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ON THE DIMENSION OF THE KERNEL OF THE LINEARIZED THERMISTOR OPERATOR

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ABSTRACT. The elliptic system of partial differential equations of the thermistor problem is linearized to obtain the system

$$\nabla\cdot(\sigma(\bar{u})\nabla\Phi+\sigma'(\bar{u})U\nabla\bar{\varphi})=0\quad\text{in }\Omega,\quad \Phi=0\quad\text{on }\Gamma$$

$$\Delta U+\sigma'(\bar{u})|\nabla\bar{\varphi}|^2U+2\sigma(\bar{u})\nabla\bar{\varphi}\cdot\nabla\Phi=0\quad\text{in }\Omega,\quad U=0\quad\text{on }\Gamma.$$

We study the existence of nontrivial solutions for this linear boundary-value problem, which is useful in the study of the thermistor problem.

1. Introduction

The name "thermistor" refers to a three-dimensional body made up of substances conducting both heat and electricity (typically a mixture of semiconductors) for which the electrical conductivity depends sharply on the temperature. We shall represent the body of the thermistor by Ω, an open and bounded subset of $\mathbb{R}^3$. The regular boundary Γ of Ω consists of two disjoint surfaces Γ_1 and Γ_2, the electrodes, to which a difference of potential $2V$ is applied.

Under stationary conditions the electric potential $\varphi(\mathbf{x})$, $\mathbf{x}=(x_1,x_2,x_3)$ and the temperature $u(\mathbf{x})$ inside Ω are determined by the following nonlinear elliptic boundary-value problem

$$\begin{gathered}\nabla\cdot(\sigma(u)\nabla\varphi)=0\quad\text{in }\Omega,\\ \varphi=-V\quad\text{on }\Gamma_1,\quad\varphi=V\quad\text{on }\Gamma_2,\\ \Delta u+\sigma(u)|\nabla\varphi|^2=0\quad\text{in }\Omega,\\ u=u_b\quad\text{on }\Gamma,\end{gathered}\tag{1.1}$$

where V is a given constant and u_b a given function on Γ. If u_b is an arbitrary boundary data, many papers give results of existence of both classical and weak solutions (see [1, 8, 6] and references therein). However, the nonlinear structure of (1.1) seems to be, in full generality, an open problem. We quote, in this respect, the following result [5].

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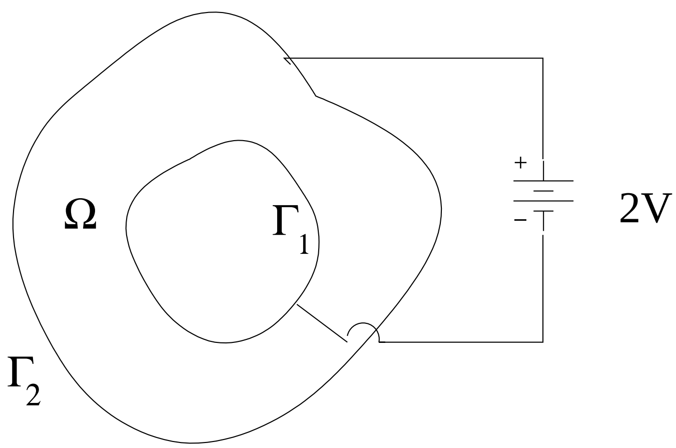

FIGURE 1. Thermistor and its circuit

Theorem 1.1. *Let $\sigma(u) \in C^0(\mathbb{R}^1)$, $\sigma(u) > 0$ satisfy*

$$\int_0^\infty \frac{dt}{\sigma(t)} = \infty$$

and suppose in the problem (1.1),

$$u = 0 \quad \text{on } \Gamma,$$

then problem (1.1) *has one and only one classical solution.*

For more comprehensive results a first step is certainly to linearize the equations and to study the corresponding linear boundary value problem. Thus we consider the following linear problem in the unknowns $(\Phi(\mathbf{x}), U(\mathbf{x}))$

$$\nabla \cdot (\sigma(\bar{u})\nabla\Phi + \sigma'(\bar{u})U\nabla\bar{\varphi}) = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\Delta U + \sigma'(\bar{u})|\nabla\bar{\varphi}|^2 U + 2\sigma(\bar{u})\nabla\bar{\varphi} \cdot \nabla\Phi = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\Phi = 0 \quad \text{on } \Gamma, \quad U = 0 \quad \text{on } \Gamma, \tag{1.4}$$

where $(\bar{\varphi}, \bar{u})$ is a solution of (1.1). We have the following result.

Lemma 1.2. *Let $(\bar{\varphi}(\mathbf{x}), \bar{u}(\mathbf{x})) \in (C^1(\bar{\Omega}))^2$ and assume*

$$\sigma_M \geq \sigma(u) \geq \sigma_m > 0. \tag{1.5}$$

Define

$$\alpha = \sup\{|2\sigma(\bar{u}(\mathbf{x})) - \sigma'(\bar{u}(\mathbf{x}))|\,|\nabla\bar{\varphi}|,\ \mathbf{x} \in \Omega\},$$
$$\beta = \sup\{\sigma'(\bar{u}(\mathbf{x}))|\nabla\bar{\varphi}|^2,\ \mathbf{x} \in \Omega\}$$

and suppose that

$$\sigma_m - \frac{\alpha}{2} > 0, \quad 1 - \frac{\alpha}{2\lambda_0} - \frac{\beta}{\lambda_0} > 0, \tag{1.6}$$

where λ_0 is the first eigenvalue of the laplacian with zero boundary conditions. Then the problem (1.2)–(1.4) *has only the trivial solution.*

Proof. Let us multiply (1.2) by Φ and (1.3) by U. Integrating by parts over Ω and adding we obtain

$$\int_\Omega (\sigma(\bar{u})|\nabla\Phi|^2 + |\nabla U|^2)dx = \int_\Omega (2\sigma(\bar{u}) - \sigma'(\bar{u}))U\nabla\bar{\varphi}\cdot\nabla\Phi dx + \int_\Omega \sigma'(\bar{u})U^2|\nabla\bar{\varphi}|^2 dx.$$

Hence, by (1.5) we have

$$\sigma_m \int_\Omega |\nabla\Phi|^2 + |\nabla U|^2 dx \le \alpha\int_\Omega |U||\nabla\Phi|dx + \beta\int_\Omega U^2 dx.$$

Using the Cauchy-Schwartz and the Poincarè inequalities we obtain

$$(\sigma_m - \frac{\alpha}{2})\int_\Omega |\nabla\Phi|^2 dx + (1 - \frac{\alpha}{2\lambda_0} - \frac{\beta}{\lambda_0})\int_\Omega |\nabla U|^2 dx \le 0.$$

This implies $\Phi = 0$ and $U = 0$ by (1.6). □

As an application of Lemma 1.2 we have the following lemma.

Lemma 1.3. *Assume $\sigma(u) \in C^1(\mathbb{R}^1)$ and*

$$\sigma_M \ge \sigma(u) \ge \sigma_m > 0. \tag{1.7}$$

Let $(\bar{\varphi}, \bar{u})$ be the unique corresponding solution of (1.1) when $u_b = 0$. Suppose that

$$\Phi = 0, \quad U = 0 \tag{1.8}$$

is the only solution of (1.2)-(1.4).

Let $u_b \in C^{0,\alpha}(\Gamma)$. Then there exists $\mu_0 > 0$ such that, if $\|u_b\|_{C^{0,\alpha}(\Gamma)} \le \mu_0$, the problem

$$\nabla\cdot(\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega, \quad \varphi = -V \quad \text{on } \Gamma_1, \quad \varphi = V \quad \text{on } \Gamma_2$$
$$\Delta u + \sigma(u)|\nabla\varphi|^2 = 0 \quad \text{in } \Omega, \quad u = u_b \quad \text{on } \Gamma$$

has one and only one solution.

Proof. Let $F : X \to Y$, where

$$X = \{(\varphi(\mathbf{x}), u(\mathbf{x})) \in (C^{2,\alpha}(\bar{\Omega}))^2,\ \varphi = -V \text{ on } \Gamma_1,\ \varphi = V \text{ on } \Gamma_2\},$$
$$Y = (C^{0,\alpha}(\bar{\Omega}))^2 \times C^{2,\alpha}(\Gamma), F((\varphi, u)) = (\nabla\cdot(\sigma(u)\nabla\varphi),\ \Delta u + \sigma(u)|\nabla\varphi|^2,\ u|_\Gamma).$$

We apply the local inversion theorem at $(\bar{\varphi}, \bar{u})$ [2]. We have

$$\begin{aligned} &F'((\bar{\varphi}, \bar{u}))[\Phi, U] \\ &= (\nabla\cdot(\sigma(\bar{u})\nabla\Phi + \sigma'(\bar{u})U\nabla\bar{\varphi}), \Delta U + \sigma'(\bar{u})|\nabla\bar{\varphi}|^2 U + 2\sigma(\bar{u})\nabla\bar{\varphi}\cdot\nabla\Phi,\ u|_\Gamma). \end{aligned}$$

We claim that the problem

$$\nabla\cdot(\sigma(\bar{u})\nabla\Phi + \sigma'(\bar{u})U\nabla\bar{\varphi}) = f$$
$$\Delta U + \sigma'(\bar{u})|\nabla\bar{\varphi}|^2 U + 2\sigma(\bar{u})\nabla\bar{\varphi}\cdot\nabla\Phi = g$$
$$\Phi = 0 \quad \text{on } \Gamma, \quad U = U_b \quad \text{on } \Gamma$$

has one and only one solution if $((f, g), U_b) \in Y$. If (Φ_1, U_1) and (Φ_2, U_2) are two solutions we set $(\Psi, W) = (\Phi_1 - \Phi_2, U_1 - U_2)$ and use (1.8). This gives $(\Phi_1, U_1) = (\Phi_2, U_2)$. To prove existence we use a continuity method (se e.g. [4] page 336). We construct a one-parameter family of problems depending on the parameter $t \in [0, 1]$. Let $\mathbf{U} = (\Phi, U)$ and define

$$\mathbf{L}_t[\mathbf{U}] = \begin{bmatrix} (1-t)\Delta\Phi + t(\nabla\cdot(\sigma(\bar{u})\nabla\Phi + \sigma'(\bar{u})U\nabla\bar{\varphi})) \\ (1-t)\Delta U + t(\sigma'(\bar{u})|\nabla\bar{\varphi}|^2 U + 2\sigma(\bar{u})\nabla\bar{\varphi}\cdot\nabla\Phi) \end{bmatrix}.$$

By the Schauder's estimates [7] any solution of the problem

$$\mathbf{L}_t[\mathbf{U}] = \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{f} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 \\ U_b \end{bmatrix} \quad \text{on } \Gamma \tag{1.9}$$

satisfies

$$\|\mathbf{U}\|_{C^{2,\alpha}} \le K_1\Big(\|\mathbf{f}\|_{C^\alpha} + \|U_b\|_{C^\alpha}\Big). \tag{1.10}$$

We call T the set of those value of t in the unit interval $[0,1]$ for which problem (1.9) is uniquely solvable. T is not empty since it contains $t = 0$. We prove that T is an open set; i.e., for every $t_0 \in T$ there exists $\epsilon(t_0) > 0$ such that every $t \in [0,1]$, for which $|t - t_0| < \epsilon(t_0)$, belongs to T. This can be seen with a contraction mapping argument as follows. We rewrite (1.9) in the form

$$\mathbf{L}_{t_0}[\mathbf{U}] = \mathbf{L}_{t_0}[\mathbf{U}] - \mathbf{L}_t[\mathbf{U}] + \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{U} = \begin{bmatrix} 0 \\ U_b \end{bmatrix} \quad \text{on } \Gamma$$

or

$$\mathbf{L}_{t_0}[\mathbf{U}] = (1-t)(\Delta\mathbf{U} - \mathbf{L}_1[\mathbf{U}]) + \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{U} = \begin{bmatrix} 0 \\ U_b \end{bmatrix} \quad \text{on } \Omega. \tag{1.11}$$

Substituting any function $\mathbf{U} \in C^{2,\alpha}$ on the right hand side of (1.11) we obtain a function $F \in C^\alpha$. Since $t_0 \in T$ there exists $\mathbf{W} \in C^\alpha$ such that

$$\mathbf{L}_{t_0}[\mathbf{W}] = \mathbf{F} \quad \text{in } \Omega, \quad \mathbf{W} = \begin{bmatrix} 0 \\ U_b \end{bmatrix} \quad \text{on } \Gamma. \tag{1.12}$$

The problem (1.12) defines a transformation

$$\mathbf{W} = \mathbf{A}(\mathbf{U}). \tag{1.13}$$

We claim that there exists a fixed point of (1.13) if $|t - t_0|$ is sufficiently small. From (1.11) we have

$$\|\mathbf{F}\|_{C^\alpha} \le \Big(|t - t_0|\|\mathbf{U}\|_{C^{2,\alpha}} + \|\mathbf{f}\|_{C^\alpha}\Big).$$

Using again the Schauder's estimates, we obtain

$$\|\mathbf{W}\|_{C^{2,\alpha}} \le K_1K_2|t - t_0|\|\mathbf{U}\|_{C^{2,\alpha}} + K_1\|\mathbf{f}\|_{C^\alpha} + K_1\|U_b\|_{C^\alpha}. \tag{1.14}$$

Hence, if we assume $2K_1K_2|t - t_0| \le 1$, an inequality of the form

$$\|\mathbf{U}\|_{C^{2,\alpha}} \le 2K_1\big(\|\mathbf{f}\|_{C^\alpha} + \|U_b\|_{C^\alpha}\big)$$

would imply

$$\|\mathbf{W}\|_{C^{2,\alpha}} = \|\mathbf{A}(\mathbf{U})\|_{C^{2,\alpha}} \le K_1\big(\|\mathbf{f}\|_{C^\alpha} + \|U_b\|_{C^\alpha}\big).$$

Moreover, if $\mathbf{W}_1 = \mathbf{A}(\mathbf{U}_1)$ and $\mathbf{W}_2 = \mathbf{A}(\mathbf{U}_2)$, $\mathbf{W}_1$-$\mathbf{W}_2$ is a solution of

$$\mathbf{L}_{t_0}[\mathbf{W}_1 - \mathbf{W}_2] = (t - t_0)\big(\Delta(\mathbf{U}_1 - \mathbf{U}_2) - \mathbf{L}_1[\mathbf{U}_1 - \mathbf{U}_2]\big), \quad \mathbf{W}_1 - \mathbf{W}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Recalling (1.14) we conclude that, if

$$2K_1K_2|t - t_0| \le 1, \tag{1.15}$$

then

$$\|\mathbf{W}_1 - \mathbf{W}_2\|_{C^{2,\alpha}} \le \frac{1}{2}\|\mathbf{U}_1 - \mathbf{U}_2\|_{C^{2,\alpha}}. \tag{1.16}$$

Therefore, if (1.15) holds the transformation $\mathbf{A}(\mathbf{U})$ maps the set of functions satisfying

$$\|\mathbf{U}\|_{C^{2,\alpha}} \le 2K_2\|\mathbf{f}\|_{C^\alpha}$$

into itself and, by (1.16), is a contraction. Thus (1.13) has a fixed point $\mathbf{U}$ which gives the desired solution of (1.9) if $|t - t_0| \le (1/2)K_1K_2$. Hence T is open. Moreover, T is a closed set. For, let $\tilde{t}$ be a cluster point of a sequence $\{t_n\}$ in T. Consider any $\mathbf{f}$ in C^α and let $\{\mathbf{U}_n\}$ be the corresponding sequence of solutions in $C^{2,\alpha}$ such that

$$\mathbf{L}_{t_n}[\mathbf{U}_n] = \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{U}_n = \begin{bmatrix} 0 \\ U_b \end{bmatrix} \quad \text{on } \Gamma.$$

By (1.10) we have

$$\|\mathbf{U}_n\|_{C^{2,\alpha}} \le K_1(\|\mathbf{f}\|_{C^\alpha} + \|U_b\|_{C^\alpha}).$$

Thus the sequence $\{\mathbf{U}_n\}$ and their first and second derivatives are equibounded and equicontinuous in $\bar{\Omega}$. Let $\{\mathbf{U}_{n_j}\}$ be a subsequence converging with first and second derivatives. If $\tilde{\mathbf{U}}$ is the limit function it gives a solution to the problem

$$\mathbf{L}_{\tilde{t}}[\tilde{\mathbf{U}}] = \mathbf{f} \quad \text{in } \Omega, \quad \tilde{\mathbf{U}} = \begin{bmatrix} 0 \\ U_b \end{bmatrix} \quad \text{on } \Gamma.$$

Hence $\tilde{t} \in T$, therefore $T = [0, 1]$. □

We may also consider the problem

$$\begin{gathered} \nabla \cdot (\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega \\ \varphi = -V \quad \text{on } \Gamma_1, \quad \varphi = V \quad \text{on } \Gamma_2 \\ \Delta u + \sigma(u)|\nabla\varphi|^2 + \mu R(u, \varphi) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma, \end{gathered} \tag{1.17}$$

where $R(u, \varphi) \in C^0(\mathbb{R}^2)$ is a temperature depending source and μ a numerical parameter.

Lemma 1.4. *Assume* $\sigma(u) \in C^1(\mathbb{R}^1)$ *and*

$$\sigma_M \ge \sigma(u) \ge \sigma_m > 0.$$

Let $(\bar{\varphi}, \bar{u})$ *be the solution (unique by Theorem 1.1) of the problem* (1.1) *when* $u_b = 0$. *Suppose that the problem* (1.2)-(1.4) *has only the trivial solution. Then there exists* $\mu_0 > 0$ *such that the problem* (1.17) *has one and only one solution if* $|\mu| < \mu_0$.

Proof. We apply the implicit function theorem. Let $\mathcal{F} : \mathcal{X} \times \mathbb{R}^1 \to \mathcal{Y}$, where

$$\mathcal{X} = \{(\varphi(\mathbf{x}), u(\mathbf{x})) \in (C^{2,\alpha}(\bar{\Omega}))^2,\ \varphi = -V \text{ on } \Gamma_1,\ \varphi = V \text{ on } \Gamma_2,\ u = 0 \text{ on } \Gamma\},$$
$$\mathcal{Y} = (C^{0,\alpha}(\bar{\Omega}))^2,$$
$$\mathcal{F}((\varphi, u), \mu) = (\nabla \cdot (\sigma(u)\nabla\varphi), \Delta u + \sigma(u)|\nabla\varphi|^2 + \mu R(u, \varphi)), \quad (\varphi, u) \in \mathcal{X},\ \mu \in \mathbb{R}^1.$$

We have $\mathcal{F}((\bar{\varphi}, \bar{u}), 0) = ((0, 0), 0)$. Moreover, the partial derivative of $\mathcal{F}$ with respect to (φ, u) at $((\bar{\varphi}, \bar{u}), 0)$ is

$$\begin{aligned} &\mathcal{F}_{(\varphi,u)}((\bar{\varphi}, \bar{u}), 0)[\Phi, U] \\ &= (\nabla \cdot (\sigma(\bar{u})\nabla\Phi + \sigma'(\bar{u})U\nabla\bar{\varphi}), \Delta U + \sigma'(\bar{u})|\nabla\bar{\varphi}|^2 U + 2\sigma(\bar{u})\nabla\bar{\varphi} \cdot \nabla\Phi). \end{aligned}$$

Proceeding, with minor changes, as in Lemma 1.2 we can prove that the problem

$$\begin{gathered} \nabla \cdot (\sigma(\bar{u})\nabla\Phi + \sigma'(\bar{u})U\nabla\bar{\varphi}) = f, \quad \Phi = 0 \quad \text{on } \Gamma \\ \Delta U + \sigma'(\bar{u})|\nabla\bar{\varphi}|^2 U + 2\sigma(\bar{u})\nabla\bar{\varphi} \cdot \nabla\Phi = g, \quad U = 0 \quad \text{on } \Gamma \end{gathered}$$

has one and only one solution for every $(f, g) \in \mathcal{Y}$. Thus $\mathcal{F}_{(\varphi,u)}((\bar{\varphi}, \bar{u}), 0)$ is invertible and therefore there exists $\mu_0 > 0$ such that the thesis holds. □

2. The one-dimensional case

In this Section we study the one-dimensional version of the problem (1.1); i.e.,

$$(\sigma(u)\varphi')' = 0, \tag{2.1}$$

$$\varphi(-L) = -V, \quad \varphi(L) = V, \quad L > 0, \tag{2.2}$$

$$u'' + \sigma(u)\varphi'^2 = 0, \tag{2.3}$$

$$u(-L) = 0, \quad u(L) = 0. \tag{2.4}$$

In the next Lemma we collect certain elementary properties of the solution of (2.1)–(2.4).

Lemma 2.1. *Let $\sigma(u) \in C^1(\mathbb{R}^1)$ and $\sigma(u) > 0$ for all $u \in \mathbb{R}^1$. Suppose*

$$\int_0^\infty \frac{dt}{\sigma(t)} = \infty. \tag{2.5}$$

Under these hypotheses there exists one and only one solution $(\varphi(x), u(x))$ of the problem (2.1)–(2.4), and the solution satisfies

$$\varphi'(x) > 0, \tag{2.6}$$

$$\varphi(x) = -\varphi(-x), \; u(x) = u(-x). \tag{2.7}$$

Moreover, if we define

$$F(u) = \int_0^u \frac{dt}{\sigma(t)}, \tag{2.8}$$

$$\xi = G(\varphi) = \int_0^\varphi \sigma(F^{-1}(\frac{V^2}{2} - \frac{t^2}{2}))dt \tag{2.9}$$

we have

$$\frac{d\varphi}{dx}(x) = \frac{G(V)}{L\sigma(u(x))}. \tag{2.10}$$

Proof. Let us define the transformation

$$\theta = \frac{1}{2}\varphi^2 + \frac{V}{2}\varphi + F(u). \tag{2.11}$$

Therefore, by (2.8),

$$\sigma(u)\theta' = \sigma(u)\varphi\varphi' + \frac{V}{2}\sigma(u)\varphi' + u'.$$

Recalling (2.1) and (2.3) we have

$$(\sigma(u)\theta')' = 0.$$

Hence in terms of φ and θ the problem (2.1)–(2.4) can be restated as

$$\begin{gathered}(\sigma(u)\varphi')' = 0,\\ \varphi(-L) = -V, \quad \varphi(L) = V,\\ (\sigma(u)\theta')' = 0,\\ \theta(-L) = 0, \quad \theta(L) = V^2.\end{gathered} \tag{2.12}$$

This suggests the existence of a functional relation between θ and φ, of the form

$$\theta = \frac{V}{2}\varphi + \frac{V^2}{2}.$$

Hence, by (2.11), we have

$$F(u)=\frac{V^2}{2}-\frac{\varphi^2}{2}.$$

By (2.5), F is globally invertible and

$$u=F^{-1}(\frac{V^2}{2}-\frac{\varphi^2}{2}). \tag{2.13}$$

Thus we can write (2.12) in the form

$$(\sigma(F^{-1}(\frac{V^2}{2}-\frac{\varphi^2}{2}))\varphi')'=0.$$

Using (2.9), we have

$$\xi''=0,$$
$$\xi(-L)=G(-V)=-G(V),\quad \xi(L)=G(V).$$

Thus we obtain

$$\xi(x)=\frac{G(V)}{L}x.$$

The potential $\varphi(x)$ can be computed from

$$G(\varphi(x))=\frac{G(V)}{L}x \tag{2.14}$$

which gives

$$\varphi(x)=G^{-1}(\frac{G(V)}{L}x).$$

Finally the temperature $u(x)$ is obtained from (2.13). The solution $(\varphi(x),u(x))$ of problem (2.1)–(2.4) obtained in this way is also unique [5]. Now we prove (2.10). From (2.14) we have

$$\frac{G(V)}{L}=\frac{dG}{d\varphi}(\varphi(x))\varphi'(x)$$

and by (2.13) and (2.9)

$$\frac{dG}{d\varphi}=\sigma(u).$$

Hence (2.10) follows. From (2.1) we have $\sigma(u)\varphi'=c$ with $c>0$ by (2.2), thus we obtain (2.6). To prove (2.7) we define

$$\tilde\varphi(x)=-\varphi(-x),\quad \tilde u(x)=u(-x).$$

As it is easily verified $(\tilde\varphi(x),\tilde u(x))$ satisfy (2.1)-(2.4). Therefore, by the uniqueness of the solution of the problem (2.1)–(2.4) we obtain (2.7). □

The linearized problem corresponding, in the present one-dimensional case, to (1.2)-(1.4) reads

$$(\sigma(\bar u)\Phi'+\sigma'(\bar u)U\bar\varphi')'=0,\quad \big(\sigma'=\frac{d\sigma}{du},\ \varphi'=\frac{d\varphi}{dx}\big), \tag{2.15}$$
$$\Phi(-L)=0,\quad \Phi(L)=0, \tag{2.16}$$
$$U''+\sigma'(\bar u)\bar\varphi'^2U+2\sigma(\bar u)\bar\varphi'\Phi'=0, \tag{2.17}$$
$$U(-L)=0,\quad U(L)=0, \tag{2.18}$$

where $(\bar{\varphi}(x), \bar{u}(x))$ is a solution of the problem (2.1)–(2.4). Here we proceed by direct integration the linear problem (2.15)–(2.18). From (2.15) we have

$$\sigma(\bar{u})\Phi' = c_1 - \sigma'(\bar{u})U\bar{\varphi}'. \tag{2.19}$$

Substituting (2.19) in (2.17) we obtain, as a problem equivalent to (2.15)–(2.18),

$$\sigma'(\bar{u})\bar{\varphi}'U + \sigma(\bar{u})\Phi' = c_1, \tag{2.20}$$

$$\Phi(-L) = 0, \quad \Phi(L) = 0, \tag{2.21}$$

$$U'' - \sigma'(\bar{u})\bar{\varphi}'^2 U = -2c_1\bar{\varphi}', \tag{2.22}$$

$$U(-L) = 0, \quad U(L) = 0. \tag{2.23}$$

If $\mathcal{V}(x)$ is a solution of the auxiliary problem

$$\mathcal{V}'' - \sigma'(\bar{u})\bar{\varphi}'^2\mathcal{V} = -2\bar{\varphi}', \tag{2.24}$$

$$\mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0, \tag{2.25}$$

then the function

$$U(x) = c_1\mathcal{V}(x) \tag{2.26}$$

solves (2.22) and (2.23) and vice versa. Substituting (2.26) into (2.20) we obtain

$$\Phi'(x) = \frac{c_1}{\sigma(\bar{u})}(1 - \sigma'(\bar{u})\bar{\varphi}'\mathcal{V}).$$

Integrating, we have

$$\Phi(x) = c_1\int_{-L}^{x}\frac{1 - \sigma'(\bar{u}(t))\bar{\varphi}(t)\mathcal{V}(t)}{\sigma(\bar{u}(t))}dt. \tag{2.27}$$

The condition $\Phi(L) = 0$ becomes

$$c_1\int_{-L}^{L}\frac{1 - \sigma'(\bar{u}(t))\bar{\varphi}(t)\mathcal{V}(t)}{\sigma(\bar{u}(t))}dt = 0. \tag{2.28}$$

Let us assume that

(H0) the number 1 is not an eigenvalue of the problem

$$\mathcal{V}'' - \sigma'(\bar{u})\bar{\varphi}'^2\mathcal{V} = 0, \quad \mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0. \tag{2.29}$$

When (H0) holds, the auxiliary problem (2.24), (2.25) has one and only one solution $\mathcal{V}(x)$ and two possibilities occur: a generic case,

$$\int_{-L}^{L}\frac{1 - \sigma'(\bar{u}(t))\bar{\varphi}'(t)\mathcal{V}(t)}{\sigma(\bar{u}(t))}dt \neq 0, \tag{2.30}$$

and a special case

$$\int_{-L}^{L}\frac{1 - \sigma'(\bar{u}(t))\bar{\varphi}'(t)\mathcal{V}(t)}{\sigma(\bar{u}(t))}dt = 0. \tag{2.31}$$

Assume (H0) and (2.30) hold. Then, from (2.28), $c_1 = 0$ and (2.27) imply

$$\Phi(x) = 0. \tag{2.32}$$

Moreover, from (2.22) and (2.23) we have

$$U'' - \sigma'(\bar{u})\bar{\varphi}'^2 U = 0, \quad U(-L) = 0, \quad U(L) = 0. \tag{2.33}$$

On the other hand, by (H0), the value 1 is not an eigenvalue of (2.33), hence $U(x) = 0$. Therefore the problem (2.15)-(2.18) has only the trivial solution and the one-dimensional version of Lemma 1.4 applies.

We consider next the special case in which the assumption (H0) holds, but

$$\int_{-L}^{L} \frac{1-\sigma'(\bar{u}(t))\bar{\varphi}'(t)\mathcal{V}(t)}{\sigma(\bar{u}(t))}dt = 0, \tag{2.34}$$

where in (2.34) $\mathcal{V}(x)$ is the unique solution of problem (2.24)-(2.25). We have

$$\Phi(x) = c_1 \int_{-L}^{x} \frac{1-\sigma'(\bar{u}(t))\bar{\varphi}(t)\mathcal{V}(t)}{\sigma(\bar{u}(t))}dt,$$

where c_1 is an arbitrary constant and, by (2.26), $U(x) = c_1\mathcal{V}(x)$. Thus in this case the linear problem (2.20)-(2.23) has nontrivial solutions, more precisely the space of its solutions has dimension 1.

Example 2.2. The problem (2.29) can be solved only in special cases and it is therefore difficult to check the condition (H0). However, this can be done for the physical relevant conductivity law

$$\sigma(u) = \frac{K}{au+b}, \ K > 0, \ a > 0, \ b > 0 \tag{2.35}$$

which is quite accurate for metals. If (2.35) holds, we have, using the notation of Lemma 2.1,

$$\xi = F(u) = \frac{1}{K}(\frac{au^2}{2} + bu), \quad u = F^{-1}(\xi) = \frac{-b+\sqrt{b^2+2a\xi K}}{a}.$$

Moreover,

$$\sigma(F^{-1}(\frac{V^2}{2} - \frac{t^2}{2})) = \frac{K}{\sqrt{b^2+aK(V^2-t^2)}}$$

and

$$G(V) = \frac{\sqrt{K}}{\sqrt{a}} \arctan\frac{\sqrt{aK}V}{b}. \tag{2.36}$$

Problem (2.29) can be restated, in view of (2.10), in the form

$$\mathcal{V}'' - \frac{\sigma'(\bar{u})(G(V))^2}{L^2(\sigma(\bar{u}))^2}\mathcal{V} = 0, \quad \mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0. \tag{2.37}$$

If (2.35) holds, we have

$$\frac{\sigma'(\bar{u})}{(\sigma(\bar{u}))^2} = -\frac{a}{K}.$$

Hence, by (2.36), the equation in (2.37) becomes

$$\mathcal{V}'' + \frac{1}{L^2}\big(\arctan\frac{\sqrt{aK}V}{b}\big)^2\mathcal{V} = 0.$$

Recalling that $\mu_0 = \frac{\pi^2}{4L^2}$ is the first eigenvalue of the problem

$$\mathcal{V}'' + \mu\mathcal{V} = 0, \quad \mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0$$

and taking into account that

$$\frac{1}{L^2}\big(\arctan\frac{\sqrt{aK}V}{b}\big)^2 < \frac{\pi^2}{4L^2} \tag{2.38}$$

we conclude that 1 is not an eigenvalue of the problem (2.29) if (2.35) holds. Hence the condition (H0) is certainly verified. Moreover, in view of (2.38) the operator

$$\frac{d^2}{dx^2} + \frac{1}{L^2}\big(\arctan\frac{\sqrt{aK}V}{b}\big)^2$$

is a "maximum principle operator". Thus the unique solution of the problem

$$\mathcal{V}'' + \frac{1}{L^2}(\arctan \frac{\sqrt{aK}V}{b})^2\mathcal{V} = -2\bar{\varphi}'(x),\ \mathcal{V}(-L) = 0,\ \mathcal{V}(L) = 0$$

is positive in $(-L, L)$ since $\varphi'(x) > 0$ by (2.6). It follows

$$\int_{-L}^{L} \frac{1 - \sigma'(\bar{u}(t))\bar{\varphi}'(t)\mathcal{V}(t)}{\sigma(\bar{u}(t))} dt > 0.$$

Therefore, the condition (2.30) is satisfied. It follows that the problem (2.15)-(2.18) has only the trivial solution and the one-dimensional version of Lemma 1.4 applies.

Example 2.3. If $\sigma'(u) \geq 0$ the problem (2.29) has only the trivial solution $\mathcal{V}(x) = 0$ [2], therefore (H0) is verified. However, in this case, we have by the maximum principle, from (2.24)-(2.25) and, in view of (2.6), $\mathcal{V}(x) > 0$. Thus the cases (2.30) and (2.31) are, in principle, both possible.

To treat the case in which 1 is an eigenvalue of (2.29), we recall [3] the following result on the eigenvalues and eigenfunctions of the problem

$$v'' + \lambda p(x)v = 0, \quad v(L) = 0, \quad (L) = 0. \tag{2.39}$$

Lemma 2.4. *If $p(x) \in C^0([-L, L])$ and $p(x) > 0$, then the eigenvalues λ_n, $n = 0, 1, 2, \dots$ of problem (2.39) are all simple. When the eigenvalues are arranged in increasing order, the eigenfunctions $v_n(x)$ (determined except for a constant multiplier) possess exactly n zeros in $(-L, L)$. In particular, the first eigenvalue $v_0(x)$ has constant sign.*

Lemma 2.5. *Let $p(x) \in C^0([-L, L])$ be even and $p(x) > 0$. Then the eigenfunctions $v_n(x)$ of (2.39) with an even index are even, and the eigenfunctions with an odd index are odd.*

Proof. All eigenfunctions of (2.39) are either even or odd. Let $v(x)$ be an eigenfunction corresponding to the eigenvalue λ. Let $v(0) \neq 0$ and define

$$W(x) = v(-x). \tag{2.40}$$

It is easily seen that $W(x)$ is also an eigenfunction corresponding to λ. Thus $W(x) = Cv(x)$ and $W(0) = v(0)$ by (2.40). Hence $C = 1$ and therefore $v(-x) = v(x)$. Let $v(0) = 0$. We have $v'(0) = \alpha \neq 0$ since $v'(0) = 0$ would imply $v(x) = 0$. Define $W(x) = -v(-x)$. $W(x)$ is an eigenfunction corresponding to λ. On the other hand, $W(0) = -v(0) = 0$. Therefore $W(x) = v(x)$ and $v(x) = -v(-x)$. To prove that $v_0(x)$ is even we simply note that $v_0(x) \neq 0$. We prove that $v_1(x)$ is odd. By Lemma 2.4, $v_1(x)$ has only one zero x^* in $(-L, L)$ with $v_1'(x^*) \neq 0$. We claim that $x^* = 0$. Let $x^* \neq 0$, thus either $v_1(x^*) = 0$ and $v_1(-x^*) = 0$ or $v(x^*) = 0$ and $-v_1(-x^*) = 0$ and this cannot be since $v_1(x)$ has only one zero in $(-L, L)$. Suppose, by contradiction, $v_1(x)$ to be even. This implies

$$v_1'(0) = 0. \tag{2.41}$$

But $v_1(0) = 0$ and that, together with (2.41), would imply $v_1(x) = 0$. Hence $v_1(x)$ is odd. In a similar vein we can prove the general result: $v_n(x)$ is even if n is even and $v_n(x)$ is odd if n is odd. □

Lemma 2.6. *Let $p(x) \in C^0([-L, L])$, $f(x) \in C^0([-L, L])$ be even functions and $p(x) > 0$. Consider the two-point problem*

$$v'' + \lambda p(x)v = f(x), \quad v(L) = 0, \quad v(L) = 0. \tag{2.42}$$

Let λ be an eigenvalue of odd index of the problem (2.39) and $\tilde{v}(x)$ the corresponding (odd) eigenfunction. Then the solutions of (2.42) can be written as follows

$$v(x) = C\tilde{v}(x) + w(x),$$

where C is an arbitrary constant and $w(x)$ is the only solution of (2.42) which is even and satisfies the condition

$$\int_{-L}^{L} w(x)\tilde{v}(x)dx = 0. \tag{2.43}$$

Proof. The condition of solvability of problem (2.42), i. e., $\int_{-L}^{L} f(x)\tilde{v}(x)dx = 0$ is satisfied in view of the assumptions on $f(x)$ and on the eigenvalue λ. Let us normalize the eigenfunction $\tilde{v}(x)$ assuming $\int_{-L}^{L} \tilde{v}^2 dx = 1$. The solutions of problem (2.42) are given by

$$v(x) = C\tilde{v}(x) + v^*(x),$$

where $v^*(x)$ is an arbitrary function which satisfies

$$\frac{d^2v^*}{dx^2} + \lambda v^* = f(x), \ v^*(-L) = 0, \ v^*(L) = 0.$$

Define

$$w(x) = v^*(x) - \int_{-L}^{L} v^*(t)\tilde{v}(t)dt \ \tilde{v}(x).$$

We have

$$\int_{-L}^{L} w(x)\tilde{v}(x)dx = 0.$$

On the other hand, if $w_1(x)$ and $w_2(x)$ both satisfy (2.42) and (2.43) and if we define $h(x) = w_1(x) - w_2(x)$, we have $h(x) = C\tilde{v}(x)$. If $C = 0$ we have done. If $C \neq 0$ we have

$$\int_{-L}^{L} h(x)\tilde{v}(x)dx = 0, \quad C\int_{-L}^{L} \tilde{v}^2(x)dx = 0$$

which cannot be. Thus there exists only one solution of (2.42) which satisfies (2.43). We claim that $w(x)$ is even. Define $z(x) = w(-x)$. Since $\tilde{v}(x)$ is odd we have

$$\int_{-L}^{L} z(x)\tilde{v}(x)dx = \int_{-L}^{L} w(x)\tilde{v}(x)dx = 0.$$

Also we have

$$\frac{d^2z}{dx^2} + \lambda p(x)z = f(x), \quad z(-L) = 0, \quad z(L) = 0$$

since $p(x)$ and $f(x)$ are even functions. By uniqueness we conclude that $w(x)$ is even. □

We assume now

(H1) $\sigma'(\bar{u}(x)) < 0$ and the number 1 is the first eigenvalue of the problem

$$\mathcal{V}'' - \sigma'(\bar{u})\bar{\varphi}'^2\mathcal{V} = 0, \quad \mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0.$$

Denote by $\mathcal{V}_0(x)$ the corresponding eigenfunction normalized with the condition $\int_{-L}^{L} \mathcal{V}^2(x)dx = 1$. Then we have

$$\int_{-L}^{L} \bar{\varphi}'(x)\mathcal{V}_0(x)dx \neq 0$$

since $\mathcal{V}_0(x) \neq 0$ by Lemma 2.4 and $\bar{\varphi}'(x) > 0$ by Lemma 2.1. Hence the auxiliary problem

$$\mathcal{V}'' - \sigma'(\bar{u})\bar{\varphi}'^2\mathcal{V} = -2\bar{\varphi}', \quad \mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0$$

has no solution. Therefore the problem

$$U'' - \sigma'(\bar{u})\bar{\varphi}'^2 U = -2c_1\bar{\varphi}', \quad U(-L) = 0, \quad U(L) = 0$$

has solutions only when

$$c_1 = 0 \tag{2.44}$$

and these solutions are

$$U(x) = \gamma\mathcal{V}_0(x), \quad \gamma \in \mathbb{R}^1. \tag{2.45}$$

From (2.19), (2.44) and (2.45) we have

$$\sigma(\bar{u})\Phi' = -\gamma\sigma'(\bar{u})\mathcal{V}_0(x)\bar{\varphi}'(x), \tag{2.46}$$

$$\Phi(-L) = 0, \quad \Phi(L) = 0. \tag{2.47}$$

The condition of solvability of (2.46) and (2.47) is thus given by

$$\gamma \int_{-L}^{L} \frac{\sigma'(\bar{u}(t))\mathcal{V}_0(t)\bar{\varphi}'(t)dt}{\sigma(\bar{u}(t))} = 0.$$

On the other hand, by (H1), Lemma 2.1 and Lemma 2.4, we have

$$\int_{-L}^{L} \frac{\sigma'(\bar{u}(t))\mathcal{V}_0(t)\bar{\varphi}'(t)dt}{\sigma(\bar{u}(t))} \neq 0$$

which implies $\gamma = 0$ and $U(x) = 0$ and, from (2.46) and (2.47), $\Phi(x) = 0$. Therefore, the problem (2.20)-(2.23) has only the trivial solution and Lemma 1.4 applies.

Next we examine the case when

(H2) $\sigma'(\bar{u}(x)) < 0$ and the number 1 is the second eigenvalue of the problem

$$\mathcal{V}'' - \sigma'(\bar{u})\bar{\varphi}'^2\mathcal{V} = 0, \quad \mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0.$$

Let $\mathcal{V}_1(x)$ be the corresponding eigenvalue which is normalized with the condition $\int_{-L}^{L} \mathcal{V}_1^2(x)dx = 1$. By Lemma 2.5, $\mathcal{V}_1(x)$ is an odd function. Thus we have, recalling that $\bar{\varphi}'(x)$ is an even function,

$$\int_{-L}^{L} \bar{\varphi}'(x)\mathcal{V}_1(x)dx = 0.$$

Thus, by Lemma 2.6, the solutions of

$$\mathcal{V}'' - \sigma'(\bar{u})\bar{\varphi}'^2\mathcal{V} = -2\bar{\varphi}', \quad \mathcal{V}(-L) = 0, \quad \mathcal{V}(L) = 0 \tag{2.48}$$

are given by

$$\mathcal{V}(x) = C\mathcal{V}_1(x) + \tilde{\mathcal{V}}(x),$$

where C is an arbitrary constant and $\tilde{\mathcal{V}}(x)$ the only solution of (2.48) which is even and satisfies

$$\int_{-L}^{L} \tilde{\mathcal{V}}(x)\mathcal{V}_1(x)dx = 0.$$

Therefore, the solutions of

$$U'' - \sigma'(\bar{u})\bar{\varphi}'^2 U = -2c_1\bar{\varphi}', \quad U(-L) = 0, \quad U(L) = 0$$

are given by

$$U(x) = c_1 C\mathcal{V}_1(x) + c_1\tilde{\mathcal{V}}(x) \tag{2.49}$$

or, if we put $K = c_1 C$,

$$U(x) = K\mathcal{V}_1(x) + c_1\tilde{\mathcal{V}}(x). \tag{2.50}$$

From (2.19), using (2.50), we have

$$\Phi'(x) = \frac{c_1}{\sigma(\bar{u})}(1 - \sigma'(\bar{u})\bar{\varphi}'\tilde{\mathcal{V}}) - \frac{K}{\sigma(\bar{u})}\sigma'(\bar{u})\bar{\varphi}'\mathcal{V}_1.$$

Hence

$$\Phi(x) = c_1\int_{-L}^{x}\frac{1}{\sigma(\bar{u})}(1 - \sigma'(\bar{u})\bar{\varphi}'\tilde{\mathcal{V}})dt - K\int_{-L}^{x}\frac{1}{\sigma(\bar{u})}\sigma'(\bar{u})\bar{\varphi}'\mathcal{V}_1 dt. \tag{2.51}$$

The condition $\Phi(L) = 0$ gives

$$c_1\int_{-L}^{L}\frac{1}{\sigma(\bar{u})}(1 - \sigma'(\bar{u})\bar{\varphi}'\tilde{\mathcal{V}})dt = 0$$

if we take into account that $\frac{\sigma'(\bar{u}(x))\bar{\varphi}'\mathcal{V}_1(x)}{\sigma(\bar{u}(x))}$ is an odd function of x . Thus we need to distinguish a generic case when

$$\int_{-L}^{L}\frac{1}{\sigma(\bar{u})}(1 - \sigma'(\bar{u})\bar{\varphi}'\tilde{\mathcal{V}})dt \neq 0.$$

This implies $c_1 = 0$. From (2.50) we have

$$U(x) = K\mathcal{V}_1(x)$$

and from (2.51),

$$\Phi(x) = -K\int_{-L}^{x}\frac{1}{\sigma(\bar{u})}\sigma'(\bar{u})\bar{\varphi}'\mathcal{V}_1 dt.$$

Therefore, in the generic case the kernel of the linearized operator has dimension 1. On the other hand, if

$$\int_{-L}^{L}\frac{1}{\sigma(\bar{u})}(1 - \sigma'(\bar{u})\bar{\varphi}'\tilde{\mathcal{V}})dt = 0$$

the kernel has dimension 2, since (2.49) and (2.51) hold.

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