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EXISTENCE OF SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS INVOLVING CRITICAL EXPONENTS AND HARDY TERMS

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Abstract. Using variational methods, including the Ljusternik-Schnirelmann theory, we prove the existence of solutions for quasilinear elliptic systems with critical Sobolev exponents and Hardy terms.

1. Introduction and statement of main results

We consider the critical quasilinear elliptic system

$$
\begin{gathered}
-\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{1}{p^*} F_u(u,v) + G_u(u,v), \quad x \in \Omega, \\
-\Delta_p v - \mu \frac{|v|^{p-2}v}{|x|^p} = \frac{1}{p^*} F_v(u,v) + G_v(u,v), \quad x \in \Omega, \\
u = v = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $N \geq p^2, 2 \leq p \leq q < p^*$, $p^* = \frac{Np}{N-p}$ denotes the Sobolev critical exponent, $F, G \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ are homogeneous functions of degrees p^* and q, respectively. $\mathbb{R}^+ = [0, +\infty)$, $(F_u(u,v), F_v(u,v)) = \nabla F, (G_u(u,v), G_v(u,v)) = \nabla G$, $0 \leq \mu < \bar{\mu}$, $\bar{\mu} = (\frac{N-p}{p})^p$ is the best constant of the Hardy inequality [4]:

$$
\bar{\mu} \int_\Omega \frac{|u|^p}{|x|^p} dx \leq \int_\Omega |\nabla u|^p dx,
$$

for all $u \in W_0^{1,p}(\Omega)$, where $W_0^{1,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega |\nabla u|^p dx)^{1/p}$. For $\mu \in [0, \bar{\mu})$, it follows from the Hardy inequality that

$$
\|u\|_\mu = \Big(\int_\Omega |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} dx \Big)^{1/p}
$$

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defines a norm in $W_0^{1,p}(\Omega)$ equivalent to its usual norm. The best Sobolev constant is defined as

$$S_\mu = \inf_{u\in\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p})dx}{(\int_{\mathbb{R}^N}|u|^{p^*}dx)^{p/p^*}}, \quad \mu\in[0,\bar{\mu}). \tag{1.2}$$

In recent years, much attention has been focused on singular problems involving both the Hardy potential and the Sobolev critical term. For example, see [7, 13, 16, 18, 19, 20, 23, 26] and the references therein. In [9], Ding and Xiao consider the p-Laplacian system

$$\begin{gathered} -\Delta_p u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta + \lambda|u|^{q-2}u, \quad x\in\Omega,\\ -\Delta_p v = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \delta|v|^{q-2}v, \quad x\in\Omega,\\ u=v=0, \quad x\in\partial\Omega, \end{gathered} \tag{1.3}$$

where $p\le q<p^*, \alpha,\beta>1, \alpha+\beta=p^*$. Using standard tools of the variational theory and the Ljusternik-Schnirelmann category theory, in [9] sufficient conditions on λ,δ are given for (1.3) to have at least $\operatorname{cat}_\Omega(\Omega)$ positive solutions. This result extended the result of Alves and Ding in [2] where the single equation case was studied. Hsu [17] obtained the existence of two positive solutions for (1.3) including a sublinear perturbation of $1<q<p<N$. Recently, Shen and Zhang extended the results in [25] to a general class of homogeneous functions and obtained similar results. For similar problems, we refer the reader to [3, 6, 8, 10, 11, 12, 14, 15, 21, 22, 24] and the references therein.

In this paper, motivated by [2, 9, 17, 25], we shall extend these results to the case containing a general class of homogeneous nonlinearities and Hardy terms. To the best of our knowledge, problem (1.1) has not been considered before. Thus, it is necessary for us to investigate the related singular critical systems.

The following assumptions are used in this article:

- (F0) $F\in C^1(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$ and $F(tu,tv)=t^{p^*}F(u,v)(t>0)$ holds for all $(u,v)\in\mathbb{R}^+\times\mathbb{R}^+$,
- (F1) $F_u(0,1)=F_v(1,0)=0$,
- (F2) $F_u(u,v)\ge 0, F_v(u,v)\ge 0$ for all $u,v\ge 0$,
- (F3) the 1-homogeneous function $(u,v)\mapsto F(u^{\frac{1}{p^*}}, v^{\frac{1}{p^*}})$ is concave for all $(u,v)\in\mathbb{R}^+\times\mathbb{R}^+$.
- (G0) G is q-homogeneous for some $p\le q<p^*$,
- (G1) $G_u(0,1)=G_v(1,0)=0$.

To present our results, we define

$$\lambda = \max\{G(u,v): u,v\ge 0, u^q+v^q=1\}, \tag{1.4}$$
$$\delta = \min\{G(u,v): u,v\ge 0, u^q+v^q=1\}. \tag{1.5}$$

If Y is a closed subset of a topological space X, we denote, by $\operatorname{cat}_X(Y)$, the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y. We say that a weak solution $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ of problem (1.1) is nonnegative if $u,v\ge 0$ in Ω.

The main results of this paper are stated in the following two theorems whose conclusions are new (to the best of our knowledge).

Theorem 1.1. *Suppose* (F0)–(F3), (G0)–(G1) *are satisfied, and one of the following two conditions holds:*

(I) $\bar{p} < q < p^*$ *with* $\bar{p} = \max\left\{p, \frac{N}{b(\mu)}, \frac{p(2N-pb(\mu)-p)}{N-p}\right\}$, $0 \le \mu < \bar{\mu}$ *and* $\lambda, \delta > 0$*;*

(II) $q = p$, $0 \le \mu \le \frac{N^{p-1}(N-p^2)}{p^p}$ *and* $\lambda, \delta \in (0, \frac{1}{p}\Lambda_1)$*, where* Λ_1 *is the first eigenvalue of* $(-\Delta_p, W_0^{1,p}(\Omega))$.

Then problem (1.1) *has at least one nonnegative solution.*

Theorem 1.2. *Suppose* (F0)–(F3), (G0)–(G1) *are satisfied, and one of the following two conditions holds:*

(I) $\bar{p} < q < p^*$ *with* $\bar{p} = \max\left\{p, \frac{N}{b(\mu)}, \frac{p(2N-pb(\mu)-p)}{N-p}\right\}$, $0 \le \mu < \bar{\mu}$*;*

(II) $q = p$, $0 \le \mu \le \frac{N^{p-1}(N-p^2)}{p^p}$.

Then there exists $\Lambda > 0$ *such that problem* (1.1) *has at least* $\operatorname{cat}_\Omega(\Omega)$ *distinct nonnegative solutions for* $\lambda, \delta \in (0, \Lambda)$.

Remark 1.3. Our Theorem 1.1 is a generalization of [16, Theorem 1.1] from quasilinear elliptic equations to quasilinear elliptic systems.

Remark 1.4. Theorem 1 in [9] is the special case of our Theorem 1.2 corresponding to $\mu = 0, F(u,v) = 2|u|^\alpha |v|^\beta, \alpha + \beta = p^*$ and $G(u,v) = \lambda |u|^q + \delta |v|^q$. In this paper, different from [25], we can deal with $F(u,v)$ which possesses both coupled and uncoupled terms. For example, let

$$F(u,v) = au^{p^*} + \sum_{i=1}^k b_i u^{\alpha_i} v^{\beta_i} + cv^{p^*},$$

where $a, b_i, c \ge 0$, $\alpha_i, \beta_i > 1$, $\alpha_i + \beta_i = p^*$. $F(u,v)$ obviously satisfies (F0)–(F3).

This article is organized as follows. In Section 2, some notation and the mountain pass levels are established and Theorem 1.1 is proven. We present some technical lemmas which are crucial in the proof of Theorem 1.2 in Section 3. Theorem 1.2 is proven in Section 4.

2. Preliminaries and proof of Theorem 1.1

Throughout this paper, C, C_i will denote various positive constants whose exact values are not important. And $\to$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence. $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \le C, o_m(1)$ denotes $o_m(1) \to 0$ as $m \to \infty$. $L^s(\Omega), for (1 \le s < +\infty)$, denotes Lebesgue spaces, the norm L^s is denoted by $|\cdot|_s$ for $1 \le s < +\infty$. Let $B_r(x)$ denote a ball centered at x with radius r. The dual space of a Banach space E will be denoted by E^{-1}. We define the product space $E := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ endowed with the norm $\|(u,v)\|_E = \left(\|u\|_\mu^p + \|v\|_\mu^p\right)^{1/p}$.

In view of (F1), (G1), we can extend the function $F(u,v)$ and $G(u,v)$ to the whole $\mathbb{R}^2$ by considering $F(u,v) = F(u^+, v^+)$, $G(u,v) = G(u^+, v^+)$, where $u^+ = \max\{u, 0\}$ and $v^+ = \max\{v, 0\}$. It is easy to check that $F(u,v)$ and $G(u,v) \in C^1(\mathbb{R}^2)$. Therefore, we always consider $F(u,v)$ and $G(u,v)$ as these extensions.

A pair of functions $(u,v) \in E$ is said to be a weak solution of problem (1.1) if

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla \varphi_1 - \mu \frac{|u|^{p-2}u\varphi_1}{|x|^p} + |\nabla v|^{p-2}\nabla v \nabla \varphi_2 - \mu \frac{|v|^{p-2}v\varphi_2}{|x|^p})dx$$

$$- \frac{1}{p^*}\int_\Omega (F_u(u,v)\varphi_1 + F_v(u,v)\varphi_2)dx - \int_\Omega (G_u(u,v)\varphi_1 + G_v(u,v)\varphi_2)dx = 0,$$

for all $(\varphi_1, \varphi_2) \in E$. Using (F0)-(G1) and well-known arguments, we know that the weak solutions of (1.1) are precisely the critical points of the C^1-functional $I_{\lambda,\delta} : E \to \mathbb{R}$ given by

$$\begin{aligned} &I_{\lambda,\delta}(u,v)\\ &= \frac{1}{p}\int_\Omega (|\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu\frac{|v|^p}{|x|^p})dx - \frac{1}{p^*}\int_\Omega F(u,v)dx - \int_\Omega G_{\lambda,\delta}(u,v)dx. \end{aligned}$$

We notice that, in the definition of $I_{\lambda,\delta}$, we are denoting $G_{\lambda,\delta}(u,v) := G(u,v)$ for $(u,v) \in \mathbb{R}^2$. We shall write $G_{\lambda,\delta}$ instead of G to emphasize that the main theorems depend on the value of the parameters λ and δ defined in (1.4) and (1.5), respectively.

The functional $I \in C^1(E,\mathbb{R})$ is said to satisfy the $(PS)_c$ condition if any sequence $\{z_m\} \subset E$ such that as $m \to \infty$, $I(z_m) \to c$, $I'(z_m) \to 0$ strongly in E^{-1} contains a subsequence converging in E to a critical point of I. In this paper, we will take $I = I_{\lambda,\delta}$ and $E = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

In this section, we will find the range of c where the $(PS)_c$ condition holds for the functional $I_{\lambda,\delta}$. First, let us define

$$S_F = \inf_{(u,v)\in E\setminus\{(0,0)\}} \Big\{ \frac{\int_\Omega |\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu\frac{|v|^p}{|x|^p}dx}{(\int_\Omega F(u,v)dx)^{p/p^*}} : \int_\Omega F(u,v)dx > 0\Big\}. \tag{2.1}$$

Lemma 2.1. *Suppose* (F0)–(F3), (G0)–(G1) *are satisfied, then the functional $I_{\lambda,\delta}$ satisfies the $(PS)_c$ condition for all $c < \frac{1}{N}S_F^{N/p}$, provided either $p < q < p^*$ or $q = p$ and the parameter λ defined in* (1.4) *belongs to $(0, \frac{1}{p}\Lambda_1)$, where $\Lambda_1 > 0$ denotes the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$.*

Proof. Let $\{(u_m, v_m)\} \subset E$ such that $I'_{\lambda,\delta}(u_m, v_m) \to 0$ and $I_{\lambda,\delta}(u_m, v_m) \to c < \frac{1}{N}S_F^{N/p}$. Now, we firstly prove that $\{(u_m, v_m)\}$ is bounded in E. If $p < q < p^*$, it suffices to use the definition of $I_{\lambda,\delta}$ to obtain $C_1 > 0$ such that

$$\begin{aligned} c + C_1\|(u_m,v_m)\|_E + o_m(1) &\geq I_{\lambda,\delta}(u_m,v_m) - \frac{1}{q}\langle I'_{\lambda,\delta}(u_m,v_m),(u_m,v_m)\rangle\\ &= \Big(\frac{1}{p} - \frac{1}{q}\Big)\|(u_m,v_m)\|_E^p + \Big(\frac{1}{q} - \frac{1}{p^*}\Big)\int_\Omega F(u_m,v_m)dx\\ &\geq \frac{q-p}{pq}\|(u_m,v_m)\|_E^p, \end{aligned}$$

which implies that $\{(u_m, v_m)\} \subset E$ is bounded. When $q = p$, in this case, it follows that

$$\int_\Omega G_{\lambda,\delta}(u_m,v_m)dx \leq \lambda\int_\Omega (|u_m|^p + |v_m|^p)dx \leq \frac{\lambda}{\Lambda_1}\|(u_m,v_m)\|_E^p,$$

and therefore,

$$\begin{aligned} c + C_1\|(u_m,v_m)\|_E + o_m(1) &\geq I_{\lambda,\delta}(u_m,v_m) - \frac{1}{p^*}\langle I'_{\lambda,\delta}(u_m,v_m),(u_m,v_m)\rangle\\ &= \frac{1}{N}\|(u_m,v_m)\|_E^p - \frac{p}{N}\int_\Omega G(u_m,v_m)dx \end{aligned}$$

$$\geq \frac{1}{N}\Big(1-\frac{p\lambda}{\Lambda_1}\Big)\|(u_m,v_m)\|_E^p.$$

Since $p\lambda < \Lambda_1$, the boundedness of $\{(u_m,v_m)\}$ follows as in the first case.

So $\{(u_m,v_m)\}$ is bounded in E. Going if necessary to a subsequence, we can assume that

$$\begin{gathered}(u_m,v_m)\rightharpoonup (u,v), \quad \text{in } E,\\ (u_m,v_m)\to (u,v), \quad \text{a.e. in } \Omega,\\ (u_m,v_m)\to (u,v), \quad \text{in } L^s(\Omega)\times L^s(\Omega), 1\leq s<p^*,\end{gathered}$$

as $m\to\infty$. Clearly, we have that

$$\int_\Omega G_{\lambda,\delta}(u_m,v_m)dx=\int_\Omega G_{\lambda,\delta}(u,v)dx+o_m(1). \tag{2.2}$$

Moreover, a standard argument shows that $I'_{\lambda,\delta}(u,v)=0$. Thus, we obtain

$$\begin{aligned} I_{\lambda,\delta}(u,v)&=\frac{1}{p}\|(u,v)\|_E^p-\frac{1}{p^*}\int_\Omega F(u,v)dx-\int_\Omega G_{\lambda,\delta}(u,v)dx\\ &=\Big(\frac{1}{p}-\frac{1}{q}\Big)\|(u,v)\|_E^p+\Big(\frac{1}{q}-\frac{1}{p^*}\Big)\int_\Omega F(u,v)dx\geq 0.\end{aligned} \tag{2.3}$$

Let $(\tilde u_m,\tilde v_m)=(u_m-u,v_m-v)$. Then by the Brezis-Lieb Lemma [5], we have

$$\|(\tilde u_m,\tilde v_m)\|_E^p=\|(u_m,v_m)\|_E^p-\|(u,v)\|_E^p+o_m(1). \tag{2.4}$$

By the same method as in [11, Lemma 8], we obtain

$$\int_\Omega F(u_m,v_m)dx=\int_\Omega F(u,v)dx+\int_\Omega F(\tilde u_m,\tilde v_m)dx+o_m(1). \tag{2.5}$$

By (2.2),(2.3),(2.4),(2.5) and the weak convergence of (u_m,v_m), we have

$$\begin{aligned} c+o_m(1)&=I_{\lambda,\delta}(u,v)+\frac{1}{p}\|(\tilde u_m,\tilde v_m)\|_E^p-\frac{1}{p^*}\int_\Omega F(\tilde u_m,\tilde v_m)dx\\ &\geq \frac{1}{p}\|(\tilde u_m,\tilde v_m)\|_E^p-\frac{1}{p^*}\int_\Omega F(\tilde u_m,\tilde v_m)dx.\end{aligned} \tag{2.6}$$

Using that $I'_{\lambda,\delta}(u_m,v_m)\to 0$ and (2.2), (2.4), (2.5), we obtain

$$\begin{aligned} o_m(1)&=\langle I'_{\lambda,\delta}(u_m,v_m),(u_m,v_m)\rangle\\ &=\|(u_m,v_m)\|_E^p-\int_\Omega F(u_m,v_m)dx-q\int_\Omega G_{\lambda,\delta}(u_m,v_m)dx\\ &=\langle I'_{\lambda,\delta}(u,v),(u,v)\rangle+\|(\tilde u_m,\tilde v_m)\|_E^p-\int_\Omega F(\tilde u_m,\tilde v_m)dx.\end{aligned}$$

Recalling that $I'_{\lambda,\delta}(u,v)=0$, we can use the above equality and (2.6) to obtain

$$\lim_{m\to\infty}\|(\tilde u_m,\tilde v_m)\|_E^p=k=\lim_{m\to\infty}\int_\Omega F(\tilde u_m,\tilde v_m)dx,\quad c\geq\Big(\frac{1}{p}-\frac{1}{p*}\Big)k=\frac{1}{N}k,$$

where $k\geq 0$.

In view of the definition of S_F, we deduce that

$$\|(\tilde u_m,\tilde v_m)\|_E^p\geq S_F\Big(\int_\Omega F(\tilde u_m,\tilde v_m)dx\Big)^{p/p^*}.$$

Taking the limit, we obtain $k \geq S_F k^{p/p^*}$. So, if $k > 0$, we conclude that $k \geq S_F^{N/p}$ and therefore

$$\frac{1}{N} S_F^{N/p} \leq \frac{1}{N} k \leq c < \frac{1}{N} S_F^{N/p},$$

which is a contradiction. Hence $k = 0$ and therefore $(u_m, v_m) \to (u, v)$ in E. □

For all $\mu \in [0, \bar{\mu})$, we consider the limiting problem

$$\begin{gathered} -\Delta_p U - \mu \frac{U^{p-1}}{|x|^p} = U^{p^*-1}, \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\ U > 0, \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\ U \to 0, \quad \text{as } |x| \to +\infty. \end{gathered} \tag{2.7}$$

From [1], we know that problem (2.7) has a ground state $U_{p,\mu}$, which is unique up to scaling. That is, all ground states must be of the form

$$V_{p,\mu,\varepsilon}(x) = \varepsilon^{-\frac{N-p}{p}} U_{p,\mu}\Big(\frac{x}{\varepsilon}\Big) = \varepsilon^{-\frac{N-p}{p}} U_{p,\mu}\Big(\frac{|x|}{\varepsilon}\Big), \quad \varepsilon > 0, \tag{2.8}$$

that satisfy

$$\int_{\mathbb{R}^N} (|\nabla V_{p,\mu,\varepsilon}(x)|^p - \mu \frac{|V_{p,\mu,\varepsilon}(x)|^p}{|x|^p}) dx = \int_{\mathbb{R}^N} |V_{p,\mu,\varepsilon}(x)|^{p^*} dx = S_\mu^{N/p}, \tag{2.9}$$

where S_μ is the best Sobolev constant given in (1.2).

Moreover, the ground state $U_{p,\mu}$ is radially symmetric and decreasing, and the following asymptotic properties at the origin and infinity for $U_{p,\mu}(r)$ and $U'_{p,\mu}(r)$ hold:

$$\begin{gathered} \lim_{r \to 0^+} r^{a(\mu)} U_{p,\mu}(r) = c_1 > 0, \quad \lim_{r \to 0^+} r^{a(\mu)+1} |U'_{p,\mu}(r)| = c_1 a(\mu) \geq 0, \\ \lim_{r \to +\infty} r^{b(\mu)} U_{p,\mu}(r) = c_2 > 0, \quad \lim_{r \to +\infty} r^{b(\mu)+1} |U'_{p,\mu}(r)| = c_2 b(\mu) > 0, \end{gathered}$$

where c_1 and c_2 are positive constants depending only on N, p, μ, and $a(\mu), b(\mu)$, the zeros of the function $h(t) = (p-1)t^p - (N-p)t^{p-1} + \mu$, $t \geq 0$, which satisfy $0 \leq a(\mu) < b(\mu) \leq \frac{N-p}{p-1}$.

After a direct calculation, we infer that $t_{\min} = \frac{N-p}{p}$ is the unique minimal point of $h(t)$ and $h(\frac{N-p}{p}) = -\bar{\mu} + \mu < 0$. Moreover, $h'(t) < 0$ for all $0 < t < t_{\min}$ and $h'(t) > 0$ for all $t > t_{\min}$. That is, $h(t)$ is decreasing on the interval $(0, t_{\min})$ and increasing on the interval $(t_{\min}, +\infty)$. Thus $0 \leq a(\mu) < \frac{N-p}{p} < b(\mu)$.

In addition, using [11, Lemma 3] and the homogeneity of F, we obtain $A, B > 0$ such that

$$S_F = \frac{\|(AV_{p,\mu,\varepsilon}, BV_{p,\mu,\varepsilon}\|_E^p}{(\int_{\mathbb{R}^N} F(AV_{p,\mu,\varepsilon}, BV_{p,\mu,\varepsilon}) dx)^{p/p^*}} = \frac{A^p + B^p}{(F(A,B))^{p/p^*}} \cdot \frac{S_\mu^{N/p}}{|V_{p,\mu,\varepsilon}|_{p^*}^p},$$

from this and (2.9), we have

$$S_F = \frac{A^p + B^p}{(F(A,B))^{p/p^*}} S_\mu. \tag{2.10}$$

We define a cut-off function $\phi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ if $|x| \leq R$; $\phi(x) = 0$ if $|x| \geq 2R$ and $0 \leq \phi(x) \leq 1$, where $B_{2R}(0) \subset \Omega$ and set $u_\varepsilon = \frac{\phi(x) V_{p,\mu,\varepsilon}}{|\phi V_{p,\mu,\varepsilon}|_{p^*}}$, where

$V_{p,\mu,\varepsilon}$ was defined in (2.8). So, $|u_\varepsilon|_{p^*} = 1$. Thus, we can get the following results from [26, Lemma 2.2] (or [16]):

$$\|u_\varepsilon\|_\mu^p = S_\mu + O(\varepsilon^{pb(\mu)+p-N}), \tag{2.11}$$

$$\int_\Omega |u_\varepsilon|^\xi dx \approx \begin{cases} \varepsilon^{(b(\mu)-\frac{N-p}{p})\xi}, & \text{if } 1 \le \xi < \frac{N}{b(\mu)}, \\ \varepsilon^{N-\frac{N-p}{p}\xi}|\ln\varepsilon|, & \text{if } \xi = \frac{N}{b(\mu)}, \\ \varepsilon^{N-\frac{N-p}{p}\xi}, & \text{if } \frac{N}{b(\mu)} < \xi < p^*, \end{cases} \tag{2.12}$$

where $A \approx B$ means $C_1 B \le A \le C_2 B$.

As $I_{\lambda,\delta}$ is not bounded below on E, we need to study $I_{\lambda,\delta}$ on the Nehari manifold:

$$\mathcal{N}_{\lambda,\delta} = \big\{(u,v) \in E \setminus \{(0,0)\} : \langle I'_{\lambda,\delta}(u,v),(u,v)\rangle = 0\big\}.$$

Note that $\mathcal{N}_{\lambda,\delta}$ contains every nonzero solution of problem (1.1), and we define the minimax $c_{\lambda,\delta}$ as

$$c_{\lambda,\delta} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\delta}} I_{\lambda,\delta}(u,v).$$

Next, we present some properties of $c_{\lambda,\delta}$ and $\mathcal{N}_{\lambda,\delta}$. Their proofs can be done as [27, Theorem 4.2]. First of all, we note that there exists $\rho > 0$ such that

$$\|(u,v)\|_E \ge \rho > 0, \ \forall\ (u,v) \in \mathcal{N}_{\lambda,\delta}. \tag{2.13}$$

It is standard to check that $I_{\lambda,\delta}$ satisfies the mountain pass geometry, so we can use the homogeneity of F and G to prove that $c_{\lambda,\delta}$ can be alternatively characterized by

$$c_{\lambda,\delta} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\lambda,\delta}(\gamma(t)) = \inf_{(u,v)\in E\setminus\{(0,0)\}}\max_{t\ge 0} I_{\lambda,\delta}(t(u,v)) > 0, \tag{2.14}$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, I_{\lambda,\delta}(\gamma(1)) < 0\}$. Moreover, for each $(u,v) \in E\setminus\{(0,0)\}$, there exists a unique $t^* > 0$ such that $t^*(u,v) \in \mathcal{N}_{\lambda,\delta}$. The maximum of the function $t \mapsto I_{\lambda,\delta}(t(u,v))$, for $t \ge 0$, is achieved at $t = t^*$.

Lemma 2.2. *Suppose that $(F_0)-(F_3)$ and $(G_0)-(G_1)$ hold, $\bar p < q < p^*$ with $\bar p = \max\Big\{p, \frac{N}{b(\mu)}, \frac{p(2N-pb(\mu)-p)}{N-p}\Big\}$, $0 \le \mu < \bar\mu$ and λ,δ defined in (1.4), (1.5) are positive, then $c_{\lambda,\delta} < \frac{1}{N}S_F^{N/p}$. The same result holds if $q = p, 0 \le \mu \le \frac{N^{p-1}(N-p^2)}{p^p}$ and $\lambda,\delta \in (0,\frac{1}{p}\Lambda_1)$.*

Proof. We can use the homogeneity of F and G to get, for any $t \ge 0$,

$$I_{\lambda,\delta}(tAu_\varepsilon, tBu_\varepsilon) = \frac{t^p}{p}(A^p+B^p)\|u_\varepsilon\|_\mu^p - \frac{t^{p^*}}{p^*}F(A,B) - t^q G_{\lambda,\delta}(A,B)|u_\varepsilon|_q^q.$$

We shall denote the right-hand side of the above equality by $h(t)$ and consider two distinct cases.

Case 1: $\bar p < q < p^*$ with $\bar p = \max\Big\{p, \frac{N}{b(\mu)}, \frac{p(2N-pb(\mu)-p)}{N-p}\Big\}$. From the fact that $\lim_{t\to+\infty} h(t) = -\infty$ and $h(t) > 0$ when t is close to 0, there exists $t_\varepsilon > 0$ such that

$$h(t_\varepsilon) = \max_{t\ge 0} h(t). \tag{2.15}$$

Let

$$g(t) = \frac{t^p}{p}(A^p+B^p)\|u_\varepsilon\|_\mu^p - \frac{t^{p^*}}{p^*}F(A,B), \quad t \ge 0,$$

and notice that the maximum value of $g(t)$ occurs at the point

$$\tilde{t_\varepsilon} = \Big(\frac{(A^p+B^p)\|u_\varepsilon\|_\mu^p}{F(A,B)}\Big)^{\frac{1}{p^*-p}}.$$

So, for each $t \geq 0$,

$$g(t) \leq g(\tilde{t_\varepsilon}) = \frac{1}{N}\Big(\frac{(A^p+B^p)\|u_\varepsilon\|_\mu^p}{(F(A,B))^{p/p^*}}\Big)^{N/p},$$

and therefore

$$h(t_\varepsilon) \leq \frac{1}{N}\Big(\frac{(A^p+B^p)\|u_\varepsilon\|_\mu^p}{(F(A,B))^{p/p^*}}\Big)^{N/p} - t_\varepsilon^q G_{\lambda,\delta}(A,B)|u_\varepsilon|_q^q. \tag{2.16}$$

We claim that, for some $C_2 > 0$, there holds

$$t_\varepsilon^q G_{\lambda,\delta}(A,B) \geq C_2.$$

Indeed, if this is not the case, we have that $t_{\varepsilon_m} \to 0$ for some sequence $\varepsilon_m \to 0^+$, then

$$0 < c_{\lambda,\delta} \leq \sup_{t\geq 0} I_{\lambda,\delta}(tAu_{\varepsilon_m}, tBu_{\varepsilon_m}) = I_{\lambda,\delta}(t_{\varepsilon_m}Au_{\varepsilon_m}, t_{\varepsilon_m}Bu_{\varepsilon_m}) \to 0,$$

which is a contradiction. So, the claim holds, and we infer from (2.16), (2.10), (2.11) and (2.12) that

$$\begin{aligned} h(t_\varepsilon) &\leq \frac{1}{N}\Big(\frac{A^p+B^p}{(F(A,B))^{p/p^*}}\Big(S_\mu + O(\varepsilon^{pb(\mu)+p-N})\Big)\Big)^{N/p} - C_2|u_\varepsilon|_q^q \\ &\leq \frac{1}{N}S_F^{N/p} + O(\varepsilon^{pb(\mu)+p-N}) - C_2|u_\varepsilon|_q^q \\ &\leq \frac{1}{N}S_F^{N/p} + O(\varepsilon^{pb(\mu)+p-N}) - O(\varepsilon^{N-\frac{N-p}{p}q}). \end{aligned} \tag{2.17}$$

By $\bar{p} < q < p^*$, we obtain $pb(\mu)+p-N > N - \frac{N-p}{p}q$. Thus, from the above inequality we conclude that, for each $\varepsilon > 0$ small, there holds

$$c_{\lambda,\delta} \leq \sup_{t\geq 0} I_{\lambda,\delta}(tAu_\varepsilon, tBu_\varepsilon) = h(t_\varepsilon) < \frac{1}{N}S_F^{N/p}.$$

Case 2: $q = p$ and $0 \leq \mu \leq \frac{N^{p-1}(N-p^2)}{p^p}$. In this case, we have that $h'(t) = 0$ if and only if

$$(A^p+B^p)\|u_\varepsilon\|_\mu^p - pG_{\lambda,\delta}(A,B)|u_\varepsilon|_p^p = t^{p^*-p}F(A,B).$$

Since we suppose $\lambda < \frac{1}{p}\Lambda_1$, we can use Poincaré inequality to obtain

$$\begin{aligned} pG_{\lambda,\delta}(A,B)|u_\varepsilon|_p^p &\leq p\lambda(A^p+B^p)|u_\varepsilon|_p^p \\ &< \Lambda_1(A^p+B^p)|u_\varepsilon|_p^p \\ &\leq (A^p+B^p)\|u_\varepsilon\|_\mu^p. \end{aligned}$$

Thus, there exists $t_\varepsilon > 0$ satisfying (2.15).

Arguing, as in the first case, we conclude that, from (2.17), for $\varepsilon > 0$ small, there holds

$$\begin{aligned} h(t_\varepsilon) &\leq \frac{1}{N}S_F^{N/p} + O(\varepsilon^{pb(\mu)+p-N}) - C_2|u_\varepsilon|_p^p \\ &= \begin{cases} \frac{1}{N}S_F^{N/p} + O(\varepsilon^{pb(\mu)+p-N}) - O(\varepsilon^p|\ln\varepsilon|), & b(\mu) = \frac{N}{p}, \\ \frac{1}{N}S_F^{N/p} + O(\varepsilon^{pb(\mu)+p-N}) - O(\varepsilon^p), & b(\mu) > \frac{N}{p}. \end{cases} \end{aligned}$$

If $b(\mu) = N/p$, then $pb(\mu) + p - N = p$, so $\varepsilon^{pb(\mu)+p-N} = o(\varepsilon^p |\ln \varepsilon|)$. If $b(\mu) > N/p$, then $pb(\mu) + p - N > p$, so $\varepsilon^{pb(\mu)+p-N} = o(\varepsilon^p)$. Choosing $\varepsilon > 0$ small enough, we have

$$c_{\lambda,\delta} \le \sup_{t\ge 0} I_{\lambda,\delta}(tAu_\varepsilon, tBu_\varepsilon) = h(t_\varepsilon) < \frac{1}{N} S_F^{N/p}.$$

On the other hand, it is easy to verify that the function

$$g(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \ge 0$$

has the only minimal point $\bar{t} = \frac{N-p}{p}$ and is increasing on the interval $(\bar{t}, +\infty)$. Thus, for $N \ge p^2$ we deduce that

$$\frac{N}{p} \le b(\mu) \Leftrightarrow g(\frac{N}{p}) \le g(b(\mu)) = 0 \Leftrightarrow 0 \le \mu \le \frac{N^{p-1}(N-p^2)}{p^p}.$$

This concludes the proof. □

Using Lemmas 2.1 and 2.2, we can prove our first result.

Proof of Theorem 1.1. Since $I_{\lambda,\delta}$ satisfies the geometric conditions of the mountain pass theorem, there exists $\{(u_m, v_m)\} \subset E$ such that $I_{\lambda,\delta}(u_m, v_m) \to c_{\lambda,\delta}$, and $I'_{\lambda,\delta}(u_m, v_m) \to 0$. It follows from Lemmas 2.1 and 2.2 that $\{(u_m, v_m)\}$ converges, along a subsequence, to a nonzero critical point $(u, v) \in E$ of $I_{\lambda,\delta}$. If we then denote, by $u^- = \max\{-u, 0\}$ and $v^- = \max\{-v, 0\}$, the negative part of u and v, respectively, we obtain

$$\begin{aligned}
0 &= \langle I'_{\lambda,\delta}(u,v), (u^-, v^-)\rangle \\
&= -\|(u^-, v^-)\|_E^p - \frac{1}{p^*}\int_\Omega (F_u(u,v)u^- + F_v(u,v)v^-)dx \\
&\quad - \int_\Omega (G_u(u,v)u^- + G_v(u,v)v^-)dx \\
&\le -\|(u^-, v^-)\|_E^p.
\end{aligned}$$

It thus follows that $(u^-, v^-) = (0, 0)$. Hence, $u, v \ge 0$ in Ω. The theorem 1.1 is thus proven. □

We finalize this section with the study of the asymptotic behavior of the minimax level $c_{\lambda,\delta}$ as both the parameters λ, δ approach zero.

Lemma 2.3. $\lim_{\lambda,\delta\to 0^+} c_{\lambda,\delta} = c_{0,0} = \frac{1}{N} S_F^{N/p}$.

Proof. We first prove the second equality. It follows from $\lambda = \delta = 0$ that $G_{0,0} \equiv 0$. If $A, B, u_\varepsilon, g_\varepsilon$ and t_ε are the same as those in the proof of Lemma 2.2, we have that $(t_\varepsilon A u_\varepsilon, t_\varepsilon B u_\varepsilon) \in \mathcal{N}_{0,0}$. Thus

$$\begin{aligned}
c_{0,0} &\le I_{0,0}(t_\varepsilon A u_\varepsilon, t_\varepsilon B u_\varepsilon) \\
&= \frac{1}{N}\Big(\frac{A^p + B^p}{(F(A,B))^{p/p^*}} \|u_\varepsilon\|_\mu^p\Big)^{N/p} \\
&= \frac{1}{N}\Big(\frac{A^p + B^p}{(F(A,B))^{p/p^*}} \Big(S_\mu + O\big(\varepsilon^{pb(\mu)+p-N}\big)\Big)\Big)^{N/p}.
\end{aligned}$$

Taking the limit as $\varepsilon \to 0^+$ and using (2.10), we conclude that $c_{0,0} \le \frac{1}{N} S_F^{N/p}$.

In order to obtain the reverse inequality, we consider $\{(u_m, v_m)\} \subset E$ such that $I_{0,0}(u_m, v_m) \to c_{0,0}$ and $I'_{0,0}(u_m, v_m) \to 0$. It is easy to show that the sequence $\{(u_m, v_m)\}$ is bounded in E and therefore

$$\langle I'_{0,0}(u_m, v_m), (u_m, v_m)\rangle = \|(u_m, v_m)\|_E^p - \int_\Omega F(u_m, v_m)dx = o_m(1).$$

It follows that

$$\lim_{m\to\infty} \|(u_m, v_m)\|_E^p = l = \lim_{m\to\infty} \int_\Omega F(u_m, v_m)dx.$$

Taking the limit in the inequality $S_F(\int_\Omega F(u_m, v_m)dx)^{p/p^*} \leq \|(u_m, v_m)\|_E^p$, we conclude, as in the proof of Lemma 2.1, that $Nc_{0,0} = l \geq S_F^{N/p}$. Hence,

$$\begin{aligned} c_{0,0} = \lim_{m\to\infty} I_{0,0}(u_m, v_m) &= \lim_{m\to\infty} \Big(\frac{1}{p}\|(u_m, v_m)\|_E^p - \frac{1}{p^*}\int_\Omega F(u_m, v_m)dx\Big) \\ &= \frac{1}{N} l \geq \frac{1}{N} S_F^{N/p}, \end{aligned}$$

and therefore $c_{0,0} = \frac{1}{N} S_F^{N/p}$.

We proceed now to the calculation of $\lim_{\lambda,\delta\to 0^+} c_{\lambda,\delta}$. Let $\{\lambda_m\}, \{\delta_m\} \subset \mathbb{R}^+$ such that $\lambda_m, \delta_m \to 0^+$. Since δ_m, defined in (1.5), is positive, we have that $G_{\lambda_m,\delta_m}(u, v) \geq 0$ whenever (u, v) is nonnegative. Thus, for this kind of function, we have that $I_{\lambda_m,\delta_m}(u, v) \leq I_{0,0}(u, v)$. It follows that

$$\begin{aligned} c_{\lambda_m,\delta_m} &= \inf_{(u,v)\neq(0,0)} \max_{t\geq 0} I_{\lambda_m,\delta_m}(t(u, v)) \\ &\leq \inf_{(u,v)\neq(0,0),\,(u,v)\geq 0} \max_{t\geq 0} I_{\lambda_m,\delta_m}(t(u, v)) \\ &\leq \inf_{(u,v)\neq(0,0),\,(u,v)\geq 0} \max_{t\geq 0} I_{0,0}(t(u, v)) = c_{0,0}. \end{aligned}$$

In the last equality above, we used the infimum $c_{0,0}$, which can be attained at a nonnegative solution. The above inequality implies that

$$\limsup_{m\to\infty} c_{\lambda_m,\delta_m} \leq c_{0,0}. \tag{2.18}$$

On the other hand, it follows from Theorem 1.1 that there exists $\{(u_m, v_m)\} \subset E$ such that

$$I_{\lambda_m,\delta_m}(u_m, v_m) = c_{\lambda_m,\delta_m}, \quad I'_{\lambda_m,\delta_m}(u_m, v_m) \to 0.$$

Since c_{λ_m,δ_m} is bounded, the same argument performed in the proof of Lemma 2.1 implies that $\{(u_m, v_m)\}$ is bounded in E. Since $(u_m, v_m) \geq 0$, we obtain $0 \leq \int_\Omega G_{\lambda_m,\delta_m}(u_m, v_m)dx \leq \lambda_m \int_\Omega(|u_m|^q + |v_m|^q)dx$, from which it follows that

$$\lim_{m\to\infty} \int_\Omega G_{\lambda_m,\delta_m}(u_m, v_m)dx = 0. \tag{2.19}$$

Let $t_m > 0$ be such that $t_m(u_m, v_m) \in \mathcal{N}_{0,0}$. Since $(u_m, v_m) \in \mathcal{N}_{\lambda_m,\delta_m}$, we have that

$$\begin{aligned} c_{0,0} &\leq I_{0,0}(t_m(u_m, v_m)) \\ &= I_{\lambda_m,\delta_m}(t_m(u_m, v_m)) + t_m^q \int_\Omega G_{\lambda_m,\delta_m}(u_m, v_m)dx \\ &\leq I_{\lambda_m,\delta_m}(u_m, v_m) + t_m^q \int_\Omega G_{\lambda_m,\delta_m}(u_m, v_m)dx \end{aligned}$$

$$= c_{\lambda_m,\delta_m} + t_m^q \int_\Omega G_{\lambda_m,\delta_m}(u_m, v_m)dx.$$

If $\{t_m\}$ is bounded, we can use the above estimate and (2.19) to obtain

$$c_{0,0} \le \liminf_{m\to\infty} c_{\lambda_m,\delta_m}.$$

Using this and (2.18), we obtain

$$c_{0,0} \le \liminf_{m\to\infty} c_{\lambda_m,\delta_m} \le \limsup_{m\to\infty} c_{\lambda_m,\delta_m} \le c_{0,0}.$$

Thus, $c_{0,0} = \lim_{m\to\infty} c_{\lambda_m,\delta_m}$.

It remains to check that $\{t_m\}$ is bounded. A straightforward calculation shows that

$$t_m = \Big(\frac{\|(u_m, v_m)\|_E^p}{\int_\Omega F(u_m, v_m)dx}\Big)^{\frac{1}{p^*-p}}. \tag{2.20}$$

Since $(u_m, v_m) \in \mathcal{N}_{\lambda_m,\delta_m}$, we obtain

$$\begin{aligned}&\|(u_m, v_m)\|_E^p\\ &= \int_\Omega F(u_m, v_m)dx + q\int_\Omega G_{\lambda_m,\delta_m}(u_m, v_m)dx \le S_F^{-\frac{p^*}{p}} \|(u_m, v_m)\|_E^{p^*} + o_m(1).\end{aligned}$$

Hence $\|(u_m, v_m)\|_E^p \ge C_3 > 0$, and therefore from the above expression, it follows that $\int_\Omega F(u_m, v_m)dx \ge C_4 > 0$. Thus, the boundedness of $\{(u_m, v_m)\}$ and (2.20) imply that $\{t_m\}$ is bounded. This completes the proof. □

3. Some technical lemmas

In this section, we will recall and prove some lemmas which are crucial in the proof of Theorem 1.2. The first lemma is standard, and its proof follows adapting arguments found in [27].

Lemma 3.1. *Let $\{(u_m, v_m)\} \subset E$ such that $\int_\Omega F(u_m, v_m)dx = 1$ and*

$$\lim_{m\to\infty} \|(u_m, v_m)\|_E^p = S_F.$$

Then there exist $\{r_m\} \subset (0, +\infty)$ and $\{y_m\} \subset \mathbb{R}^N$ such that

$$\omega_m(x) = (\omega_m^1(x), \omega_m^2(x)) = r_m^{\frac{N-p}{p}} (u_m(r_m x + y_m), v_m(r_m x + y_m)) \tag{3.1}$$

contains a convergent subsequence, denoted again by $\{\omega_m\}$, such that $\omega_m \to \omega$ in $\mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,p}(\mathbb{R}^N)$. Moreover, as $m \to \infty$, we have $r_m \to 0$ and $y_m \to y \in \overline{\Omega}$.

Up to translations, we may assume that $0 \in \Omega$. Since Ω is a smooth bounded domain of $\mathbb{R}^N$, we can choose $r > 0$ small enough such that $B_r = B_r(0) = \{x \in \mathbb{R}^N : d(x, 0) < r\} \subset \Omega$ and the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < r\}, \quad \Omega_r^- = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \partial\Omega) > r\}$$

are homotopically equivalent to Ω. Let

$$\begin{gathered}W_{0,rad}^{1,p}(B_r) = \big\{u \in W_0^{1,p}(B_r) : u \text{ is radial}\big\},\\ E_{\rm rad}(B_r) = W_{0,\rm rad}^{1,p}(B_r) \times W_{0,\rm rad}^{1,p}(B_r).\end{gathered}$$

We thus define the functional

$$I_{B_r}(u, v) = \frac{1}{p}\int_{B_r} (|\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu\frac{|v|^p}{|x|^p})dx$$

$$- \frac{1}{p^*}\int_{B_r} F(u,v)dx - \int_{B_r} G_{\lambda,\delta}(u,v)dx$$

for $(u,v)\in E_{\rm rad}(B_r)$, and set

$$m_{\lambda,\delta} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\delta}^{B_r}} I_{B_r}(u,v),$$

where

$$\mathcal{N}_{\lambda,\delta}^{B_r} := \{(u,v)\in E_{\rm rad}(B_r)\setminus\{(0,0)\} : \langle I'_{B_r}(u,v),(u,v)\rangle = 0\}.$$

Clearly, $m_{\lambda,\delta}$ is nonincreasing in λ,δ. Note that $m_{\lambda,\delta}>0$ for all $\lambda,\delta>0$.

Arguing, as in the proof of Lemma 2.3 and Theorem 1.1, we obtain the following result.

Lemma 3.2. *Suppose* (F0)-(F3), (G0)–(G1) *are satisfied. Then the infimum* $m_{\lambda,\delta}$ *is attained by a positive radial function* $(u_{\lambda,\delta},v_{\lambda,\delta})\in E_{\rm rad}$ *whenever* $\bar p<q<p^*$ *with* $\bar p=\max\left\{p,\frac{N}{b(\mu)},\frac{p(2N-pb(\mu)-p)}{N-p}\right\}$, $0\le\mu<\bar\mu$ *and* $\lambda,\delta>0$, *or* $q=p, 0\le\mu\le\frac{N^{p-1}(N-p^2)}{p^p}$ *and* $\lambda,\delta\in(0,\frac{1}{p}\Lambda_{1,rad})$, *and where* $\Lambda_{1,rad}>0$ *is the first eigenvalue of the operator* $(-\Delta_p u, W^{1,p}_{0,rad}(B_r))$. *Moreover,*

$$m_{\lambda,\delta} < \frac{1}{N}S_F^{N/p}, \quad \lim_{\lambda,\delta\to 0^+} m_{\lambda,\delta} = \frac{1}{N}S_F^{N/p}.$$

We introduce the barycenter map $\beta:\mathcal{N}_{\lambda,\delta}\to\mathbb{R}^N$ as

$$\beta(u,v) = S_F^{-N/p}\int_\Omega F(u,v)x\,dx.$$

This map has the following property.

Lemma 3.3. *If* (F0)–(F3), (G0)–(G1), *then there exists* $\lambda^*>0$ *such that* $\beta(u,v)\in\Omega_r^+$ *whenever* $(u,v)\in\mathcal{N}_{\lambda,\delta},\lambda,\delta\in(0,\lambda^*)$ *and* $I_{\lambda,\delta}(u,v)\le m_{\lambda,\delta}$.

Proof. Arguing by contradiction, we suppose that there exist $\{\lambda_m\},\{\delta_m\}\subset\mathbb{R}^+$ and $\{(u_m,v_m)\}\subset\mathcal{N}_{\lambda_m,\delta_m}$ such that $\lambda_m,\delta_m\to 0^+$ as $m\to\infty, I_{\lambda_m,\delta_m}(u_m,v_m)\le m_{\lambda_m,\delta_m}$, but $\beta(u_m,v_m)\notin\Omega_r^+$.

From $\{(u_m,v_m)\}\subset\mathcal{N}_{\lambda_m,\delta_m}$ and $I_{\lambda_m,\delta_m}(u_m,v_m)\le m_{\lambda_m,\delta_m}$, it follows that $\{(u_m,v_m)\}$ is bounded in E. Moreover,

$$\begin{aligned}
0 &= \langle I'_{\lambda_m,\delta_m}(u_m,v_m),(u_m,v_m)\rangle\\
&= \|(u_m,v_m)\|_E^p - \int_\Omega F(u_m,v_m)dx - q\int_\Omega G_{\lambda_m,\delta_m}(u_m,v_m)dx.
\end{aligned}$$

Since $\lambda_m\to 0$, we can use the boundedness of $\{(u_m,v_m)\}$ to get

$$0\le\int_\Omega G_{\lambda_m,\delta_m}(u_m,v_m)dx \le \lambda_m\int_\Omega(|u_m|^q+|v_m|^q)dx\to 0,$$

from which it follows that

$$\lim_{m\to\infty}\|(u_m,v_m)\|_E^p = \lim_{m\to\infty}\int_\Omega F(u_m,v_m)dx = k\ge 0.$$

Notice that

$$\begin{aligned}
c_{\lambda_m,\delta_m} &\le I_{\lambda_m,\delta_m}(u_m,v_m)\\
&= \frac{1}{p}\|(u_m,v_m)\|_E^p - \frac{1}{p^*}\int_\Omega F(u_m,v_m)dx - \int_\Omega G_{\lambda_m,\delta_m}(u_m,v_m)dx
\end{aligned}$$

$$\le m_{\lambda_m,\delta_m}.$$

Recalling that c_{λ_m,δ_m} and m_{λ_m,δ_m} both converge to $\frac{1}{N}S_F^{N/p}$, we can use the above expression and $\int_\Omega G_{\lambda_m,\delta_m}(u_m,v_m)dx \to 0$ again to conclude that $k = S_F^{N/p}$. That is,

$$\lim_{m\to\infty} \|(u_m,v_m)\|_E^p = S_F^{N/p} = \lim_{m\to\infty} \int_\Omega F(u_m,v_m)dx. \tag{3.2}$$

Let $t_m = (\int_\Omega F(u_m,v_m)dx)^{-1/p^*} > 0$ and notice that $t_m(u_m,v_m)$ satisfies the hypotheses of Lemma 3.1. Using Lemma 3.1, there exist sequences $\{r_m\} \subset (0,+\infty)$ and $\{y_m\} \subset \mathbb{R}^N$ satisfying $r_m \to 0, y_m \to y \in \overline{\Omega}$. We thus have that $\omega_m \to \omega$ in $\mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,p}(\mathbb{R}^N)$.

The definition of $\beta(u,v)$, (3.2), the strong convergence of $\{\omega_m\}$, and Lebesgue's Theorem provide

$$\begin{aligned}
\beta(u_m,v_m) &= t_m^{-p^*} S_F^{-N/p} \int_\Omega F(t_m(u_m,v_m))x dx \\
&= (1+o_m(1)) \int_\Omega F(t_m u_m, t_m v_m) x dx \\
&= (1+o_m(1)) \int_\Omega F(\omega_m)(r_m x + y_m) dx \\
&= (1+o_m(1)) \Big(\int_\Omega F(\omega)\bar{y} dx + o_m(1) \Big).
\end{aligned}$$

Since $\bar{y} \in \overline{\Omega}$ and $\int_\Omega F(\omega)dx = 1$, the above expression implies that

$$\lim_{m\to\infty} \text{dist } (\beta(u_m,v_m), \overline{\Omega}) = 0.$$

Such contradicts $\beta(u_m,v_m) \notin \Omega_r^+$. $\square$

According to Lemma 3.2, for each $\lambda,\delta > 0$ small, the infimum $m_{\lambda,\delta}$ is attained by a nonnegative radial function $\sigma_{\lambda,\delta} = (u_{\lambda,\delta}, v_{\lambda,\delta}) \in \mathcal{N}_{\lambda,\delta}^{B_r}$. We consider

$$I_{\lambda,\delta}^{m_{\lambda,\delta}} = \{(u,v) \in E : I(u,v) \le m_{\lambda,\delta}\}$$

and define the function $\gamma : \Omega_r^- \to I_{\lambda,\delta}^{m_{\lambda,\delta}}$ by setting, for each $y \in \Omega_r^-$,

$$\gamma(y) = \begin{cases} \sigma_{\lambda,\delta}(x-y), & \text{if } x \in B_r(y), \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

A change of variables and straightforward calculations show that the map γ is well defined. Since $\sigma_{\lambda,\delta}$ is radial, we have that $\int_{B_r} F(u_{\lambda,\delta}, v_{\lambda,\delta}) x dx = 0$. Hence, for each $y \in \Omega_r^-$, we obtain

$$\begin{aligned}
(\beta\circ\gamma)(y) &= S_F^{-N/p} \int_\Omega F(u_{\lambda,\delta}(x-y), v_{\lambda,\delta}(x-y)) x dx \\
&= S_F^{-N/p} \int_\Omega F(u_{\lambda,\delta}(t), v_{\lambda,\delta}(t))(t+y) dt \\
&= S_F^{-N/p} \int_\Omega F(u_{\lambda,\delta}(t), v_{\lambda,\delta}(t)) y dt = y\alpha_{\lambda,\delta},
\end{aligned}$$

where $\alpha_{\lambda,\delta} = S_F^{-N/p} \int_\Omega F(u_{\lambda,\delta}(t), v_{\lambda,\delta}(t)) dt$.

Along the way of proving Lemma 3.3, we can check easily the following.

Lemma 3.4. *If $\lambda, \delta \to 0^+$, then $\alpha_{\lambda,\delta} \to 1$.*

Proof. By Lemma 3.2, we have

$$\begin{aligned} m_{\lambda,\delta} &= \frac{1}{p}\int_{B_r}\Big(|\nabla u_{\lambda,\delta}|^p + |\nabla v_{\lambda,\delta}|^p - \mu\frac{|u_{\lambda,\delta}|^p + |v_{\lambda,\delta}|^p}{|x|^p}\Big)dx \\ &\quad - \frac{1}{p^*}\int_{B_r} F(u_{\lambda,\delta}, v_{\lambda,\delta})dx - \int_{B_r} G_{\lambda,\delta}(u_{\lambda,\delta}, v_{\lambda,\delta})dx \\ &< \frac{1}{N}S_F^{N/p}. \end{aligned}$$

As before, $\int_{B_r} G_{\lambda,\delta}(u_{\lambda,\delta}, v_{\lambda,\delta})dx \to 0$. Thus, $I'_{B_r}(u_{\lambda,\delta}, v_{\lambda,\delta}) = 0$, and the above expression and the same arguments used in the proof of Lemma 3.2 imply that

$$\int_\Omega F(u_{\lambda,\delta}, v_{\lambda,\delta})dx \to S_F^{N/p}.$$

The above equality and the definition of $\alpha_{\lambda,\delta}$ imply that $\alpha_{\lambda,\delta} \to 1$. The lemma is thus proven. □

Next we define $H_{\lambda,\delta} : [0,1] \times (\mathcal{N}_{\lambda,\delta} \cap I_{\lambda,\delta}^{m_{\lambda,\delta}}) \to \mathbb{R}^N$ by

$$H_{\lambda,\delta}(t,(u,v)) = \Big(t + \frac{1-t}{\alpha_{\lambda,\delta}}\Big)\beta(u,v).$$

Lemma 3.5. *Suppose* (F0)–(F3), (G0)–(G1) *are satisfied. There then exists $\lambda^{**} > 0$ such that*

$$H_{\lambda,\delta}\big([0,1] \times (\mathcal{N}_{\lambda,\delta} \cap I_{\lambda,\delta}^{m_{\lambda,\delta}})\big) \subset \Omega_r^+ \tag{3.4}$$

for all $\lambda, \delta \in (0, \lambda^{**})$.

Proof. Arguing by contradiction, we suppose that there exist sequences $\{\lambda_m\}$, $\{\delta_m\} \subset \mathbb{R}^+$ and $t_m \in [0,1], (u_m, v_m) \in (\mathcal{N}_{\lambda,\delta} \cap I_{\lambda,\delta}^{m_{\lambda,\delta}})$ such that $\lambda_m, \delta_m \to 0^+$ as $m \to \infty$ and $H_{\lambda_m,\delta_m}(t_m,(u_m,v_m)) \notin \Omega_r^+$ for all m, up to a subsequence $t_m \to t_0 \in [0,1]$. Moreover, the compactness of $\overline{\Omega}$ and Lemma 3.3 imply that, up to a subsequence, $\beta(u_m,v_m) \to y \in \overline{\Omega}$. From Lemma 3.4 $\alpha_{\lambda_m,\delta_m} \to 1$, so we can use the definition of $H_{\lambda,\delta}$ to conclude that $H_{\lambda_m,\delta_m}(t_m,(u_m,v_m)) \to y \in \overline{\Omega}$, which is a contradiction. The lemma is proven. □

4. Proof of Theorem 1.2

We begin with the following lemma.

Lemma 4.1. *If (u,v) is a critical point of $I_{\lambda,\delta}$ on $\mathcal{N}_{\lambda,\delta}$, then it is a critical point of $I_{\lambda,\delta}$ in E.*

Proof. The proof is almost the same as [22, Lemma 3.2] and is thus omitted here. □

Lemma 4.2. *Suppose (F0)–(F3), (G0)–(G1) are satisfied. Then any sequence $\{(u_m,v_m)\} \subset \mathcal{N}_{\lambda,\delta}$ such that $I_{\lambda,\delta}(u_m,v_m) \to c < \frac{1}{N}S_F^{N/p}$ and $I'_{\lambda,\delta}(u_m,v_m) \to 0$ contains a convergent subsequence for $\lambda, \delta > 0$ if $q > p$ and $\lambda, \delta \in (0, \lambda^*)$ if $q = p$ for some small $\lambda^* > 0$.*

Proof. By hypothesis, there exists a sequence $\theta_m \in \mathbb{R}$ such that $\|I'_{\lambda,\delta}(u_m,v_m) - \theta_m J'_{\lambda,\delta}(u_m,v_m)\|_E \to 0$ as $m \to \infty$, where

$$J_{\lambda,\delta}(u,v) = \int_\Omega (|\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu\frac{|v|^p}{|x|^p})dx - \int_\Omega F(u,v)dx - q\int_\Omega G_{\lambda,\delta}(u,v)dx.$$

Thus,

$$I'_{\lambda,\delta}(u_m,v_m) = \theta_m J'_{\lambda,\delta}(u_m,v_m) + o_m(1).$$

Recall that for all $(u_m,v_m) \in \mathcal{N}_{\lambda,\delta}$,

$$\langle J'_{\lambda,\delta}(u_m,v_m),(u_m,v_m)\rangle = (p-p^*)\int_\Omega F(u_m,v_m)dx + (p-q)\int_\Omega G_{\lambda,\delta}(u_m,v_m)dx \le 0.$$

If $\langle J'_{\lambda,\delta}(u_m,v_m),(u_m,v_m)\rangle \to 0$, we have

$$\int_\Omega F(u_m,v_m)dx \to 0, \quad \int_\Omega G_{\lambda,\delta}(u_m,v_m)dx \to 0.$$

Consequently, $\|(u_m,v_m)\|_E \to 0$. On the other hand, if $(u_m,v_m) \in \mathcal{N}_{\lambda,\delta}$, it follows that

$$1 \le C(\lambda\|(u_m,v_m)\|_E^{q-p} + \delta\|(u_m,v_m)\|_E^{q-p} + \|(u_m,v_m)\|_E^{p^*-p})$$

for some $C > 0$. Hence we arrive at a contradiction if $\lambda,\delta > 0$ and $q > p$ or $\lambda,\delta \in (0,\lambda^*)$ for small $\lambda^* > 0$ when $q = p$. We may thus assume that $\langle J'_{\lambda,\delta}(u_m,v_m),(u_m,v_m)\rangle \to \ell < 0$. Since $\langle I'_{\lambda,\delta}(u_m,v_m),(u_m,v_m)\rangle = 0$, we conclude that $\theta_m = 0$ and, consequently, $I'_{\lambda,\delta}(u_m,v_m) \to 0$. Using this fact, we have

$$I'_{\lambda,\delta}(u_m,v_m) \to c < \frac{1}{N}S_F^{N/p} \quad \text{and} \quad I'_{\lambda,\delta}(u_m,v_m) \to 0.$$

By Lemma 2.1 the proof is completed. □

Hereafter, we denote the restriction of $I_{\lambda,\delta}$ on $\mathcal{N}_{\lambda,\delta}$ by $I_{\mathcal{N}_{\lambda,\delta}}$.

Lemma 4.3. *If* (F0)–(F3), (G0)–(G1) *are satisfied. Let* $\Lambda = \min\{\lambda^*,\lambda^{**}\} > 0$, $\lambda,\delta \in (0,\Lambda)$. *Then* $\operatorname{cat}_{I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}}}(I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}}) \ge \operatorname{cat}_\Omega(\Omega)$, *where* λ^*,λ^{**} *are given by Lemma 3.3 and 3.5, respectively.*

Proof. Suppose that $I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}} = A_1 \cup A_2 \cup \cdots \cup A_m$, where A_j, $j = 1,2,\cdots,m$, are closed and contractible sets in $I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}}$, this means that there exists $h_j \in C([0,1]\times A_j, I_{\mathcal{N}_{\lambda,\delta}}^{m_{\lambda,\delta}})$ such that

$$h_j(0,z) = z, \quad h_j(1,z) = \vartheta, \quad \text{for all } z \in A_j,$$

where $\vartheta \in A_j$ is fixed. Consider $B_j = \gamma^{-1}(A_j)$, $1 \le j \le m$. The sets B_j are closed and

$$\Omega_r^- = B_1 \cup B_2 \cup \cdots \cup B_m.$$

We define the deformation $g_j : [0,1] \times B_j$ by setting

$$g_j(t,y) = H_{\lambda,\delta}(t, h_j(t,\gamma(y)))$$

for $\lambda,\delta \in (0,\Lambda)$. Note that

$$g_j(0,y) = H_{\lambda,\delta}(0, h_j(0,\gamma(y))) = \frac{(\beta\circ\gamma)(y)}{\alpha_{\lambda,\delta}}$$

implies

$$g_j(0,y) = \frac{\alpha_{\lambda,\delta} y}{\alpha_{\lambda,\delta}} = y, \quad \text{for all } y \in B_j,$$

and $g_j(1,y) = H_{\lambda,\delta}(1, h_j(1,\gamma(y))) = \beta(h_j(1,\gamma(y)))$ implies

$$g_j(1,y) = \beta(\vartheta) \in \Omega_r^+.$$

Thus B_j are contractible in Ω_r^+. Hence $\mathrm{cat}_\Omega(\Omega) = \mathrm{cat}_{\Omega_r^+}(\Omega_r^+) \le m$. □

Proof of Theorem 1.2. Using Lemma 2.1, Lemma 2.2, and Lemma 3.2 we know that $c_{\lambda,\delta}, m_{\lambda,\delta} < \frac{1}{N} S_F^{N/p}$ for $\lambda, \delta \in (0,\Lambda)$. Moreover, by Lemma 4.2, $I_{\mathcal{N}_{\lambda,\delta}}$ satisfies the $(PS)_c$ condition for all $c < \frac{1}{N} S_F^{N/p}$. Therefore, by Lemma 4.3, a standard deformation argument implies that for $\lambda, \delta \in (0,\Lambda)$, $I_{\mathcal{N}_{\lambda,\delta}}$ contains at least $\mathrm{cat}_\Omega(\Omega)$ critical points of the restriction of $I_{\lambda,\delta}$ on $\mathcal{N}_{\lambda,\delta}$. Now, Lemma 4.1 implies that $I_{\lambda,\delta}$ has at least $\mathrm{cat}_\Omega(\Omega)$ critical points, and therefore at least $\mathrm{cat}_\Omega(\Omega)$ nontrivial solutions of (1.1). As Theorem 1.1, the obtained solutions are nonnegative in Ω. The proof is completed. □

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