Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 29, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

HABIB MAAGLI

Abstract. Using the Schauder fixed point theorem, we prove an existence of positive solutions for the fractional differential problem in the half line $\mathbb{R}^+ = (0, \infty)$:

$$D^\alpha u = f(x, u), \quad \lim_{x\to 0^+} u(x) = 0,$$

where $\alpha \in (1, 2]$ and f is a Borel measurable function in $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying some appropriate conditions.

1. Introduction

Recently, fractional differential equations have been studied extensively. The motivation for these studies stems from the fact that fractional differential equations serve as an excellent tool to describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetic, etc (see[7, 9, 11, 12, 15]). Therefore, the theory of fractional differential equations has been developed very quickly and the investigation for the existence of solutions of fractional differential equations has attracted a considerable attention from researches (see [1, 2, 3, 4, 5, 6, 8, 10, 13, 16, 17] and the references therein).

To the best of our knowledge, most of the related results focus on developing the existence and uniqueness of solutions on the finite interval $[0, 1]$. In this note, we consider the following fractional differential problem in the half line $\mathbb{R}^+ = (0, \infty)$:

$$\begin{gathered} D^\alpha u = f(x, u), \\ u > 0 \quad \text{in } \mathbb{R}^+, \\ \lim_{x\to 0^+} u(x) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$, $f(x, y)$ is a Borel measurable function in $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying the following hypotheses:

(H1) f is continuous with respect to the second variable.
(H2) There exists a nonnegative measurable function q defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that
  (i) $|f(x, y)| \leq yq(x, y)$ for all $x, y \in \mathbb{R}^+$.

2000 *Mathematics Subject Classification.* 34A08.
Key words and phrases. Riemann-Liouville fractional derivative; fixed point theorem.

Submitted November 25, 2012. Published January 28, 2013.

(ii) The function $y \to q(x,y)$ is nondecreasing and $\lim_{y\to 0^+} q(x,y) = 0$.
(iii) The integral $\int_0^\infty t^{\alpha-1} q(t, t^{\alpha-1})\,dt$ converges.

We recall that for a measurable function v, the Riemann-Liouville fractional integral $I_\beta v$ and the Riemann-Liouville derivative $D^\beta v$ of order $\beta > 0$ are respectively defined by

$$I_\beta v(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} v(t)\,dt$$

and

$$D^\beta v(x) = \frac{1}{\Gamma(n-\beta)} \big(\frac{d}{dx}\big)^n \int_0^x (x-t)^{n-\beta-1} v(t)\,dt = \big(\frac{d}{dx}\big)^n I_{n-\beta} v(x)\,,$$

provided that the right hand sides are pointwise defined for $x > 0$. Here $n = [\beta]+1$ and $[\beta]$ means the integral part of the number β and Γ is the Euler Gamma function.

Our main result is the following.

Theorem 1.1. *Assume* (H1)–(H2)*, then problem* (1.1) *has infinitely many solutions. More precisely, there exists a number $b > 0$ such that for each $c \in (0, b]$, problem* (1.1) *has a continuous solution u satisfying*

(i) $u(x) = cx^{\alpha-1} + \int_0^\infty \Big(x^{\alpha-1} - \big((x-t)^+\big)^{\alpha-1}\Big) f(t, u(t))\,dt$.
(ii) $\lim_{x\to\infty} x^{1-\alpha} u(x) = c$.
(iii) $\frac{c}{2} x^{\alpha-1} \le u(x) \le \frac{3}{2} x^{\alpha-1}$, *for* $x > 0$.

Note that Theorem 1.1 generalizes a result established by Zhao [18] in the case $\alpha = 2$ (see also [14]). As a special but important case of the above general setting is the problem

$$\begin{gathered} D^\alpha u = k(x) u^p, \\ u > 0 \quad \text{in } \mathbb{R}^+, \\ \lim_{x\to 0^+} u(x) = 0, \end{gathered} \tag{1.2}$$

where $p > 1$ and k is a Borel measurable function in $\mathbb{R}^+$ satisfying

$$\int_0^\infty t^{p(\alpha-1)}\, |k(t)|\,dt < \infty. \tag{1.3}$$

An immediate consequence of Theorem 1.1 is the following.

Corollary 1.2. *Let k be a Borel measurable function satisfying* (1.3)*, then the conclusion of Theorem 1.1 holds for problem* (1.2).

In the sequel, we denote by $C([0,\infty])$ the set of continuous functions v on $\mathbb{R}^+$ such that $\lim_{x\to 0^+} v(x)$ and $\lim_{x\to\infty} v(x)$ exist. It is easy to see that $C([0,\infty])$ is a Banach space with the norm $\|v\|_\infty = \sup_{x>0} |v(x)|$. Finally, for $\lambda \in \mathbb{R}$, we put $\lambda^+ = \max(\lambda, 0)$.

2. Proof of Theorem 1.1

Let $\mathcal{F} = \{v \in C([0,\infty]) : \|v\|_\infty \le 1\}$. To prove Theorem 1.1, we need the following Lemma.

Lemma 2.1. *Assume* (H1)–(H2)*, then the family of functions*

$$\big\{x \to \int_0^x (1 - \frac{t}{x}) f(t, t^{\alpha-1} v(t))\,dt : v \in \mathcal{F}\big\}$$

is relatively compact in $C([0,\infty])$.

Proof. For $v \in \mathcal{F}$ and $x > 0$, put $Sv(x) = \int_0^x (1 - \frac{t}{x}) f(t, t^{\alpha-1} v(t))\, dt$. By (H2), we have for all $v \in \mathcal{F}$ and $x > 0$,

$$\begin{aligned} |Sv(x)| &\leq \int_0^\infty |f(t, t^{\alpha-1} v(t))|\, dt \\ &\leq \int_0^\infty t^{\alpha-1} q(t, t^{\alpha-1})\, dt < \infty. \end{aligned}$$

Thus the family $S(\mathcal{F})$ is uniformly bounded.

Now,we prove the equicontinuity of $S(\mathcal{F})$ in $[0, \infty]$. Let $x, x' \in \mathbb{R}^+$ and $v \in \mathcal{F}$, then we have

$$|Sv(x) - Sv(x')| \leq \int_0^\infty |((1 - \frac{t}{x})^+)^{\alpha-1} - ((1 - \frac{t}{x'})^+)^{\alpha-1}| t^{\alpha-1} q(t, t^{\alpha-1})\, dt\,,$$

$$|Sv(x)| \leq \int_0^x t^{\alpha-1} q(t, t^{\alpha-1})\, dt\,,$$

$$\Big| Sv(x) - \int_0^\infty f(t, t^{\alpha-1} v(t))\, dt \Big| \leq \int_0^\infty \Big(1 - ((1 - \frac{t}{x})^+)^{\alpha-1}\Big) t^{\alpha-1} q(t, t^{\alpha-1})\, dt.$$

Using the Lebesgue's theorem, we deduce from the above inequalities that $S(\mathcal{F})$ is equicontinuous in $[0, \infty]$. Hence, by Ascoli's theorem, we conclude that $S(\mathcal{F})$ is relatively compact in $C([0, \infty])$. $\square$

Proof of Theorem 1.1. By (H2) and Lebesgue's theorem, it follows that

$$\lim_{\beta \to 0} \int_0^\infty t^{\alpha-1} q(t, \beta t^{\alpha-1})\, dt = 0.$$

Hence we can fix a number $\beta > 0$ such that

$$\int_0^\infty t^{\alpha-1} q(t, \beta t^{\alpha-1})\, dt \leq \frac{1}{3}.$$

Let $b = \frac{2}{3}\beta$ and $c \in (0, b]$. In order to apply a fixed point argument , set

$$\Lambda = \{v \in C([0, \infty]) : \frac{c}{2} \leq v(x) \leq \frac{3}{2} c\,, \ \text{ for all } x > 0\}\,.$$

Then Λ is a nonempty closed bounded and convex set in $C([0, \infty])$. Define the operator T on Λ by

$$Tv(x) = c + \int_0^\infty \Big(1 - ((1 - \frac{t}{x})^+)^{\alpha-1}\Big) f(t, t^{\alpha-1} v(t))\, dt\,, \quad \text{for } x \in \mathbb{R}^+.$$

First, we shall prove that the operator T maps Λ into itself. Let $v \in \Lambda$, then for any $x \in \mathbb{R}^+$, we have

$$\begin{aligned} |Tv(x) - c| &\leq \int_0^\infty t^{\alpha-1} v(t) q(t, t^{\alpha-1} v(t))\, dt \\ &\leq \frac{3}{2} c \int_0^\infty t^{\alpha-1} q(t, \beta t^{\alpha-1})\, dt \leq \frac{c}{2}. \end{aligned}$$

It follows that $\frac{c}{2} \leq Tv \leq \frac{3}{2} c$ and since by Lemma 2.1, $T(\Lambda) \subset C([0, \infty])$, we deduce that $T(\Lambda) \subset \Lambda$.

Next, we shall prove the continuity of T in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges uniformly to v in Λ. It follows by (H1), (H2) and Lebesgue's theorem that $Tv_k(x) \to Tv(x)$ as $k \to \infty$, for $x \in \mathbb{R}^+$. Since $T(\Lambda)$

is relatively compact in $C([0,\infty])$, the pointwise convergence implies the uniform convergence. Thus we have proved that T is a compact mapping from Λ to itself.

Now, the Schauder fixed point theorem implies the existence of $\omega \in \Lambda$ such that $T\omega = \omega$. That is

$$\omega(x) = c + \frac{1}{x^{\alpha-1}} \int_0^\infty \left(x^{\alpha-1} - ((x-t)^+)^{\alpha-1}\right) f(t, t^{\alpha-1}\,\omega(t))\,dt\,, \quad \text{for } x > 0.$$

Put $u(x) = x^{\alpha-1}\omega(x)$, for $x > 0$. Then we have

$$u(x) = cx^{\alpha-1} + \int_0^\infty \left(x^{\alpha-1} - ((x-t)^+)^{\alpha-1}\right) f(t, u(t))\,dt.$$

Moreover, for $x > 0$, we have

$$\frac{c}{2}x^{\alpha-1} \le u(x) \le \frac{3}{2}x^{\alpha-1},$$
$$\lim_{x\to\infty} x^{1-\alpha}u(x) = c.$$

It remains to show that u is a solution of problem 1.1. Indeed, for $x > 0$, u satisfies

$$u(x) = \Big(c + \int_0^\infty f(t, u(t))\,dt\Big)x^{\alpha-1} - I_\alpha(f(.,u))(x).$$

So $D^\alpha u(x) = -f(x, u(x))$, for $x > 0$. $\square$

Example 2.2. Let $\beta > 0$, $\gamma \in \mathbb{R}$ and $p > \gamma$. Let k be a Borel measurable function in $\mathbb{R}^+$ such that $\int_0^\infty t^{(p-\gamma+1)(\alpha-1)}\,|k(t)|\,dt < \infty$. Then there exists $b > 0$ such that for each $c \in (0, b]$, the problem

$$D^\alpha u = \frac{k(x)u^{p+1}}{x^\beta + u^\gamma},$$
$$u > 0 \quad \text{in } \mathbb{R}^+,$$
$$\lim_{x\to 0^+} u(x) = 0, \quad \lim_{x\to\infty} x^{1-\alpha}\,u(x) = c$$

has a continuous solution u in $\mathbb{R}^+$.

References

[1] Agarwal, Ravi P; O'Regan, Donal; Staněk, Svatoslav; *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations.* J. Math. Anal. Appl. **371** (2010), no.1, 57-68.

[2] Agarwal, Ravi P; Benchohra, Mouffak; Hamani, Samira; Pinelas, Sandra. *Boundary value problems for differential equations involving Riemann-Liouville fractional derivative on the half line.* Dyn. Contin. Discrete Impuls. Syst. Ser. A. Math. Anal. **18** (2011), no.2, 235-244.

[3] Bai, Zhanbing; Lü, Haishen; *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), no.2, 495-505.

[4] Ahmad, Bashir; *Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations*, Appl. Math. Lett. **23** (2010), no.4, 390-394.

[5] Ahmad, Bashir ; Nieto, Juan J.; *Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions*, Bound. Value Probl. (2011), 2011:36, 9 pp.

[6] Caballero, J; Harjani, J; Sadarangani, K.; *Positive solutions for a class of singular fractional boundary value problems.* Comput. Math. Appl. **62** (2011), no.3, 1325-1332.

[7] K, Diethelm; A. D, Freed; *On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity*, in "Scientifice Computing in Chemical Engi-neering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties", (F. Keil, W. Mackens, H. Voss, and J.Werther, Eds), pp 217-224, Springer-Verlag, Heidelberg, 1999.

[8] Deng, Jiqin; Ma, Lifeng; *Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations*, Appl. Math. Lett. **23** (2010), no. 6, 676-680.

[9] Koeller, R. C.; *Application of fractional calculus to the theory of viscoelasticity.* Trans. ASME J. Appl. Mech. **51** (1984), no. 2, 299-307.

[10] Kosmatov, Nickolai; *A singular boundary value problem for nonlinear differential equations of fractional order.* J. Appl. Math. Comput. **29** (2009), no. 1-2, 125-135.

[11] Kilbas, Anatoly A.; Srivastava, Hari M.; Trujillo, Juan J.; *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[12] Lin, Wei; *Global existence theory and chaos control of fractional differential equations.* J. Math. Anal. Appl. **332** (2007), no. 1, 709-726.

[13] Liu, Yang; Zhang, weiguo; Liu, Xiping; *A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the RiemannLiouville derivative.* Appl. Math. Lett. **25** (2012), no. 11, 1986-1992.

[14] Maagli, Habib; Masmoudi, Syrine; *Existence theorem of nonlinear singular boundary value problem.* Nonlinear Anal. **46** (2001), no. 4, 465-473.

[15] Podlubny, Igor; *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal **5** (2002), no. 4, 367-386.

[16] Qiu, Tingting; Bai, Zhanbing; *Existence of positive solutions for singular fractional differential equations.* Electron. J. Differential Equations (2008) , no. 146, pp. 19.

[17] Zhao, Yige; Sun, Shurong; Han, Zhenlai; Li, Qiuping; *Positive Solutions to Boundary Value Problems of Nonlinear Fractional Differential Equations.* Abstr. Appl. Anal. 2011, Article ID 390543, 16 pp.

[18] Z. Zhao; *Positive solutions of nonlinear second order ordinary differential equations*, Proc. Amer. Math. Soc. **121** (1994) , no. 2, p.465-469.

Habib Maagli

King Abdulaziz University, College of Sciences ant Arts, Rabigh Campus, Department of Mathematics P.O. Box 344, Rabigh 21911, Kingdom of Saudi Arabia

E-mail address: habib.maagli@fst.rnu.tn