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POSITIVE SOLUTIONS FOR A $2n$TH-ORDER p-LAPLACIAN BOUNDARY VALUE PROBLEM INVOLVING ALL DERIVATIVES

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Abstract. In this work, we are mainly concerned with the positive solutions for the $2n$th-order p-Laplacian boundary-value problem

$$-(((-1)^{n-1}x^{(2n-1)})^{p-1})' = f(t, x, x', \dots, (-1)^{n-1}x^{(2n-2)}, (-1)^{n-1}x^{(2n-1)}),$$
$$x^{(2i)}(0) = x^{(2i+1)}(1) = 0, \quad (i = 0, 1, \dots, n-1),$$

where $n \geq 1$ and $f \in C([0,1] \times \mathbb{R}_+^{2n}, \mathbb{R}_+)(\mathbb{R}_+ := [0,\infty))$. To overcome the difficulty resulting from all derivatives, we first convert the above problem into a boundary value problem for an associated second order integro-ordinary differential equation with p-Laplacian operator. Then, by virtue of the classic fixed point index theory, combined with a priori estimates of positive solutions, we establish some results on the existence and multiplicity of positive solutions for the above problem. Furthermore, our nonlinear term f is allowed to grow superlinearly and sublinearly.

1. Introduction

In this paper, we investigate the existence and multiplicity of positive solutions for the following $2n$th-order p-Laplacian boundary value problem involving all derivatives

$$\begin{gathered} -(((-1)^{n-1}x^{(2n-1)})^{p-1})' = f(t, x, x', \dots, (-1)^{n-1}x^{(2n-2)}, (-1)^{n-1}x^{(2n-1)}), \\ x^{(2i)}(0) = x^{(2i+1)}(1) = 0, \quad (i = 0, 1, \dots, n-1), \end{gathered} \tag{1.1}$$

where $f \in C([0,1] \times \mathbb{R}_+^{2n}, \mathbb{R}_+)$. Here, by a positive solution (1.1) we mean a function $u \in C^{2n}[0,1]$ that solves (1.1) and satisfies $u(t) > 0$ for all $t \in (0,1]$.

We are here interested in the case where f depends explicitly on all derivatives. When f involves all even derivatives explicitly, many researchers [1, 2, 4, 12, 15] study the Lidstone boundary value problem

$$\begin{gathered} (-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1}u^{(2n-2)}), \quad n \geq 2, \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, 2, \dots, n-1. \end{gathered} \tag{1.2}$$

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Yang [19] considered the existence and uniqueness of positive solutions for the following generalized Lidstone boundary value problem

$$\begin{gathered}(-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1} u^{(2n-2)}),\\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = 0, \alpha_1 u^{(2i)}(1) - \beta_1 u^{(2i+1)}(1) = 0,\ i = 0, 1, 2, \dots, n-1,\end{gathered} \tag{1.3}$$

where $\alpha_j \geq 0$, $\beta_j \geq 0$ $(j = 0, 1)$ and $\alpha_0\alpha_1 + \alpha_0\beta_1 + \alpha_1\beta_0 > 0$. In view of the symmetry, the results in [1, 2, 4, 12, 15, 19] demonstrate that problems (1.2) and (1.3) are essentially identical with second-order Dirichlet problem and Sturm-Liouville problem (the case $n = 1$), respectively.

Yang, O'Regan and Agarwal [20] studied the existence and multiplicity of positive solutions for the second-order boundary value problem depending on the first-order derivative u'

$$\begin{gathered}u'' + f(t, u, u') = 0,\\ u(0) = u'(1) = 0.\end{gathered} \tag{1.4}$$

In order to overcome the difficulty resulting from the first-order derivative, they imposed the Bernstein-Nagumo condition [3, 13] on the nonlinear term f to establish several existence theorems for (1.4).

Yang and O'Regan [21] studied the existence, multiplicity and uniqueness of positive solutions for the $2n$th-order boundary value problem involving all derivatives of odd orders

$$\begin{gathered}(-1)^n u^{(2n)} = f(t, u, u', -u''', \dots, (-1)^{n-1} u^{(2n-1)}),\\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0, \quad i = 0, 1, 2, \dots, n-1,\end{gathered} \tag{1.5}$$

where $n \geq 2$ and $f \in C([0, 1] \times \mathbb{R}_+^{n+1}, \mathbb{R}_+)$ depends on u and all derivatives of odd orders. As application, they utilized their results to discuss the positive symmetric solutions for a Lidostone problem involving an open question posed by Eloe [5]. Yang [22] discussed a $2n$th-order ordinary differential equation involving all derivatives, and the results improved and extended the corresponding ones in [19, 20, 21].

Equations of the p-Laplacian form occur in the study of non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium. Since 1980s, there exist a very large number of papers devoted to the existence of solutions for differential equations with p-Laplacian, for instance, see [6, 7, 10, 11, 16, 17, 18, 23, 24, 25] and the references therein.

Yang and his coauthors [17, 23, 24] studied some boundary value problems with the p-Laplacian operator. Yang and O'Regan [23] studied the existence and multiplicity of positive solutions for the focal problem involving both the p-Laplacian and the first order derivative

$$\begin{gathered}((u')^{p-1})' + f(t, u, u') = 0, \quad t \in (0, 1),\\ u(0) = u'(1) = 0,\end{gathered} \tag{1.6}$$

where $p > 1$ and $f \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$. Moreover, they applied their main results obtained here to establish the existence of positive symmetric solutions to the Dirichlet problem

$$\begin{gathered}(|u'|^{p-2}u')' + f(u, u') = 0, \quad t \in (-1, 0) \cup (0, 1),\\ u(-1) = u(1) = 0.\end{gathered} \tag{1.7}$$

However, the existence of positive solutions for p-Laplacian equation with the nonlinear term involving the derivatives, such as Lidstone problem, has not been extensively studied yet. To the best of our knowledge, only [8, 14, 26] is devoted to this direction. Guo and Ge [8] considered the following boundary-value problem

$$\begin{gathered}(\Phi(y^{(2n-1)}))' = f(t, y, y'', \dots, y^{(2n-2)}), \quad 0 \le t \le 1, \\ y^{(2i)}(0) = y^{(2i)}(1) = 0, \quad i = 0, 1, 2, \dots, n-1,\end{gathered} \tag{1.8}$$

where $f \in C([0,1] \times \mathbb{R}^n, \mathbb{R})(\mathbb{R} := (-\infty, +\infty))$. Some growth conditions are imposed on f which yield the existence of at least two symmetric positive solutions by using a fixed point theorem in cones. An interesting feature in [8] is that the nonlinearity f may be sign-changing.

Motivated by the works mentioned above, in particular [17, 19, 20, 21, 22, 23, 24], in this work, we discuss the existence and multiplicity of positive solutions for (1.1). To overcome the difficulty resulting from all derivatives, we first transform (1.1) into a boundary value problem for an associated second order integro-ordinary differential equation. Then, we will use fixed point index theory to establish our main results based on a priori estimates achieved by utilizing some properties of concave functions, properties including Jensen's inequalities and our inequality (2.4) below. The results obtained here improve some existing results in the literature.

2. Preliminaries

Let $E := C^1[0,1]$, $\|u\| := \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 := \max_{t\in[0,1]} |u(t)|$. Furthermore, let $P := \{u \in E : u(t) \ge 0, u'(t) \ge 0, \forall t \in [0,1]\}$. Then E is a real Banach space and P a cone on E. For any positive integer $i \ge 2$, we denote

$$k_1(t,s) := \min\{t,s\}, \quad k_i(t,s) := \int_0^1 k_{i-1}(t,\tau)k_1(\tau,s)\,\mathrm{d}\tau, \forall t,s \in [0,1].$$

Define

$$(B_i u)(t) := \int_0^1 k_i(t,s)u(s)\,\mathrm{d}s, \quad h_i(t,s) := \partial k_i(t,s)/\partial t, \quad i = 1, 2, \dots,$$

Then

$$((B_i u)(t))' := \int_0^1 h_i(t,s)u(s)\,\mathrm{d}s, \quad i = 1, 2, \dots,$$

and $B_i, B_i' : E \to E$ are completely continuous linear operators and B_i, B_i' are also positive operators.

Let $(-1)^{n-1}x^{(2n-2)} := u$, it is easy to see that (1.1) is equivalent to the following system of integro-ordinary differential equations

$$\begin{gathered}-((u')^{p-1})' = f(t, (B_{n-1}u)(t), ((B_{n-1}u)(t))', \dots, u, u'), \\ u(0) = u'(1) = 0.\end{gathered} \tag{2.1}$$

Furthermore, the above system can be written in the form

$$u(t) = \int_0^t \Big(\int_s^1 f(\tau, (B_{n-1}u)(\tau), ((B_{n-1}u)(\tau))', \dots, u(\tau), u'(\tau))\,\mathrm{d}\tau \Big)^{\frac{1}{p-1}} \mathrm{d}s. \tag{2.2}$$

Denote by

$$(Au)(t) := \int_0^t \Big(\int_s^1 f(\tau, (B_{n-1}u)(\tau), ((B_{n-1}u)(\tau))', \dots, u(\tau), u'(\tau))\,\mathrm{d}\tau \Big)^{\frac{1}{p-1}} \mathrm{d}s.$$

Hence, $f \in C([0,1] \times \mathbb{R}_+^{2n}, \mathbb{R}_+)$ implies that $A : P \to P$ is a completely continuous operator, and the existence of positive solutions for (2.1) is equivalent to that of positive fixed points of A.

Lemma 2.1. *Let $\kappa := 1 - 2/e$ and $\psi(t) := te^t, t \in [0,1]$. Then $\psi(t)$ is nonnegative on $[0,1]$ and*

$$\kappa\psi(s) \le \int_0^1 k_1(t,s)\psi(t)\,\mathrm{d}t \le \psi(s). \tag{2.3}$$

Lemma 2.2. *Let u is concave, increasing and nonnegative on $[0,1]$. Then*

$$\int_0^1 u(t)\psi(t)\,\mathrm{d}t \ge \kappa e\|u\|. \tag{2.4}$$

Proof. The concavity of u and $\max_{t\in[0,1]} u(t) = u(1) = \|u\|$ imply

$$\int_0^1 u(t)\psi(t)\,\mathrm{d}t = \int_0^1 u(t\cdot 1 + (1-t)\cdot 0)\psi(t)\,\mathrm{d}t \ge u(1)\int_0^1 t\psi(t)\,\mathrm{d}t = \kappa e\|u\|.$$

This completes the proof. □

Lemma 2.3 ([21])**.** *Let $u \in P$ and $q > 0$. Then*

$$\int_0^1 \Big[(B_{n-1}u^q)(t) + 2\sum_{i=0}^{n-2}((B_{n-1-i}u^q)(t))'\Big]\psi(t)\,\mathrm{d}t = \int_0^1 u^q(t)\psi(t)\,\mathrm{d}t. \tag{2.5}$$

Lemma 2.4 ([9])**.** *Let $\Omega \subset E$ be a bounded open set and $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $v_0 \in P \setminus \{0\}$ such that $v - Av \ne \lambda v_0$ for all $v \in \partial\Omega \cap P$ and $\lambda \ge 0$, then $i(A, \Omega \cap P, P) = 0$, where i is the fixed point index on P.*

Lemma 2.5 ([9])**.** *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $v \ne \lambda Av$ for all $v \in \partial\Omega \cap P$ and $0 \le \lambda \le 1$, then $i(A, \Omega \cap P, P) = 1$.*

Lemma 2.6 (Jensen's inequalities)**.** *Let $\theta > 0$ and $\varphi \in C([0,1], \mathbb{R}^+)$. Then*

$$\Big(\int_0^1 \varphi(t)\,\mathrm{d}t\Big)^\theta \le \int_0^1 (\varphi(t))^\theta\,\mathrm{d}t, \quad \text{if } \theta \ge 1,$$
$$\Big(\int_0^1 \varphi(t)\,\mathrm{d}t\Big)^\theta \ge \int_0^1 (\varphi(t))^\theta\,\mathrm{d}t, \quad \text{if } 0 < \theta \le 1.$$

3. Main results

For brevity, we define $y = (y_1, y_2, \dots, y_{2n-1}, y_{2n}) \in \mathbb{R}_+^{2n}$, $\gamma_p := \max\{1, 2^{p-2}\}$, $p_* = \min\{1, p-1\}$, $p^* = \max\{1, p-1\}$, $\mathscr{K}_i := \max_{t,s\in[0,1]} k_i(t,s) > 0$, $\mathscr{H}_i := \max_{t,s\in[0,1]} h_i(t,s) > 0$,

$$\beta_p := \Big\{2^{p_*-1}\kappa\Big[(n-1)\Big(\sum_{i=1}^{n-1}(\mathscr{K}_i + \mathscr{H}_i)\Big)^{p_*-1} + 1\Big]\Big\}^{1-p/p_*},$$
$$\alpha_p := \Big\{2^{p^*-1}\Big[(n-1)\Big(\sum_{i=1}^{n-1}(\mathscr{K}_i + \mathscr{H}_i)\Big)^{p^*-1} + 1\Big]\Big\}^{1-p/p^*}.$$

(H1) $f \in C([0,1] \times \mathbb{R}_+^{2n}, \mathbb{R}_+)$.

(H2) There exist $a_1 > \beta_p$ and $c > 0$ such that

$$f(t,y) \geq a_1\Big(\sum_{i=1}^{n-1}(y_{2i-1}+2(n-i)y_{2i})+y_{2n-1}\Big)^{p-1}-c, \quad \text{for all } y \in \mathbb{R}_+^{2n} \text{ and } t \in [0,1].$$

(H3) For any $M_0 > 0$ there is a function $\Phi_{M_0} \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t,y) \leq \Phi_{M_0}(y_{2n}^{p-1}), \forall (t,y) \in [0,1] \times [0, M_0]^{2n-1} \times \mathbb{R}_+,$$
$$\int_\delta^\infty \frac{\mathrm{d}\xi}{\Phi_{M_0}(\xi)} = \infty \quad \text{for any } \delta > 0.$$

(H4) There exist $b_1 \in (0, \alpha_p)$ and $r > 0$ such that

$$f(t,y) \leq b_1\Big(\sum_{i=1}^{n-1}(y_{2i-1}+2(n-i)y_{2i})+y_{2n-1}\Big)^{p-1} \quad \text{for all } y \in [0,r]^{2n} \text{ and } t \in [0,1].$$

(H5) There exist $a_2 > \beta_p$ and $r > 0$ such that

$$f(t,y) \geq a_2\Big(\sum_{i=1}^{n-1}(y_{2i-1}+2(n-i)y_{2i})+y_{2n-1}\Big)^{p-1} \text{ for all } y \in [0,r]^{2n} \text{ and } t \in [0,1].$$

(H6) There exist $b_2 \in (0, \alpha_p)$ and $c > 0$ such that

$$f(t,y) \leq b_2\Big(\sum_{i=1}^{n-1}(y_{2i-1}+2(n-i)y_{2i})+y_{2n-1}\Big)^{p-1}+c \text{ for all } y \in \mathbb{R}_+^{2n} \text{ and } t \in [0,1].$$

(H7) f is increasing in y and there is a constant $\omega > 0$ such that

$$\int_0^1 f^{p^*/(p-1)}(s,\omega,\dots,\omega)\,\mathrm{d}s < \omega.$$

Remark 3.1. A function f is said to be increasing in y if $f(t,x) \leq f(t,y)$ holds for every pair $x, y \in \mathbb{R}_+^{2n}$ with $x \leq y$, where the partial ordering $\leq$ in $\mathbb{R}_+^{2n}$ is understood componentwise.

Theorem 3.2. *If* (H1)–(H4) *hold, then* (1.1) *has at least one positive solution.*

Proof. Let

$$\mathscr{M}_1 := \{u \in P : u = Au + \lambda\varphi, \text{ for some } \lambda \geq 0\},$$

where $\varphi(t) := te^{-t}$. Clearly, $\varphi(t)$ is nonnegative and concave on $[0,1]$. We claim $\mathscr{M}_1$ is bounded. We first establish the a priori bound of $\|u\|_0$ for $\mathscr{M}_1$. Indeed, $u \in \mathscr{M}_1$ implies u is concave (by the concavity of A and φ) and $u(t) \geq (Au)(t)$. By definition we obtain

$$u(t) \geq \int_0^t \Big(\int_s^1 f(\tau,(B_{n-1}u)(\tau),((B_{n-1}u)(\tau))',\dots,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}}\mathrm{d}s \quad (3.1)$$

for all $u \in \mathscr{M}_1$. Note that $p_*, p_*/p-1 \in [0,1]$. Now, by (H2), we find

$$\Big[a_1\Big(\sum_{i=1}^{n-1}(y_{2i-1}+2(n-i)y_{2i})+y_{2n-1}\Big)^{p-1}\Big]^{\frac{p_*}{p-1}} \leq (f(t,y)+c)^{\frac{p_*}{p-1}} \leq f^{\frac{p_*}{p-1}}(t,y)+c^{\frac{p_*}{p-1}}. \quad (3.2)$$

Combining this and Jensen's inequality, we obtain

$$
\begin{aligned}
&u^{p_*}(t)\\
&\geq \Big[\int_0^t \Big(\int_s^1 f(\tau,(B_{n-1}u)(\tau),((B_{n-1}u)(\tau))',\dots,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}}\,\mathrm{d}s\Big]^{p_*}\\
&\geq \int_0^t\int_s^1 f^{\frac{p_*}{p-1}}(\tau,(B_{n-1}u)(\tau),((B_{n-1}u)(\tau))',\dots,u(\tau),u'(\tau))\,\mathrm{d}\tau\,\mathrm{d}s\\
&= \int_0^1 k_1(t,s) f^{\frac{p_*}{p-1}}(s,(B_{n-1}u)(s),((B_{n-1}u)(s))',\dots,u(s),u'(s))\,\mathrm{d}s\\
&\geq a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}((B_iu)(s)+2(n-i)((B_iu)(s))')+u(s)\Big]^{p_*}\,\mathrm{d}s-\frac{c^{\frac{p_*}{p-1}}}{2}\\
&\geq 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}\Big((B_iu)(s)+2(n-i)((B_iu)(s))'\Big)\Big]^{p_*}\,\mathrm{d}s\\
&\quad + 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)u^{p_*}(s)\,\mathrm{d}s-\frac{c^{\frac{p_*}{p-1}}}{2}\\
&= 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}\int_0^1(k_i(s,\tau)+2(n-i)h_i(s,\tau))u(\tau)\,\mathrm{d}\tau\Big]^{p_*}\,\mathrm{d}s\\
&\quad + 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)u^{p_*}(s)\,\mathrm{d}s-\frac{c^{\frac{p_*}{p-1}}}{2}\\
&= 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)\Big[\int_0^1\frac{\sum_{i=1}^{n-1}(k_i(s,\tau)+2(n-i)h_i(s,\tau))}{\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)}\sum_{i=1}^{n-1}(\mathscr{K}_i\\
&\quad +\mathscr{H}_i)u(\tau)\,\mathrm{d}\tau\Big]^{p_*}\,\mathrm{d}s+2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)u^{p_*}(s)\,\mathrm{d}s-\frac{c^{\frac{p_*}{p-1}}}{2}\\
&\geq \Big(2\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)\\
&\quad\times\Big[\sum_{i=1}^{n-1}\int_0^1(k_i(s,\tau)+2(n-i)h_i(s,\tau))u^{p_*}(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s\\
&\quad + 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)u^{p_*}(s)\,\mathrm{d}s-\frac{c^{\frac{p_*}{p-1}}}{2}\\
&= \Big(2\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)\\
&\quad\times\Big[\sum_{i=1}^{n-1}((B_iu^{p_*})(s)+2(n-i)((B_iu^{p_*})(s))')\Big]\,\mathrm{d}s\\
&\quad + 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\int_0^1 k_1(t,s)u^{p_*}(s)\,\mathrm{d}s-\frac{c^{\frac{p_*}{p-1}}}{2}.
\end{aligned}
\tag{3.3}
$$

Multiply both sides of the above expression by $\psi(t)$ and integrate over [0,1] and use (2.3) and (2.5) to obtain

$$\begin{aligned}
&\int_0^1 \psi(t)u^{p_*}(t)\,\mathrm{d}t \\
&\geq \Big(2\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p_*-1} a_1^{\frac{p_*}{p-1}}\kappa\int_0^1\psi(t)\Big[\sum_{i=1}^{n-1}\big((B_iu^{p_*})(t) \\
&\quad + 2(n-i)((B_iu^{p_*})(t))'\big)\Big]\,\mathrm{d}t + 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\kappa\int_0^1\psi(t)u^{p_*}(t)\,\mathrm{d}t - \frac{c^{\frac{p_*}{p-1}}}{2} \\
&= 2^{p_*-1}a_1^{\frac{p_*}{p-1}}\kappa\Big[(n-1)\Big(\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p_*-1}+1\Big]\int_0^1\psi(t)u^{p_*}(t)\,\mathrm{d}t - \frac{c^{\frac{p_*}{p-1}}}{2}.
\end{aligned} \tag{3.4}$$

Therefore,

$$\int_0^1\psi(t)u^{p_*}(t)\,\mathrm{d}t \leq \frac{c^{\frac{p_*}{p-1}}}{2^{p_*}a_1^{\frac{p_*}{p-1}}\kappa\Big[(n-1)\Big(\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p_*-1}+1\Big]-2} := \mathscr{N}_1.$$

Recall that every $u\in\mathscr{M}_1$ is concave and increasing on $[0,1]$. So is u^{p_*} with $p_*\in(0,1]$. Now Lemma 2.2 yields

$$\|u\|_0 \leq (\kappa e)^{-1/p_*}\mathscr{N}_1^{1/p_*} \tag{3.5}$$

for all $u\in\mathscr{M}_1$, which implies the a priori bound of $\|u\|_0$ for $\mathscr{M}_1$, as claimed. It follows, from the boundedness of $\|u\|_0$ for $\mathscr{M}_1$, that there is $\lambda_0>0$ such that $\lambda\leq\lambda_0$ for all $\lambda\in\Lambda$, where

$$\Lambda := \{\lambda\geq 0:\ \text{there exists } u\in\mathscr{M}_1 \text{ such that } u = Au+\lambda\varphi\}.$$

If $u\in\mathscr{M}_1$, then

$$u'(t) = \Big(\int_t^1 f(\tau,(B_{n-1}u)(\tau),((B_{n-1}u)(\tau))',\dots,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}} + \lambda(1-t)e^{-t}$$

for some $\lambda\geq 0$, and by (H3),

$$\begin{aligned}
(u')^{p-1}(t) &\leq \gamma_p\int_t^1 f(\tau,(B_{n-1}u)(\tau),((B_{n-1}u)(\tau))',\dots,u(\tau),u'(\tau))\,\mathrm{d}\tau + \gamma_p\lambda_0^{p-1} \\
&\leq \gamma_p\int_t^1 \Phi_{M_0}((u')^{p-1}(\tau))\,\mathrm{d}\tau + \gamma_p\lambda_0^{p-1}.
\end{aligned}$$

Let $v(t) := (u')^{p-1}(t)$. Then $v(t)\in C([0,1],\mathbb{R}_+)$ and $v(1)=0$. Moreover,

$$v(t) \leq \gamma_p\int_t^1 \Phi_{M_0}(v(\tau))\,\mathrm{d}\tau + \gamma_p\lambda_0^{p-1}.$$

Let $F(t) := \int_t^1 \Phi_{M_0}(v(\tau))\,\mathrm{d}\tau$. Then

$$-F'(t) = \Phi_{M_0}(v(t)) \leq \Phi_{M_0}(\gamma_pF(t)+\gamma_p\lambda_0^{p-1}).$$

Therefore,

$$\int_{\gamma_p\lambda_0^{p-1}}^{v(t)} \frac{\mathrm{d}\xi}{\Phi_{M_0}(\xi)} \leq \int_{\gamma_p\lambda_0^{p-1}}^{\gamma_pF(t)+\gamma_p\lambda_0^{p-1}} \frac{\mathrm{d}\xi}{\Phi_{M_0}(\xi)} \leq \gamma_p(1-t).$$

Hence there is $\mathscr{N}_2 > 0$ such that

$$\|(u')^{p-1}\|_0 = \|v\|_0 = v(0) \le \mathscr{N}_2.$$

Let $\mathscr{N}_3 := \max\{(\kappa e)^{-1/p_*}\mathscr{N}_1^{1/p_*}, \mathscr{N}_2^{1/p-1}\}$. Then

$$\|u\| \le \mathscr{N}_3, \quad \forall u \in \mathscr{M}_1.$$

This proves the boundedness of $\mathscr{M}_1$. As a result of this, for every $R > \mathscr{N}_3$, we have

$$u - Au \ne \lambda\psi, \quad \forall u \in \partial B_R \cap P,\ \lambda \ge 0.$$

Now by Lemma 2.4, we obtain

$$i(A, B_R \cap P, P) = 0. \tag{3.6}$$

Let

$$\mathscr{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda A u \text{ for some } \lambda \in [0,1]\}.$$

We shall prove $\mathscr{M}_2 = \{0\}$. Indeed, if $u \in \mathscr{M}_2$, we have for any $u \in \overline{B}_r \cap P$

$$u(t) \le \int_0^t \Big(\int_s^1 f(\tau, (B_{n-1}u)(\tau), ((B_{n-1}u)(\tau))', \dots, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}}\,\mathrm{d}s. \tag{3.7}$$

Notice that $p^*, p^*/p-1 \ge 1$. Now, similar to (3.3), by Jensen's inequality and (H4), we obtain

$$\begin{aligned}
&u^{p^*}(t)\\
&\le \Big[\int_0^t \Big(\int_s^1 f(\tau, (B_{n-1}u)(\tau), ((B_{n-1}u)(\tau))', \dots, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}}\,\mathrm{d}s\Big]^{p^*}\\
&\le \int_0^1 k_1(t,s) f^{p^*/(p-1)}(s, (B_{n-1}u)(s), ((B_{n-1}u)(s))', \dots, u(s), u'(s))\,\mathrm{d}s\\
&\le b_1^{p^*/(p-1)} \int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}\Big((B_iu)(s) + 2(n-i)((B_iu)(s))'\Big) + u(s)\Big]^{p^*}\,\mathrm{d}s\\
&\le 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}\int_0^1 (k_i(s,\tau) + 2(n-i)h_i(s,\tau))u(\tau)\,\mathrm{d}\tau\Big]^{p^*}\,\mathrm{d}s\\
&\quad + 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 k_1(t,s) u^{p^*}(s)\,\mathrm{d}s\\
&= 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 k_1(t,s)\Big[\int_0^1 \frac{\sum_{i=1}^{n-1}(k_i(s,\tau) + 2(n-i)h_i(s,\tau))}{\sum_{i=1}^{n-1}(\mathscr{K}_i + \mathscr{H}_i)}\\
&\quad \times \sum_{i=1}^{n-1}(\mathscr{K}_i + \mathscr{H}_i)u(\tau)\,\mathrm{d}\tau\Big]^{p^*}\,\mathrm{d}s + 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 k_1(t,s) u^{p^*}(s)\,\mathrm{d}s\\
&\le \Big(2\sum_{i=1}^{n-1}(\mathscr{K}_i + \mathscr{H}_i)\Big)^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}((B_iu^{p^*})(s)\\
&\quad + 2(n-i)((B_iu^{p^*})(s))')\Big]\,\mathrm{d}s + 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 k_1(t,s) u^{p^*}(s)\,\mathrm{d}s.
\end{aligned} \tag{3.8}$$

Multiply both sides of the above expression by $\psi(t)$ and integrate over $[0,1]$ and use (2.3) and (2.5) to obtain

$$\begin{aligned} &\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t \\ &\le \Big(2\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p^*-1} b_1^{p^*/(p-1)}\int_0^1 \psi(t)\Big[\sum_{i=1}^{n-1}\big((B_iu^{p^*})(t) \\ &\quad + 2(n-i)((B_iu^{p^*})(t))'\big)\Big]\,\mathrm{d}t + 2^{p^*-1}b_1^{p^*/(p-1)}\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t \\ &= 2^{p^*-1}b_1^{p^*/(p-1)}\Big[(n-1)\Big(\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p^*-1}+1\Big]\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t. \end{aligned} \tag{3.9}$$

Therefore, $\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t = 0$, whence $u(t)\equiv 0, \forall u \in \mathscr{M}_2$. As a result, $\mathscr{M}_2 = \{0\}$, as claimed. Consequently,

$$u \ne \lambda Au, \quad \forall u \in \partial B_r \cap P,\ \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, B_r \cap P, P) = 1. \tag{3.10}$$

Combining this with (3.6) gives

$$i(A, (B_R\backslash \overline{B}_r)\cap P, P) = 0 - 1 = -1.$$

Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r)\cap P$ and therefore (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.3. *If* (H1), (H5), (H6) *are satisfied, then* (1.1) *has at least one positive solution.*

Proof. Let

$$\mathscr{M}_3 := \{u \in \overline{B}_r \cap P : u = Au + \lambda\psi \text{ for some } \lambda \ge 0\}.$$

We claim $\mathscr{M}_3 \subset \{0\}$. Indeed, if $u \in \mathscr{M}_3$, then we have $u \ge Au$ by definition. That is,

$$u(t) \ge \int_0^t \Big(\int_s^1 f(\tau, (B_{n-1}u)(\tau), ((B_{n-1}u)(\tau))', \dots, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}}\,\mathrm{d}s. \tag{3.11}$$

Similar to $\mathscr{M}_2 = \{0\}$, we can also obtain $\mathscr{M}_3 \subset \{0\}$. As a result of this, we have

$$u - Au \ne \lambda\psi, \forall u \in \partial B_r \cap P, \lambda \ge 0.$$

Now Lemma 2.4 gives

$$i(A, B_r \cap P, P) = 0. \tag{3.12}$$

Let

$$\mathscr{M}_4 := \{u \in P : u = \lambda Au \text{ for some } \lambda \in [0,1]\}.$$

We assert $\mathscr{M}_4$ is bounded. We first establish the a priori bound of $\|u\|_0$ for $\mathscr{M}_4$. Indeed, if $u \in \mathscr{M}_4$, then u is concave and $u \le Au$, which can be written in the form

$$u(t) \le \int_0^t \Big(\int_s^1 f(\tau, (B_{n-1}u)(\tau), ((B_{n-1}u)(\tau))', \dots, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}}\,\mathrm{d}s, \tag{3.13}$$

for all $u \in \mathscr{M}_4$. Note that $p^*, p^*/p-1 \geq 1$. Now by (H6) and Jensen's inequality, we obtain

$$\begin{aligned}
u^{p^*}(t) &\leq \Big[\int_0^t \Big(\int_s^1 f(\tau,(B_{n-1}u)(\tau),((B_{n-1}u)(\tau))',\dots,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)^{\frac{1}{p-1}}\,\mathrm{d}s\Big]^{p^*}\\
&\leq \int_0^1 k_1(t,s) f^{p^*/(p-1)}(s,(B_{n-1}u)(s),((B_{n-1}u)(s))',\dots,u(s),u'(s))\,\mathrm{d}s\\
&\leq \int_0^1 k_1(t,s)\Big\{b_2\Big[\sum_{i=1}^{n-1}\big((B_iu)(s)+2(n-i)((B_iu)(s))'\big)+u(s)\Big]^{p-1}\\
&\quad + c\Big\}^{p^*/(p-1)}\,\mathrm{d}s\\
&\leq b_3^{p^*/(p-1)}\int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}((B_iu)(s)+2(n-i)((B_iu)(s))')+u(s)\Big]^{p^*}\,\mathrm{d}s\\
&\quad + \frac{c_1^{p^*/(p-1)}}{2}\\
&\leq 2^{p^*-1}b_3^{p^*/(p-1)}\int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}((B_iu)(s)+2(n-i)((B_iu)(s))')\Big]^{p^*}\,\mathrm{d}s\\
&\quad + 2^{p^*-1}b_3^{p^*/(p-1)}\int_0^1 k_1(t,s)u^{p^*}(s)\,\mathrm{d}s+\frac{c_1^{p^*/(p-1)}}{2}\\
&= 2^{p^*-1}b_3^{p^*/(p-1)}\int_0^1 k_1(t,s)\Big[\int_0^1 \frac{\sum_{i=1}^{n-1}(k_i(s,\tau)+2(n-i)h_i(s,\tau))}{\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)}\\
&\quad\times\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)u(\tau)\,\mathrm{d}\tau\Big]^{p^*}\,\mathrm{d}s+2^{p^*-1}b_3^{p^*/(p-1)}\int_0^1 k_1(t,s)u^{p^*}(s)\,\mathrm{d}s\\
&\quad+\frac{c_1^{p^*/(p-1)}}{2}\\
&\leq \Big(2\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p^*-1}b_3^{p^*/(p-1)}\int_0^1 k_1(t,s)\Big[\sum_{i=1}^{n-1}((B_iu^{p^*})(s)\\
&\quad+2(n-i)((B_iu^{p^*})(s))')\Big]\,\mathrm{d}s+2^{p^*-1}b_3^{p^*/(p-1)}\int_0^1 k_1(t,s)u^{p_*}(s)\,\mathrm{d}s\\
&\quad+\frac{c_1^{p^*/(p-1)}}{2},
\end{aligned} \tag{3.14}$$

for all $u \in \mathscr{M}_4$, $b_3 \in (b_2, \alpha_p)$ and $c_1 > 0$ being chosen so that

$$(b_2z+c)^{p^*/(p-1)} \leq b_3^{p^*/(p-1)}z^{p^*/(p-1)}+c_1^{p^*/(p-1)}, \forall z \geq 0.$$

Multiply both sides of (3.14) by $\psi(t)$ and integrate over $[0,1]$ and use (2.3) and (2.5) to obtain

$$\begin{aligned}
&\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t \\
&\le \Big(2\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p^*-1} b_3^{p^*/(p-1)}\int_0^1 \psi(t)\Big[\sum_{i=1}^{n-1}\big((B_iu^{p^*})(t) \\
&\quad + 2(n-i)((B_iu^{p^*})(t))'\big)\Big]\,\mathrm{d}t + 2^{p^*-1}b_3^{p^*/(p-1)}\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t + \frac{c_1^{p^*/(p-1)}}{2} \\
&= 2^{p^*-1}b_3^{p^*/(p-1)}\Big[(n-1)\Big(\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\Big)^{p^*-1}+1\Big]\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t \\
&\quad + \frac{c_1^{p^*/(p-1)}}{2}.
\end{aligned} \tag{3.15}$$

Therefore,

$$\int_0^1 \psi(t)u^{p^*}(t)\,\mathrm{d}t \le \frac{c_1^{p^*/(p-1)}}{2-2^{p^*}b_3^{p^*/(p-1)}\big[(n-1)\big(\sum_{i=1}^{n-1}(\mathscr{K}_i+\mathscr{H}_i)\big)^{p^*-1}+1\big]} := \mathscr{N}_4.$$

This, together with Jensen's inequality and $\psi(t)/e\in[0,1]$ (Note that $p^*\ge 1$), leads to

$$e\int_0^1 u(t)\frac{\psi(t)}{e}\,\mathrm{d}t \le e\Big(\int_0^1 u^{p^*}(t)\big(\frac{\psi(t)}{e}\big)^{p^*}\,\mathrm{d}t\Big)^{1/p^*} \le e^{\frac{p^*-1}{p^*}}\mathscr{N}_4^{1/p^*} \tag{3.16}$$

for all $u\in\mathscr{M}_4$. From Lemma 2.2, we find

$$\|u\|_0 \le \kappa^{-1}e^{-1/p^*}\mathscr{N}_4^{1/p^*} := \mathscr{N}_5, \forall u\in\mathscr{M}_4,$$

which implies the a priori bound of $\|u\|_0$ for $\mathscr{M}_4$, as claimed. Furthermore, for any positive integer $i\ge 1$, this estimate leads to

$$\|(B_iu)\|_0 = (B_iu)(1) \le \mathscr{N}_5, \quad \forall u\in\mathscr{M}_4$$

and for each positive integer $i\ge 2$, we see

$$\|(B_iu)'\|_0 = (B_iu)'(0) = \int_0^1 (B_{i-1}u)(t)\,\mathrm{d}t \le \mathscr{N}_5, \forall u\in\mathscr{M}_4.$$

Moreover, for $i=1$, we have

$$\|(B_1u)'\|_0 = (B_1u)'(0) = \int_0^1 u(t)\,\mathrm{d}t \le \mathscr{N}_5, \forall u\in\mathscr{M}_4.$$

Combining these and (H6), we have

$$\begin{gathered}
-((u')^{p-1})' \le f(t,(B_{n-1}u)(t),((B_{n-1}u)(t))',\dots,u,u'),\\
u(0)=u'(1)=0.
\end{gathered}$$

Let $(u')^{p-1}(t) := w'(t)$. Then $w\in C([0,1],\mathbb{R}_+)$ and $u'(1)=0$ implies $w'(1)=0$. Therefore,

$$-w''(t) \le n^2\mathscr{N}_5, \forall u\in\mathscr{M}_4,$$

so that

$$\|w'\|_0 = w'(0) \le n^2\mathscr{N}_5.$$

Consequently,

$$\|u'\|_0 = \|w'\|_0^{1/p-1} \le (n^2\mathscr{N}_5)^{1/p-1}, \quad \forall u \in \mathscr{M}_4.$$

Let $\mathscr{N}_6 := \max\{\mathscr{N}_5, (n^2\mathscr{N}_5)^{1/p-1}\}$. Then

$$\|u\| \le \mathscr{N}_6, \quad \forall u \in \mathscr{M}_4.$$

This proves the boundedness of $\mathscr{M}_4$. As a result of this, for every $R > \mathscr{N}_6$, we have

$$u \ne \lambda Au, \quad \forall u \in \partial B_R \cap P, \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, B_R \cap P, P) = 1. \tag{3.17}$$

Combining this with (3.12) gives

$$i(A, (B_R \backslash \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.4. *If* (H1)–(H3), (H6), (H7) *are satisfied. Then* (1.1) *has at least two positive solutions.*

Proof. By (H2), (H3), and (H6), we know that (3.6) and (3.12) hold. Note we may choose $R > \omega > r$ in (3.6) and (3.12) (see the proofs of Theorems 3.2 and 3.3). By (H7) and Jensen's inequality, we have that for all $u \in \partial B_\omega \cap P$,

$$\begin{aligned}
&[(Au)(t)]^{p^*} \\
&\le \int_0^1 k_1(t,s) f^{p^*/(p-1)}(s, (B_{n-1}u)(s), ((B_{n-1}u)(s))', \dots, u(s), u'(s))\,\mathrm{d}s \\
&\le \int_0^1 f^{p^*/(p-1)}(s, (B_{n-1}u)(s), ((B_{n-1}u)(s))', \dots, u(s), u'(s))\,\mathrm{d}s < \omega
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&[((Au)(t))']^{p^*} \\
&= \Big(\int_t^1 f(s, (B_{n-1}u)(s), ((B_{n-1}u)(s))', \dots, u(s), u'(s))\,\mathrm{d}s\Big)^{p^*/(p-1)} \\
&\le \int_0^1 f^{p^*/(p-1)}(s, (B_{n-1}u)(s), ((B_{n-1}u)(s))', \dots, u(s), u'(s))\,\mathrm{d}s < \omega.
\end{aligned} \tag{3.19}$$

Thus we obtain

$$\|Au\| < \omega = \|u\|, \quad \forall u \in \partial B_\omega \cap P,$$

This implies

$$u \ne \lambda Au, \quad \forall u \in \partial B_\omega \cap P,\ \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, B_\omega \cap P, P) = 1. \tag{3.20}$$

Combining this with (3.6) and (3.12) gives

$$i(A, (B_R \backslash \overline{B}_\omega) \cap P, P) = 0 - 1 = -1, \quad i(A, (B_\omega \backslash \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Hence the operator A has at least two fixed points, with one on $(B_R \setminus \overline{B}_\omega) \cap P$ and the other on $(B_\omega \setminus \overline{B}_r) \cap P$. Therefore, (1.1) has at least two positive solutions. This completes the proof. □

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