Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 30, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A COMPUTATIONAL TECHNIQUE FOR SOLVING BOUNDARY VALUE PROBLEM WITH TWO SMALL PARAMETERS

DEVENDRA KUMAR

Abstract. In this article we study a singularly perturbed boundary-value problem for a delay differential equation with a small delay parameter in the first derivative term whose solution has a single boundary layer. The proposed method is shown to be stable, and its performance is confirmed with examples.

1. Introduction

Delay differential equations arise in the mathematical modelling of various practical phenomena, for instance, micro scale heat transfer [17], hydrodynamics of liquid helium [8], second-sound theory [9], thermo elasticity [6], diffusion in polymers [12], reaction-diffusion equations [2], stability [3], control including control of chaotic systems [11], a variety of model for physiological processes or diseases [4, 13] etc. As a consequence, they have received a lot of interest in recent years, especially for linear problems, for which one can obtain analytical solutions by means of, for example, the Laplace transform in time, separation of variables in finite spatial domains, etc.

A delay differential equation is said to be of retarded type if the delay argument does not occur in the highest order derivative term. If we restrict it to a class in which the highest derivative term is multiplied by a small parameter, then we obtain singularly perturbed delay differential equations of the retarded type. Frequently, delay differential equations have been reduced to differential equations with coefficients that depend on the delay by means of first order accurate Taylor's series expansions of the terms that involve delay and the resulting differential equations have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not [15].

It is well-known that the standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ϵ is small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the parameter value ϵ; i.e., the methods that are convergent ϵ-uniformly [5, 7, 16]. There are essentially two strategies to design schemes which have small truncation errors inside

2000 *Mathematics Subject Classification.* 65L03, 65L10, 65L11, 65L20.
Key words and phrases. Finite difference method; singular perturbation;
delay differential equation; negative shift.

Submitted November 2, 2012. Published January 28, 2013.

the layer region(s). The first approach which forms the class of fitted mesh methods consists in choosing a fine mesh in the layer region(s). The second approach is in the context of the fitted operator methods in which the mesh remains uniform and the difference schemes reflect the qualitative behavior of the solution inside the layer region(s). A nice discussion using one or both of the above strategies can be found in Miller *et al.* [5]. The work in this paper falls under the first category. We have derived a finite difference scheme on a non uniform mesh for the boundary value problems for a class of singularly perturbed delay differential equations.

2. Statement of the problem

Consider the following boundary-value problem for a singularly perturbed delay differential equation with a small parameter multiplying to the second derivative and containing negative shift in the first derivative term

$$\epsilon y''(x) + p(x)y'(x-\delta) - q(x)y(x) = r(x), \quad x \in [0,1], \tag{2.1}$$

under the interval and boundary conditions

$$y(x) = \phi(x), \quad -\delta \le x \le 0, \quad y(1) = \beta, \tag{2.2}$$

where ϵ is a small parameter, $0 < \epsilon \le 1$, and δ is also a small shift parameter of $o(\epsilon)$. The functions $p(x)$, $q(x)$, $r(x)$ and $\phi(x)$ are sufficiently smooth. It is assumed that $p(x) \ge p^* > 0$, $q(x) \ge q^* > 0$, for all $x \in [0,1]$ for some positive constants p^* and q^*. For $\delta = 0$ Equation (2.1) reduces to a singularly perturbed ordinary differential equation with only a single parameter ϵ, which has a boundary layer at one end or both ends depending on $p(x)$ and $q(x)$. It is well-known that if $p(x) > 0$ throughout the interval $[0,1]$, the boundary layer exists near left end $x = 0$ and if $p(x) < 0$ throughout the interval $[0,1]$, the boundary layer exists near right end $x = 1$. However, for $p(x) = 0$ the layer exists at interior of the interval $[0,1]$ and such point is called the turning point. In this paper, we consider the problems which have boundary layers (not interior layers) only. This problem has already been considered by Kadalbajoo and Kumar in [10]. There we have used B-spline collocation method with piecewise-uniform mesh and the method was shown to be parameter uniform of order almost two. In this paper we have generated a geometric mesh and used finite difference method with geometric mesh. The proposed method is not very useful in the case when the delay parameter is relatively large as the Taylor's series expansion is not valid in that case.

Since ϵ is small and $\delta = o(\epsilon)$, so taking the Taylor's series expansion for the term $y'(x-\delta)$, from (2.1)-(2.2), we obtain

$$\epsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), \tag{2.3}$$

with

$$y(0) = \phi_0 = \phi(0), \quad y(1) = \beta, \tag{2.4}$$

where

$$a(x) = \frac{p(x)}{1-(\delta/\epsilon)p(x)}, \quad b(x) = \frac{q(x)}{1-(\delta/\epsilon)p(x)}, \quad f(x) = \frac{r(x)}{1-(\delta/\epsilon)p(x)}.$$

Here we also assume that $\epsilon - \delta p(x) > 0$, then we have $a(x) \ge a^* > 0$ and $b(x) \ge b^* > 0$ for some positive constants a^* and b^*.

The operator $L_\epsilon^c = \epsilon\frac{d^2}{dx^2} + a(x)\frac{d}{dx} - b(x)I$ in (2.3) satisfies the following minimum principle:

Lemma 2.1. *Suppose $\pi(x)$ be any sufficiently smooth function satisfying $\pi(0) \geq 0$ and $\pi(1) \geq 0$. Then $L_\epsilon^c \pi(x) \leq 0$ for all $x \in (0,1)$ implies that $\pi(x) \geq 0$ for all $x \in [0,1]$.*

Proof. Let $z \in [0,1]$ be such that $\pi(z) < 0$ and $\pi(z) = \min_{0 \leq x \leq 1} \pi(x)$. Clearly $z \notin \{0,1\}$, therefore $\pi'(z) = 0$ and $\pi''(z) \geq 0$. Now we have from Eq. (2.3)

$$L_\epsilon^c \pi(z) = \epsilon \pi''(z) + a(z)\pi'(z) - b(z)\pi(z) > 0,$$

which contradict our assumption, therefore we must have $\pi(z) \geq 0$ and thus $\pi(x) \geq 0,\ \forall\, x \in [0,1]$. □

Now we can show the boundedness of the solutions of the continuous problem (2.3)-(2.4).

Lemma 2.2. *The solution $y(x)$ of (2.3)-(2.4) satisfies the inequality*

$$\|y\| \leq C \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\},$$

where $\|\cdot\|$ is the l_∞ norm given by $\|y\| = \max_{0 \leq x \leq 1} |y(x)|$.

Proof. Consider the barrier functions $\psi^{\pm}(x)$ defined by

$$\psi^{\pm}(x) = \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\} \pm y(x).$$

Then we have

$$\begin{aligned}
\psi^{\pm}(0) &= \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\} \pm y(0) \\
&= \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\} \pm \phi_0 \geq 0, \\
\psi^{\pm}(1) &= \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\} \pm y(1) \\
&= \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\} \pm \beta \geq 0.
\end{aligned}$$

Now we have for $x \in (0,1)$

$$\begin{aligned}
L_\epsilon^c \psi^{\pm}(x) &= \epsilon \frac{d^2}{dx^2}\psi^{\pm}(x) + a(x)\frac{d}{dx}\psi^{\pm}(x) - b(x)\psi^{\pm}(x) \\
&= \pm L_\epsilon^c y(x) - b(x) \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\} \\
&\leq \pm f - b^* \max\big\{|\phi_0|, |\beta|, \frac{1}{b^*}\|f\|\big\} \leq 0.
\end{aligned}$$

Using the minimum principle we obtain the required estimate. □

Lemma 2.3. *The derivatives of the solution $y(x)$ of the problem (2.3)-(2.4) satisfies*

$$\|y^{(k)}\| \leq C\epsilon^{-k}, \quad k = 1,2,3,$$

where C is a positive constant independent of ϵ.

Proof. Let $x \in (0,1)$ and let $V = (c, c+\gamma)$ be a neighborhood of x, where $\gamma > \epsilon$ is a constant, so that $V \subset (0,1)$, then by mean value theorem there exists a point $\zeta \in V$ such that

$$y'(\zeta) = \frac{y(c+\epsilon) - y(c)}{\epsilon},$$

so

$$\epsilon\|y'(\zeta)\| \le 2\|y\|. \tag{2.5}$$

Now differentiating (2.3) from ζ to x and taking the modulus from both sides, we obtain

$$\epsilon|y'(x)| \le \epsilon|y'(\zeta)| + \|f\||x-\zeta| + \int_\zeta^x |a(t)y'(t)|\,dt + \|b\|\|y\||x-\zeta|. \tag{2.6}$$

Now since

$$\int_\zeta^x |a(t)y'(t)|\,dt \le (2\|a\| + \|a'\||x-\zeta|)\|y\|, \tag{2.7}$$

with inequalities (2.5), (2.7) and using the fact $|x-\zeta| < \gamma$ and Lemma 2.2, from (2.6) we obtain

$$\|y'\| \le C\epsilon^{-1},$$

where C is a positive constant independent of ϵ. The bounds for $\|y''\|$ and $\|y'''\|$ can be obtained similarly. □

The solution $y(x)$ of the problem (2.3)-(2.4) can be decomposed into a smooth and singular components as

$$y = u + v,$$

where u and v are smooth and singular components respectively. The smooth component u can be written in three term asymptotic expansion as

$$u(x) = u_0(x) + u_1(x)\epsilon + u_2(x)\epsilon^2,$$

where u_0, u_1 and u_2 satisfies

$$\begin{gathered}
a(x)u_0'(x) + b(x)u_0(x) = f(x), \quad u_0(1) = u(1), \\
a(x)u_1'(x) + b(x)u_1(x) = -\epsilon u_0''(x), \quad u_1(1) = 0, \\
u_2''(x) = 0, \quad u_2(0) = 0,\ u_2(1) = 0.
\end{gathered}$$

The smooth component u is the solution of

$$L_\epsilon^c u(x) = f(x), \quad u(0) = u_0(0) + \epsilon u_1(0), \quad u(1) = y(1), \tag{2.8}$$

and the singular component v is the solution of the homogeneous problem

$$L_\epsilon^c v(x) = 0, \quad v(0) = y(0) - u(0), \quad v(1) = 0. \tag{2.9}$$

Now we can state the following theorem on the bounds for the solutions and derivatives of (2.8) and (2.9)

Theorem 2.4. *The solutions and the derivatives of* (2.8) *and* (2.9) *satisfy the following estimates*

$$\begin{gathered}
\|u\| \le C(1 + \exp(-a^* x/\epsilon)), \\
\|v\| \le C\exp(-a^* x/\epsilon), \\
\|u^{(k)}\| \le C(1 + \epsilon^{2-k}\exp(-a^* x/\epsilon)), \\
\|v^{(k)}\| \le C\epsilon^{-k}\exp(-a^* x/\epsilon).
\end{gathered}$$

The proof of the above theorem can be found in [14].

3. The discrete problem

Let $\Omega^N = \{x_0, x_1, x_2, \dots, x_N\}$ be the partition of $[0,1]$ such that $x_0 = 0$, $x_i = \sum_{k=0}^{i-1} h_k$, $i = 1(1)N$, $h_k = x_{k+1} - x_k$, $x_N = 1$. Let $r = \frac{h_i}{h_{i-1}}$, $i = 1(1)N$ be the common mesh ratio. Taking the Taylor's series expansion and neglecting the term of third and higher orders, we obtain the following expansions for y_{i+1} and y_{i-1},

$$y_{i+1} \simeq y_i + h_i y_i' + \frac{h_i^2}{2} y_i'', \tag{3.1}$$

$$y_{i-1} \simeq y_i - h_{i-1} y_i' + \frac{h_{i-1}^2}{2} y_i''. \tag{3.2}$$

If the boundary layer occurs at the left end then we choose $r > 1$, this gives more mesh points near the left boundary layer and if the boundary layer occurs at the right end then we choose $r < 1$, this gives more mesh points near the right boundary layer.

Multiplying (3.1) by r and adding it to (3.2), we obtain the approximation

$$y_i'' \simeq \frac{2r}{h_i^2(1+r)} \left[y_{i+1} - (1+r) y_i + r y_{i-1} \right]. \tag{3.3}$$

Similarly we can get the two terms expression for y_i' as

$$y_i' \simeq \frac{y_{i+1} - y_i}{r h_i}. \tag{3.4}$$

Here we can use the central difference formula also in place of forward difference formula. With the help of (3.3) and (3.4), Equation (2.3) can be discretized as

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{3.5}$$

with

$$y_0 = \phi_0, \quad y_N = \beta, \tag{3.6}$$

where

$$\begin{gathered}
E_i = \frac{2\epsilon r^2}{(1+r)h_i^2}, \\
F_i = \frac{2\epsilon r}{h_i^2} + \frac{a_i}{r h_i} + b_i, \\
G_i = \frac{2\epsilon r}{(1+r)h_i^2} + \frac{a_i}{r h_i}, \\
H_i = f_i, \quad i = 1, 2, \dots, N-1.
\end{gathered} \tag{3.7}$$

This can be written in matrix form as

$$AY = B, \tag{3.8}$$

where $A = [c_{i,j}]$ is a tridiagonal matrix of order $N-1$ with entries

$$\begin{gathered}
c_{i,i-1} = E_i, \quad i = 2(1)N, \\
c_{i,i} = -F_i, \quad i = 1(1)N-1, \\
c_{i,i+1} = G_i, \quad i = 1(1)N-1,
\end{gathered}$$

$Y = (y_1, y_2, \dots y_{N-1})'$ and $B = (f_1, f_2, \dots f_{N-1})'$ are the column vectors. Equation (3.8) represents a system of linear equations with $N-1$ equations in $N-1$ unknowns, $y_1, y_2, \dots y_{N-1}$. The system of equations can be easily solved by discrete invariant imbedding algorithm given in [1]. Note that one can use any other algorithm also such as Thomas algorithm.

The discrete problem (3.5) satisfies the following discrete minimum principle.

Lemma 3.1 (Discrete minimum principle)**.** *Let ψ_i be any mesh function such that ψ_0, $\psi_N \geq 0$, then $L_\epsilon^d \psi_i \leq 0$ for $1 \leq i \leq N-1$ implies that $\psi_i \geq 0$ for all $0 \leq i \leq N$.*

Proof. Suppose there is a positive integer k such that $0 > \psi_k = \min_{1\leq i\leq N-1} \psi_i$. Then we have

$$\begin{aligned} L_\epsilon^d \psi_k &= E_k \psi_{k-1} - F_k \psi_k + G_k \psi_{k+1} \\ &= \frac{2\epsilon r^2}{(1+r)h_k^2}(\psi_{k-1} - \psi_k) + \frac{2\epsilon r}{(1+r)h_k^2}(\psi_{k+1} - \psi_k) \\ &\quad + \frac{a_k}{h_{k+1}}(\psi_{k+1} - \psi_k) - b_k \psi_k > 0, \end{aligned}$$

which contradict the hypothesis and hence $\psi_i \geq 0$ for all $0 \leq i \leq N$. □

The existence, uniqueness and the stability of the solution of problem (3.5)-(3.6) are given by the following theorem.

Theorem 3.2. *The solution of the discrete problem* (3.5) *together with the boundary condition* (3.6) *exists, is unique and satisfies*

$$|y_i| \leq C \max\{|y(0)|, |y(1)|, \frac{1}{b^*} \max_{1\leq j\leq N-1} |L_\epsilon^d y_j|\}.$$

Proof. Let ψ_i be any mesh function satisfying $L_\epsilon^d(\psi_i) = f_i$. Taking absolute value on both sides, using (3.5), we obtain

$$|F_i||\psi_i| \leq |H_i| + |E_i||\psi_{i-1}| + |G_i||\psi_{i+1}|, \quad i = 1, 2, \dots, N-1.$$

This gives

$$\begin{aligned} &\frac{2\epsilon r^2}{(1+r)h_i^2}(|\psi_{i-1}| - |\psi_i|) + \frac{2\epsilon r}{h_i^2(1+r)}(|\psi_{i+1}| - |\psi_i|) \\ &+ \frac{a_i}{h_{i+1}}(|\psi_{i+1}| - |\psi_i|) - b_i|\psi_i| + |H_i| \geq 0. \end{aligned} \tag{3.9}$$

Let u_i, v_i be two solutions of the difference equation (3.5) satisfying the boundary condition (3.6). Then $w_i = u_i - v_i$ satisfies $L_\epsilon^d(w_i) = 0$, with $w_0 = w_N = 0$.

Let k be the integer such that $w_k = \max_{1\leq i\leq N-1} w_i$, then from (3.9) we have

$$\begin{aligned} &\frac{2\epsilon r^2}{(1+r)h_k^2}(|w_{k-1}| - |w_k|) + \frac{2\epsilon r}{h_k^2(1+r)}(|w_{k+1}| - |w_k|) \\ &+ \frac{a_k}{h_{k+1}}(|w_{k+1}| - |w_k|) - b_k|w_k| \geq 0. \end{aligned} \tag{3.10}$$

Since $a_k > 0$, $b_k > 0$, so the inequality (3.10) gives $w_k = 0$ and so $w_i \leq 0$ for $i = 1, 2, \dots, N-1$. Hence $u_i \leq v_i$ for $i = 1, 2, \dots, N-1$.

Again if we set $z_i = v_i - u_i$, then z_i is a mesh function satisfying $z_0 = z_N = 0$. Continuing in the same way as above, we obtain $u_i \geq v_i$ for $i = 1, 2, \dots, N-1$. Hence $u_i = v_i$ for $i = 1, 2, \dots, N-1$, which shows the uniqueness of the solution.

Now we define two mesh functions $\varphi_i^\pm$ such that

$$\varphi_i^\pm = \max\{|y(0)|, |y(1)|, \max_{1\leq j\leq N-1} |L_\epsilon^d y_j|\} \pm y_i.$$

Then $\varphi_0^{\pm} \geq 0$ and $\varphi_N^{\pm} \geq 0$ and for $1 \leq i \leq N-1$ we have

$$\begin{aligned} L_\epsilon^d \varphi_i^{\pm} &= -b_i \max\{|y(0)|, |y(1)|, \frac{1}{b^*} \max_{1\leq j\leq N-1} |L_\epsilon^d y_j|\} \pm L_\epsilon^d y_i \\ &\leq -b^* \max\{|y(0)|, |y(1)|, \frac{1}{b^*} \max_{1\leq j\leq N-1} |L_\epsilon^d y_j|\} \pm L_\epsilon^d y_i < 0. \end{aligned}$$

A consequence of Lemma 3.1 gives the required estimate. □

4. Stability Analysis

Consider a difference relation

$$y_i = S_i y_{i+1} + T_i, \tag{4.1}$$

where $S_i = S(x_i)$ and $T_i = T(x_i)$ are unknowns which are to be determined. From (4.1), we have

$$y_{i-1} = S_{i-1} y_i + T_{i-1}. \tag{4.2}$$

Using (4.2) in (3.5), we obtain

$$y_i = \frac{G_i}{F_i - E_i S_{i-1}} y_{i+1} + \frac{E_i T_{i-1} - H_i}{F_i - E_i S_{i-1}}. \tag{4.3}$$

On comparing (4.1) and (4.3), we obtain the recurrence relations

$$S_i = \frac{G_i}{F_i - E_i S_{i-1}}, \tag{4.4}$$

$$T_i = \frac{E_i T_{i-1} - H_{i-1}}{F_i - E_i S_{i-1}}. \tag{4.5}$$

To solve above recurrence relations for $i = 1, 2, \dots N-1$, we need S_0 and T_0. Now it is given that $y_0 = \phi_0$, therefore we have $S_0 y_1 + T_0 = \phi_0$. So we can choose $S_0 = 0$ and then $T_0 = \phi_0$. Now by using these initial conditions, we can compute S_i and T_i for $i = 1, 2, \dots, N-1$ and using these values of S_i and T_i in (4.1), we obtain y_i for $i = 1, 2, \dots, N-1$.

Now we give the proof of the stability of our scheme. Suppose a small error e_{i-1} has been made in the calculation of S_{i-1}, then we have, $\bar{S}_{i-1} = S_{i-1} + e_{i-1}$, and we are actually calculating

$$\bar{S}_i = \frac{G_i}{F_i - E_i \bar{S}_{i-1}}. \tag{4.6}$$

From (4.4) and (4.6), we have

$$\begin{aligned} e_i &= \frac{G_i}{F_i - E_i(S_{i-1} + e_{i-1})} - \frac{G_i}{F_i - E_i S_{i-1}} \\ &= \frac{G_i E_i e_{i-1}}{(F_i - E_i(S_{i-1} + e_{i-1}))(F_i - E_i S_{i-1})}. \end{aligned} \tag{4.7}$$

Under the assumption, the error is small initially, from (4.7) we obtain

$$e_i = \Big(\frac{S_i^2 E_i}{G_i}\Big) e_{i-1}. \tag{4.8}$$

Now we have

$$G_1 - F_1 = -\frac{2\epsilon r^2}{(1+r)h_1^2} - b_1 < 0,$$

so $\frac{G_1}{F_1} < 1$. Therefore from (4.4), we have $S_1 = \frac{G_1}{F_1} < 1$. Again from (4.4) we have

$$S_2 = \frac{G_2}{F_2 - E_2 S_1} < \frac{G_2}{F_2 - E_2} < \frac{G_2}{E_2 + G_2 - E_2} = 1.$$

Similarly we can show that $S_i < 1$ for $1 \le i \le N-1$. Now we have

$$\begin{aligned} |E_i| - |G_i| &= \frac{2\epsilon r^2}{(1+r)h_i^2} - \frac{2\epsilon r}{(1+r)h_i^2} - \frac{a_i}{rh_i}, \\ &= \frac{2\epsilon r(r-1)}{(1+r)h_i^2} - \frac{a_i}{rh_i} \\ &< 0 \quad \text{as } \epsilon \text{ is small and } r \text{ is close to 1.} \end{aligned}$$

Thus $\frac{|E_i|}{|G_i|} < 1$, it follows from (4.8) that

$$|e_i| = |S_i|^2 \Big|\frac{E_i}{G_i}\Big| |e_{i-1}| < |e_{i-1}|.$$

Which confirm the stability of the recurrence relation (4.4). Similarly we can prove that the recurrence relation (4.5) is also stable.

5. Numerical results and discussions

To validate the theoretical results, we apply the proposed numerical scheme to a test problem having a left boundary layer.

Example 5.1. Consider the problem $\epsilon y''(x) + y'(x-\delta) - y(x) = 0$, under the interval conditions $y(x) = 1$, $-\delta \le x \le 0$, $y(1) = 1$.

Table 1. Maximum absolute error for Example 5.1 for $\delta = 0.001 \times \epsilon$ using $N = 100$

ϵ	$r = 1.1$	$r = 1.01$	$r = 1.001$
10^{-2}	2.42E-02	5.65E-02	7.96E-02
10^{-4}	2.95E-02	9.05E-03	7.97E-03
10^{-6}	6.39E-02	1.65E-03	1.53E-03
10^{-8}	2.57E-02	1.66E-03	1.47E-03
10^{-10}	2.57E-02	1.66E-03	1.47E-03
10^{-12}	2.57E-02	1.66E-03	1.46E-03

A numerical method for solving singularly perturbed boundary value problem with a negative shift in the first derivative term is considered. It is a practical method and can be easily implemented on a computer to solve such problems. An example is given to demonstrate the efficiency of the proposed method. The maximum absolute errors $E_\epsilon^N = \max_i |y(x_i) - y_i|$ at the nodal points are tabulated in the table for different values of perturbation parameter ϵ and different values of mesh ratio r by using $N = 100$.

The graph of the solution of the considered example for different values of delay is plotted in Figure 1 to examine the questions on the effect of delay on the boundary layer behavior of the solution. One can observe that if $\delta = o(\epsilon)$, the layer behavior is maintained in the case of left boundary layer (the layer behavior is also maintained in the case of right boundary layer). As the delay increases, the thickness of the layer decreases in the case when the solution exhibits layer behavior on the left

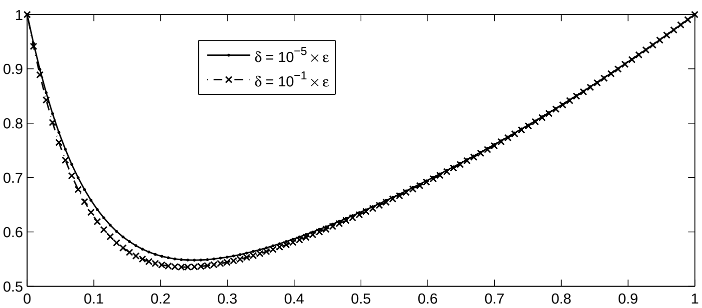

FIGURE 1. Numerical solution of Example 5.1 for $\epsilon = 0.1$

side (as shown in Figure 1) while in the case of the right side boundary layer, it increases. The delay affects more in the case when the solution of the boundary value problem exhibits layer behavior on the left side in comparison to the right side boundary layer case.

Acknowledgements. Author is very thankful to the reviewer for his/her valuable suggestions and comments for the improvement of this paper.

References

[1] E. Angel, R. Bellman; Dynamic Programming and Partial Differential Equations, Academic, New York, (1972).

[2] M. Bestehorn, E. V. Grigorieva; Formation and propagation of localized states in extended systems, *Ann. Phys.*, (Leipzig) Vol. 13 (2004) 423-431.

[3] T. A. Burton; Fixed points, stability, and exact linearization, *Nonlinear Anal.*, Vol. 61 (2005) 857-870.

[4] M. Wazewska-Czyzewska, A. Lasota; Mathematical models of the red cell system, *Mat. Stos.*, Vol. 6 (1976) 25-40.

[5] E. P. Doolan, J. J. H. Miller, W. H. A. Schilders; Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, (1980).

[6] M. A. Ezzat, M. I. Othman, A. M. S. El-Karamany; State space approach to two-dimensional generalized thermo-viscoelasticity with two relaxation times, *Int. J. Eng. Sci.*, Vol. 40 (2002) 1251-1274.

[7] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, R. E. ORiordan, G. I. Shishkin; Robust Computational Techniques for Boundary Layers, Chapman- Hall/CRC, New York, (2000).

[8] D. D. Joseph, L. Preziosi; Heat waves, *Rev. Mod. Phys.*, Vol. 61 (1989) 41-73.

[9] D. D. Joseph, L. Preziosi; Addendum to the paper “Heat waves”, *Rev. Mod. Phys.*, Vol. 62 (1990) 375-391.

[10] M. K. Kadalbajoo, D. Kumar; Fitted mesh B-spline collocation method for singularly perturbed differential-difference equations with small delay, *Appl. Math. Comput.*, Vol. 204 (2008) 90-98.

[11] X. Liao; Hopf and resonant codimension two bifurcation in van der Pol equation with two time delays, *Chaos, Solitons & Fractals*, Vol. 23 (2005) 857-871.

[12] Q. Liu, X. Wang, D. De Kee; Mass transport through swelling membranes, *Int. J. Eng. Sci.*, Vol. 43 (2005) 1464-1470.

[13] M. C. Mackey, L. Glass; Oscillations and chaos in physiological control systems, *Science*, Vol. 197 (1977) 287-289.

[14] J. J. H. Miller, E. O’Riordan, G. I. Shishkin; Fitted numerical methods for singular perturbation problems, World Scientific, (1996).

[15] J. I. Ramos; On the numerical treatment of an ordinary differential equation arising in one-dimensional non-Fickian diffusion problems, *Comput. Phys. Commun.*, Vol. 170 (2005) 231-238

[16] H. G. Roos, M. Stynes, L. Tobiska; Numerical Methods for Singularly Perturbed Differential Equations, Convection-Diffusion and Flow Problems, Springer Verlag, Berlin, (1996).

[17] D. Y. Tzou; Micro-to-macroscale heat transfer, Taylor & Francis, Washington, DC, 1997.

Devendra Kumar
Department of Mathematics, BITS Pilani Pilani - 333031, Rajasthan, India
Phone +919413487072; fax: +911596244183
E-mail address: `dkumar2845@gmail.com, dkumar@bits-pilani.ac.in`