-s)^{\alpha-1}(T-s)^{\alpha-1}ds
\end{aligned}\tag{3.8}$$

Therefore, for ε sufficiently small, the right- hand side of (3.8) tends to zero as $h\to 0$. On the other hand, the compactness of $\mathfrak{A}_\alpha(t), t>0$, implies the continuity in the uniform operator topology. Thus, the set $\{\Theta_\varepsilon z : z\in B_r\}$ is equicontinuous. □

Lemma 3.6. *Let Assumptions* (H1)–(H6) *hold. Then the set* $\{\Upsilon_\varepsilon z) : z\in B_r\}$ *is an equicontinuous family of functions on* $[0,T]$.

Proof. For $z\in B_r, 0<|h|<\delta$ and $t+h\in[0,T]$, we have

$$\|(\Upsilon_\varepsilon z)(t+h)-(\{\Upsilon_\varepsilon z)(t)\|$$

$$
\begin{aligned}
&\le \| \sum_{0<t_i<t+h} \mathfrak{T}_\alpha(t+h-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i}) - \sum_{0<t_i<t} \mathfrak{T}_\alpha(t+h-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i})\| \\
&\quad + \| \sum_{0<t_i<t} \mathfrak{T}_\alpha(t+h-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i}) - \sum_{0<t_i<t} \mathfrak{T}_\alpha(t-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i})\| \\
&\le \sum_{0<t_i<t+h} \|\mathfrak{T}_\alpha(t+h-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i})\| \\
&\quad + \sum_{0<t_i<t} \|\mathfrak{T}_\alpha(t+h-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i}) - \mathfrak{T}_\alpha(t-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i})\|,
\end{aligned}
$$

from which follows that $\{\Upsilon_\varepsilon z : z \in B_r\}$ is equicontinuous on each interval $[0,T]$ due to the equicontinuous of $\mathfrak{T}(t), t > 0$ and Hypothesis (H5). $\square$

Lemma 3.7. *Let Assumptions* (H1)-(H6) *hold. Then* $(\Theta_\varepsilon + \Upsilon_\varepsilon)$ *maps* B_r *onto a precompact set in* B_r.

Proof. Let $0 < t < T$ be fixed and ε be a real number satisfying $0 < \lambda < t$. For $\delta > 0$ define an operator $(\Theta_\varepsilon^{\lambda,\delta} z + \Upsilon_\varepsilon^{\lambda,\delta} z)$ on B_r by

$$
\begin{aligned}
&\Big(\Theta_\varepsilon^{\lambda,\delta} z + \Upsilon_\varepsilon^{\lambda,\delta} z\Big)(t) \\
&= \alpha \int_0^{t-\lambda} \int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S(t-s)^\alpha\theta)\big[Bu_\varepsilon\big(s,\tilde{\phi}+z\big) \\
&\quad + f\big(s,\tilde{\phi}_s+z_s\big)\big]d\theta ds + \sum_{0<\lambda<t} \mathfrak{T}_\alpha(t-\lambda)I_i(\tilde{\phi}_\lambda+z_\lambda) \\
&= \alpha S(\lambda^\alpha\delta) \int_0^{t-\lambda} \int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S(t-s)^\alpha\theta - \lambda^\alpha\delta)\big[Bu_\varepsilon\big(s,\tilde{\phi}+z\big) \\
&\quad + f\big(s,\tilde{\phi}_s+z_s\big)\big]d\theta ds + \sum_{0<\lambda<t} \mathfrak{T}_\alpha(t-\lambda)I_i(\tilde{\phi}_\lambda+z_\lambda)
\end{aligned}
$$

Since $S(t)$, $t > 0$ is a compact operator, the set $\Big\{\Big(\Theta_\varepsilon^{\lambda,\delta} z + \Upsilon_\varepsilon^{\lambda,\delta} z\Big)(t) : z \in B_r\Big\}$ is precompact in H for every $0 < \lambda < t, \delta > 0$. Moreover, for each $z \in B_r$, we have

$$
\begin{aligned}
&\|\big(\Theta_\varepsilon z + \Upsilon_\varepsilon z\big)(t) - \big(\Theta_\varepsilon^{\lambda,\delta} z + \Upsilon_\varepsilon^{\lambda,\delta} z\big)(t)\| \\
&\le \alpha E\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S(t-s)^\alpha\theta)\big[Bu_\varepsilon\big(s,\tilde{\phi}+z\big) + f\big(s,\tilde{\phi}_s+z_s\big)\big]d\theta ds\| \\
&\quad + \alpha E\| \int_{t-\lambda}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S(t-s)^\alpha\theta)\big[Bu_\varepsilon\big(s,\tilde{\phi}+z\big) \\
&\quad + f\big(s,\tilde{\phi}_s+z_s\big)\big]d\theta ds\| \\
&\quad + \| \sum_{0<t_i<t} \mathfrak{T}_\alpha(t-t_i)I_i(\tilde{\phi}_{t_i}+z_{t_i})\| + \| \sum_{0<\lambda<t} \mathfrak{T}_\alpha(t-\lambda)I_i(\tilde{\phi}_\lambda+z_\lambda)\| \\
&\le \alpha E\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S(t-s)^\alpha\theta)\big[Bu_\varepsilon\big(s,\tilde{\phi}+z\big) + f\big(s,\tilde{\phi}_s+z_s\big)\big]d\theta ds\| \\
&\quad + \alpha E\| \int_{t-\lambda}^t \int_\delta^\infty \theta(t-s)^{\alpha-1}\eta_\alpha(\theta)S(t-s)^\alpha\theta)\big[Bu_\varepsilon\big(s,\tilde{\phi}+z\big) \\
&\quad + f\big(s,\tilde{\phi}_s+z_s\big)\big]d\theta ds\| + 2 \sum_{0<\lambda<t_i<t} \|\mathfrak{T}_\alpha(t-t_i)I_i(\tilde{\phi} t_i+z_{t_i})\|
\end{aligned}
$$

$$=: J_1 + J_2 + J_3 \tag{3.9}$$

By a similar argument as above, we have

$$\begin{aligned}
J_1 &\le \alpha M \int_0^t (t-s)^{\alpha-1}\Big(\big\|Bu_\varepsilon\big(s,\tilde{\phi}+z\big)\big\| + \big\|f\big(s,\tilde{\phi}_s+z_s\big)\big\|\Big)ds\Big(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\Big)\\
&\le \alpha M\Big(\frac{1}{\varepsilon}M_B M_{\mathfrak{A}}\Delta\int_0^t (t-s)^{\alpha-1}(T-s)^{\alpha-1}ds + \Lambda_f(R(r))\int_0^t (t-s)^{\alpha-1}n(s)ds\Big)\\
&\quad\times\Big(\int_0^\delta \theta\eta_\alpha(\theta)d\theta\Big),\\
J_2 &\le \alpha M \int_{t-\lambda}^t (t-s)^{\alpha-1}\Big(\big\|Bu_\varepsilon\big(s,\tilde{\phi}+z\big)\big\| + \big\|f\big(s,\tilde{\phi}_s+z_s\big)\big\|\Big)ds\Big(\int_\delta^\infty \theta\eta_\alpha(\theta)d\theta\Big)\\
&\le \frac{\alpha M}{\Gamma(1+\alpha)}\Big(\frac{1}{\varepsilon}M_B M_{\mathfrak{A}}\Delta\int_{t-\lambda}^t (t-s)^{\alpha-1}(T-s)^{\alpha-1}ds\\
&\quad + \Lambda_f(R(r))\int_{t-\lambda}^t (t-s)^{\alpha-1}n(s)ds\Big),
\end{aligned}$$

and

$$J_3 \le 2M\sum_{i=1}^N N_i; \tag{3.10}$$

here we have used the equality

$$\int_0^\infty \theta^\beta\eta_\alpha(\theta)d\theta = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}.$$

From (3.9)-(3.10), one can see that for each $z \in B_r$,

$$\|\Big(\Theta_\varepsilon z + \Upsilon_\varepsilon z\Big)(t) - \Big(\Theta_\varepsilon^{\lambda,\delta} z + \Upsilon_\varepsilon^{\lambda,\delta} z\Big)(t)\| \to 0 \quad \text{as } \lambda \to 0^+,\ \delta \to 0^+.$$

Therefore, there are relatively compact sets arbitrary close to the set $\{(\Theta_\varepsilon z + \Upsilon_\varepsilon z)(t) : z \in B_r\}$; hence, the set $\big\{(\Theta_\varepsilon z + \Upsilon_\varepsilon z\big)(t) : z \in B_r\}$ is also precompact in B_r. $\square$

4. Main Results

Consider the linear impulsive fractional differential system

$$D_t^\alpha x(t) = Ax(t) + Bu(t), \quad t \in [0,T],\ t \ne t_i, \tag{4.1}$$

$$x(0) = \phi(0), \tag{4.2}$$

$$\Delta x(t_i) = I_i(x_{t_i}) \tag{4.3}$$

The approximate controllability for linear impulsive fractional differential system (4.1)-(4.3) is a natural generalization of approximate controllability of linear first order control system. It is convenient at this point to introduce the controllability operators associated with (4.1)-(4.3) as

$$L_0^T = \int_0^T (T-s)^{\alpha-1}\mathfrak{A}_\alpha(T-s)Bu(s)ds + \sum_{t_i<t}\mathfrak{T}_\alpha(T-t_i)I_i(x_{t_i})$$

$$\begin{aligned} L_0^T &= L_0^T(L_0^T)^* \\ &= \int_0^T (T-s)^{2(\alpha-1)}\mathfrak{A}_\alpha(T-s)BB^*\mathfrak{A}_\alpha^*(T-s)ds \\ &\quad + \sum_{t_i<t} \mathfrak{T}_\alpha(T-t_i)\mathfrak{T}_\alpha^*(T-t_i)I_i(x_{t_i}), \end{aligned}$$

respectively, where B^* denotes the adjoint of B, $\mathfrak{A}_\alpha^*(t)$ is the adjoint of $\mathfrak{A}_\alpha(t)$ and $\mathfrak{T}_\alpha^*(t)$ is the adjoint of $\mathfrak{T}_\alpha(t)$. It is straightforward that the operator L_0^T is a linear bounded operator for $1/2 < \alpha \leq 1$.

Theorem 4.1 ([24])**.** *The following three conditions are equivalent:*

- (i) Γ_0^T *is positive, that is,* $\langle z^*, \Gamma_0^T z^*\rangle > 0$ *for all nonzero* $z^* \in X^*$*;*
- (ii) *For all* $h \in X, J(z_\varepsilon(h))$ *coverges to the zero as* $\varepsilon \to 0^+$ *in the weak topology, where* $z_\varepsilon(h) = \varepsilon(\varepsilon I + \Gamma_0^T J)^{-1}(h)$ *is a solution of the equation* $\varepsilon z_\varepsilon + \Gamma_0^T J(z_\varepsilon(h)) = \alpha h$*;*
- (iii) *For all* $h \in X, z_\varepsilon(h) = \varepsilon(\varepsilon I + \Gamma_0^T J)^{-1}(h)$ *converges to the zero as* $\varepsilon \to 0^+$ *in the strong topology.*

Remark 4.2. It is known that Theorem 4.1(i) holds if and only if $\overline{\operatorname{Im} L_0^T} = X$. In other words, Theorem 4.1(i) holds if and only if the corresponding linear system is approximately controllable on $[0, T]$.

Theorem 4.3 ([24])**.** *Let* $p : X \to X$ *be a nonlinear operator, Assume* z_ε *is a solution of the following equation* $\varepsilon z_\varepsilon + \Gamma_0^T J(z_\varepsilon(h)) = \alpha p(z_\varepsilon)$ *and* $\|p(z_\varepsilon) - p\| \to 0$ *as* $\varepsilon \to 0^+, p \in X$*. Then there exists a subsequence of the sequence* $\{z_\varepsilon\}$ *strongly converging to zero as* $\varepsilon \to 0^+$*.*

We are now in a position to state and prove our main result.

Theorem 4.4. *Let* $1/2 < \alpha \leq 1$*. Suppose that Assumptions* (H1)–(H7) *are satisfied. Also assume that*

- (H8) $g : [0, T] \times X \to X$ *and* $A^\beta g(T, \cdot)$ *is continuous from the weak topology of* X*;*
- (H9) *There exists* $N \in L^\infty([0, T], [0, +\infty))$ *such that*

$$\sup_{x\in B_t} \|f(t, x)\| + \sup_{y\in X} \|A^\beta g(t, y)\| \leq N(t), \quad \textit{for a.e. } t \in [0, T].$$

Then system (2.1) *is approximately controllable on* $[0, T]$*.*

Proof. Let x^ε be a fixed point of Φ_ε in $B_{r(\varepsilon)}$. Then x^ε is a mild solution of (2.1) $[0, T]$ under the control given by

$$\begin{aligned} u_\varepsilon(t, x^\varepsilon) &= (T-t)^{\alpha-1}B^*S^*(T-t)J\big(((\varepsilon I + \Gamma_0^T J)^{-1}p(x^\varepsilon)\big) \\ p(x^\varepsilon) &= h - \mathfrak{T}_\alpha(T)[\phi(0) + g(0, \phi)] + g(T, x^\varepsilon(T)) \\ &\quad + \int_0^T (T-s)^{\alpha-1}A\mathfrak{A}_\alpha(T-s)g(s, x_s^\varepsilon)ds \\ &\quad - \int_0^T (T-s)^{\alpha-1}\mathfrak{A}_\alpha(T-s) + f(s, x_s^\varepsilon)ds \end{aligned}$$

and satisfies the equality

$x^\varepsilon(T)$

$$
\begin{aligned}
&= \mathfrak{T}_\alpha(T)[\phi(0)+g(0,\phi)] - g(T,x^\varepsilon(T)) \\
&\quad - \int_0^T (T-s)^{\alpha-1} A\mathfrak{A}_\alpha(T-s) g(s,x^\varepsilon_s)ds \\
&\quad + \int_0^T (T-s)^{\alpha-1}\mathfrak{A}_\alpha(T-s)[Bu_\varepsilon(s,x)+f(s,x^\varepsilon_s)]ds \\
&\quad + \sum_{t_i<T} \mathfrak{T}_\alpha(T-t_i) I_i(x^\varepsilon_{t_i}) + \sum_{t_i<T}[g(T,x^\varepsilon(T))_{|t_i^+} - g(T,x^\varepsilon(T))_{|t_i^-}] \\
&= \mathfrak{T}_\alpha(T)[\phi(0)+g(0,\phi)] - g(T,x^\varepsilon(T)) - \int_0^T (T-s)^{\alpha-1} A\mathfrak{A}_\alpha(T-s) h(s,x^\varepsilon_s)ds \\
&\quad + (-\varepsilon I + \varepsilon I + \Gamma_0^T J)\big((\varepsilon I + \Gamma_0^T J)^{-1} p(x^\varepsilon)\big) \\
&\quad + \int_0^T (T-s)^{\alpha-1}\mathfrak{A}_\alpha(T-s) + f(s,x^\varepsilon_s)ds \\
&= h - \varepsilon\big((\varepsilon I + \Gamma_0^T J)^{-1} p(x^\varepsilon) + \sum_{t_i<T} \mathfrak{T}_\alpha(T-t_i) I_i(x^\varepsilon_{t_i}) \\
&\quad + \sum_{t_i<T}[g(T,x^\varepsilon(T))_{|t_i^+} - g(T,x^\varepsilon(T))_{|t_i^-}]
\end{aligned}
$$

In other words, $z_\varepsilon = h - x^\varepsilon(T)$ is a solution of the equation $\varepsilon\big(\varepsilon I + \Gamma_0^T J(z_\varepsilon) = \varepsilon p(x^\varepsilon)$. Now it follows that

$$
\begin{gathered}
\varepsilon\langle J(z_\varepsilon), z_\varepsilon\rangle + \langle J(z_\varepsilon), \Gamma_0^T J(z_\varepsilon)\rangle = \varepsilon\langle J(z_\varepsilon), p(x^\varepsilon)\rangle, \\
\varepsilon\|z_\varepsilon\|^2 + \langle J(z_\varepsilon), \Gamma_0^T J(z_\varepsilon)\rangle = \varepsilon\langle J(z_\varepsilon), p(x^\varepsilon)\rangle, \\
\varepsilon\|z_\varepsilon\|^2 \le \varepsilon\langle J(z_\varepsilon), p(x^\varepsilon)\rangle \le \varepsilon\|z_\varepsilon\|\|p(x^\varepsilon)\|, \quad (4.4) \\
\|z_\varepsilon\| = \|J(z_\varepsilon)\| \le \|p(x^\varepsilon)\|. \quad (4.5)
\end{gathered}
$$

On the other hand, by (H9),

$$
\begin{aligned}
\|p(x^\varepsilon)\| &\le \|h\| + M\|\phi(0)\| + N(T) + \frac{M_{1-\beta}\alpha\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \int_0^T (T-s)^{\alpha\beta-1} N(s)ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} N(s)ds.
\end{aligned}
$$

From (4.4)-(4.5), it follows that $x^\varepsilon(T) \to \tilde{x}$ converges weakly as $\varepsilon \to 0^+$ and by the Assumption (H8), $A^\beta g(T,x^\varepsilon(T)) \to A^\beta g(T,\tilde{x})$ converges strongly as $\varepsilon \to 0^+$. Moreover, Assumption (H9) implies that

$$
\int_0^T \|f(s,x^\varepsilon_s)\|^2 ds + \int_0^T \|A^\beta g(s,x^\varepsilon_s)\|^2 ds \le \int_0^T N(s)ds.
$$

Consequently, the sequence $\{f(.,x^\varepsilon), A^\beta g(.,x^\varepsilon)\}$ is bonded. Then there is a subsequence denoted by $\{f(.,x^\varepsilon, A^\beta g(.,x^\varepsilon)\}$ weakly convergent to, say, $(f(.), g(.))$ in $L_2([0,T],X)$. Then

$$
\begin{aligned}
&\|p(x^\varepsilon) - p\| \\
&= \|g(T,x^\varepsilon(T)) - g(T,\tilde{x})\| + \|\int_0^T (T-s)^{\alpha-1} A^{1-\beta}\mathfrak{A}_\alpha(T-s)[A^\beta g(s,x^\varepsilon_s) - g(s)]ds\|
\end{aligned}
$$

$$
\begin{aligned}
&+\|\int_0^T (T-s)^{\alpha-1}\mathfrak{A}_\alpha(T-s)[f(s,x_s^\varepsilon)-f(s)]ds\| \\
&\leq \|A^{-\beta}(A^\beta g(T,x^\varepsilon(T))-A^\beta g(T,\tilde{x})\| \\
&\quad+\sup_{0\leq t\leq T}\|\int_0^t (t-s)^{\alpha-1}A^{1-\beta}\mathfrak{A}_\alpha(T-s)[A^\beta g(s,x_s^\varepsilon)-g(s)]ds\| \\
&\quad+\sup_{0\leq t\leq T}\|\int_0^t (T-s)^{\alpha-1}\mathfrak{A}_\alpha(t-s)[f(s,x_s^\varepsilon)-f(s)]ds\|\to 0,
\end{aligned}
$$

where

$$
\begin{aligned}
p(x) &= h-\mathfrak{T}_\alpha(T)[\phi(0)+g(0,\phi(0))]+g(T,\tilde{x}) \\
&\quad+\int_0^T (T-s)^{\alpha-1}A^{1-\beta}\mathfrak{A}_\alpha(T-s)g(s)ds-\int_0^T (T-s)^{\alpha-1}\mathfrak{A}_\alpha(T-s)f(s)ds
\end{aligned}
$$

as $\varepsilon\to 0^+$ because of compactness of an operator

$$
f(.)\to\int_0^{\cdot}(.-s)^{\alpha-1}\mathfrak{A}_\alpha(.-s)f(s)ds: \quad L_2([0,T],X)\to PC([0,T],X).
$$

Then by Theorem 4.3, $\|(x^\varepsilon(T)-h\|=\|z_\varepsilon\|\to 0$ as $\varepsilon\to 0^+$. This gives the approximate controllability. $\square$

Remark 4.5. Theorem 4.4 assumes that the operator A generates a compact semigroup and, consequently, the associated linear control system (4.1)-(4.3) is not exactly controllable. Therefore Theorem 4.4 has no analogue for the concept of exact controllability.

5. Applications

Let $X=L_2[0,\pi]$ and $Az=z''$, with domain $D(A)=\{z\in X|z,dz/d\xi$ are absolutely continuous, $d^2z/d\xi^2\in X$ and $z(0)=z(\pi)=0\}$, where A is the infinitesimal generator of a strongly continuous semigroup $S(t),t>0$, on X which is analytic compact and self-adjoint, the eigenvalues are $-n^2$, $n\in\mathbb{N}$, with corresponding normalized eigenvectors $e_n(\xi):=(2/\pi)^{1/2}\sin(n\xi)$ and

$$
S(t)e_n=e^{-n^2t}e_n,\quad n=1,2,\dots.
$$

Moreover the following statements hold:

(a) $\{e_n:n\in\mathbb{N}\}$ is an orthonormal basis of X;
(b) If $z\in D(A)$ then $A(z)=-\sum_{n=1}^\infty n^2\langle z,e_n\rangle e_n$;
(c) For $z\in\mathbb{H}$, $(-A)^{-1/2}z=\sum_{n=1}^\infty \frac{1}{n}\langle z,e_n\rangle e_n$;
(d) The operator $(-A)^{1/2}$ is given as $(-A)^{1/2}z=\sum_{n=1}^\infty n\langle z,e_n\rangle e_n$ on the space $D\Big[(-A)^{1/2}\Big]=\{z\in X:\sum_{n=1}^\infty n\langle z,e_n\rangle e_n\in X\}$

Consider the neutral system

$$
\frac{\partial^\alpha}{\partial t^\alpha}\Big[x(t,\xi)+\int_0^\pi b(\theta,\xi)x(t,\theta)d\theta\Big]=\frac{\partial^2}{\partial\xi^2}x(t,\xi)+p(t,x(t,\xi))+Bu(t,\xi), \tag{5.1}
$$

$$
x(t,0)=x(t,\pi)=0,\quad t\geq 0, \tag{5.2}
$$

$$
x(t,\xi)=\phi(\xi),\quad 0\leq\xi\leq\pi, \tag{5.3}
$$

$$
\Delta x(t_i,\cdot)=x(t_i^+,\cdot)-x(\cdot,t_i^-)=\int_0^\pi(\xi,x(t_i,s))ds, \tag{5.4}
$$

where $p : [0,T] \times R \to R$ is continuous functions and $(t_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of positive real numbers. B is a linear continuous mapping from

$$U = \Big\{ u = \sum_{n=2}^{\infty} u_n e_n | \|u\|_U^2 = \sum_{n=2}^{\infty} u_n^2 < \infty \Big\}$$

to X as follows

$$Bu = 2u_2 + \sum_{n=2}^{\infty} u_n e_n.$$

To write problem (5.1)-(5.4) in the abstract form, we assume the following:

(A1) The function b is measurable and

$$\int_0^{\pi} \int_0^{\pi} b^2(\theta,\xi) d\theta d\xi < \infty.$$

(A2) The function $\frac{\partial}{\partial\xi} b(\theta,\xi)$ is measurable, $b(\theta,0) = b(\theta,\pi) = 0$, and let

$$L_1 = \Big[\int_0^{\pi} \int_0^{\pi} \Big(\frac{\partial}{\partial\xi} b(\theta,\xi) \Big)^2 d\theta d\xi \Big]^{1/2}.$$

(A3) The functions $p_i : [0,\pi] \times \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$, are continuous and there are positive constants L_i such that

$$|p_i(\xi,s) - p_i(\xi,\bar{s})| \le L_i |s - \bar{s}|, \quad \xi \in [0,\pi],\ s,\bar{s} \in \mathbb{R}.$$

We now define the functions $g, f : [0,T] \times X \to X$, $I_i : X \to X$ by

$$\begin{gathered}
g(x)(\xi) = \int_0^{\pi} \int_0^{\pi} b(\theta,\xi) x(\theta) d\theta, \quad \xi \in [0,\pi], \\
f(t,x)(\xi) = p(t, x(\xi)), \quad t \ge 0,\ \xi \in [0,\pi], \\
I_i(\phi)\xi = \int_0^{\pi} p_i(\xi), \phi(0,s)) ds, \quad i \in \mathbb{N}, \quad \xi \in [0,\pi]
\end{gathered}$$

From (A1), it is clear that g is bounded linear operator on X. Furthermore, $g(x) \in D\big[A^{1/2}\big]$, and $\|A^{1/2}\| \le L_1$. In fact from the definition of g and (A2) it follows that

$$\begin{aligned}
\langle g(x), e_n \rangle &= \int_0^{\pi} \Big[\int_0^{\pi} b(\theta,\xi) x(\theta) d\theta \Big] e_n(\xi) d\xi \\
&= \frac{1}{n} \big(\frac{2}{\pi} \big)^{1/2} \Big\langle \int_0^{\pi} \frac{\partial}{\partial\xi} b(\theta,\xi) x(\theta) d\theta, \cos(n\xi) \Big\rangle \\
&= \frac{1}{n} \big(\frac{2}{\pi} \big)^{1/2} \langle g_1(x), \cos(n\xi) \rangle,
\end{aligned}$$

where $g_1(x) = \int_0^{\pi} b(\theta,\xi) x(\theta) d\theta$. From (A2) we know that $g_1 : X \to X$ is a bounded linear operator with $\|g_1\| \le L_1$. Hence $\|A^{1/2} g(x)\| = \|g_1(x)\|$, which implies the assertion. Moreover, assume that f and g satisfy conditions of Theorem 4.3. Thus the problem (5.1)-(5.2) can be written in the abstract form

$$\begin{gathered}
\frac{d^{\alpha}}{dt^{\alpha}} \big(x(t) + g(t, x(t)) \big) = Ax(t) + f(t, x(t)) + Bu(t), \\
x(0) = x_0, \quad t \in [0,T], \\
\Delta x(t_i) = x(t_i^+ - x(t_i^-).
\end{gathered}$$

Now consider the associated linear system

$$\frac{d^\alpha}{dt^\alpha}x(t) = Ax(t) + Bu(t), \tag{5.5}$$
$$x(0) = x_0, \quad t \in [0,T], \tag{5.6}$$
$$\Delta x(t_i) = x(t_i^+ - x(t_i^-). \tag{5.7}$$

So that it is approximately controllable on $[0,T]$ for $1/2 < \alpha < 1$. It is easy to see that if $z = \sum_{n=1}^{\infty}\langle z, e_n\rangle e_n$ then

$$B^* v = (2v_1 + v_2)e_2 + \sum_{n=3}^{\infty} v_n e_n,$$

$$\begin{aligned}
&B^*\mathfrak{A}_\alpha^*(T-s)z \\
&= B^*\alpha \int_0^\infty \theta\Psi_\alpha(\theta)S^*((T-s)^\alpha\theta)z d\theta \\
&= \alpha \int_0^\infty \theta\Psi_\alpha(\theta)\Big(\Big(2\langle z, e_1\rangle e^{-(T-s)^{\alpha\theta}} e_1 + \langle z, e_2\rangle e^{-4(T-s)^{\alpha\theta}} \\
&\quad + \sum_{n=3}^{\infty} e^{-n^2(T-s)^{\alpha\theta}}\langle z, e_n\rangle e_n\Big) d\theta \\
&= \Big(2\langle z, e_1\rangle\alpha \int_0^\infty \theta\Psi_\alpha(\theta)e^{-(T-s)^{\alpha\theta}} d\theta + \langle z, e_2\rangle\alpha \int_0^\infty \theta\Psi_\alpha(\theta)e^{-4(T-s)^{\alpha\theta}} d\theta\Big)e_2 \\
&\quad + \alpha\sum_{n=3}^{\infty}\int_0^\infty \theta\Psi_\alpha(\theta)e^{-n^2(T-s)^{\alpha\theta}} d\theta\langle z, e_n\rangle e_n,
\end{aligned}$$

$$\begin{aligned}
&\big\|(T-s)^{\alpha-1}B^*\mathfrak{A}_\alpha^*(T-s)z\big\|^2 \\
&= (T-s)^{2(\alpha-1)}\Big(2\alpha \int_0^\infty \theta\Psi_\alpha(\theta)e^{-(T-s)^{\alpha\theta}} d\theta\langle z, e_1\rangle \\
&\quad + \alpha \int_0^\infty \theta\Psi_\alpha(\theta)e^{-4(T-s)^{\alpha\theta}} d\theta\langle z, e_2\rangle\Big)^2 \\
&\quad + (T-s)^{2(\alpha-1)}\sum_{n=3}^{\infty}\Big(\alpha\sum_{n=3}^{\infty}\int_0^\infty \theta\Psi_\alpha(\theta)e^{-n^2(T-s)^{\alpha\theta}} d\theta\Big)^2\langle z, e_2\rangle^2 = 0.
\end{aligned}$$

It follows that $\langle z, e_1\rangle = \langle z, e_2\rangle = \cdots = \langle z, e_n\rangle = 0$, consequently $z = 0$, which means that system (5.5)-(5.7) is approximaely controllable on $[0,T]$. Therefore, from Theorem 4.4, system (5.1)-(5.4) is approximately controllable on $[0,T]$.

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