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EXISTENCE OF TRAVELING WAVES FOR DIFFUSIVE-DISPERSIVE CONSERVATION LAWS

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Abstract. In this work we show the existence existence and uniqueness of traveling waves for diffusive-dispersive conservation laws with flux function in $C^1(\mathbb{R})$, by using phase plane analysis. Also we estimate the domain of attraction of the equilibrium point attractor corresponding to the right-hand state. The equilibrium point corresponding to the left-hand state is a saddle point. According to the phase portrait close to the saddle point, there are exactly two semi-orbits of the system. We establish that only one semi-orbit come in the domain of attraction and converges to $(u_-, 0)$ as $y \to -\infty$. This provides the desired saddle-attractor connection.

1. Introduction

In this article, we investigate the existence and uniqueness of traveling wave solutions, which are smooth functions of the form

$$u(x,t) = s(x - ct)$$

where c is a constant, for the partial differential equation

$$u_t + f(u)_x = (a(u)u_x)_x + (b(u)u_x)_{xx} \quad x \in \mathbb{R}, t > 0, \tag{1.1}$$

where the diffusion function $a : \mathbb{R} \to \mathbb{R}$ and dispersion function $b : \mathbb{R} \to \mathbb{R}$ are given smooth functions. Furthermore, we assume $a(u), b(u) > 0$ for $u \in \mathbb{R}$ and the flux function $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

For the case $f(u) = u^2/2$, where a and b real constants with $ab \neq 0$, equation (1.1) reduces to the Korteweg-de Vries-Burgers equation

$$u_t + uu_x = au_{xx} + bu_{xxx}. \tag{1.2}$$

It is usually considered as a combination of the Burgers equation and KdV equation since in the limit $b \to 0$ the equation reduces to the Burgers equation

$$u_t + uu_x = au_{xx} \tag{1.3}$$

which is named after its use by Burgers [4] for studying the turbulence, and if the limit $a \to 0$ is taken, then the equation becomes the KdV equation

$$u_t + uu_x = bu_{xxx} \tag{1.4}$$

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which was first suggested by Korteweg and de Vries [9], who used it as a nonlinear model to study the change of forms of long waves advancing in a rectangular channel.

The Korteweg-de Vries-Burgers equation (1.2) is the simplest form of the wave equation in which the non-linearity (uu_x), the dispersion u_{xxx} and the dissipation u_{xx} occur.

The existence of traveling waves with linear diffusion and dispersion were studied by Bona and Schonbek [3]; and Jacobs, McKinney and Shearer [6]. In [3] was interested in the limiting behaviour of these waves when the coefficients a, b tends to zero while the ratio $\frac{b}{a^2}$ remains bounded. In 1993, Jacobs, McKinney, and Shearer [6] rigorously characterized all week solutions profiles of the single conservation law $u_t + (u^3)_x = 0$.

Bedjaoui and LeFloch [1]-[2] and Thanh [13] studied equations of the type

$$u_t + f(u)_x = (R(u, \beta u_x))_x + \gamma(c_1(u)(c_2(u)u_x)_x)_x. \tag{1.5}$$

In [2] the authors considered

$$R(u, v) = b(u, v)|v|^p \quad \text{for } p > 0.$$

Thanh [13] studied the case when $R = R(u, v)$ satisfies

$$R_v(u, 0) = R_u(u, 0) = 0, \quad R(u, v)v > 0, \quad \forall v \neq 0, \ \forall\ u.$$

This assumption is not satisfied in our case. Bedjaoui and LeFloch [1] studied the case $R(u, v) = b(u, v)v$ with $b(u, v) = a(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth mapping satisfying

$$uf''(u) > 0 \quad \text{for all } u \neq 0, \quad \lim_{\pm\infty} f' = +\infty. \tag{1.6}$$

The associated non-linear system (1.5) admits exactly three equilibrium points for a certain speed interval that depends the kinetic function (see [10]), thanks the hypothesis (1.6) on the flux-function $f : \mathbb{R} \to \mathbb{R}$. For such equilibrium points, the existence and uniqueness of traveling waves were investigated when a certain speed wave is fixed in this interval; however the connected points do not satisfy the Oleinik entropy criterion, see [1, Theorem 3.3]. Observe that the work [1] establishes the existence of traveling waves associated with nonclassical shocks. When the equilibrium point satisfies such criterion this implies the existence of the trajectories between them for each speed wave c in a certain interval, see [1, Theorem 5.1].

In this article, we are interested in traveling waves associated with a classical shock (see Definition 2.5). Given two states u_- and u_+, both of them arbitrary constants, we investigate the existence and uniqueness of traveling waves when the speed wave is given by $c = \frac{f(u_+) - f(u_-)}{u_+ - u_-}$, under only the hypothesis that f is smooth. In this context, the associated system has at least two equilibrium points $(u_-, 0)$ and $(u_+, 0)$ satisfying the Oleinik entropy criterion. This differs from [1, Theorem 3.3].

In our main result (Theorem 6.1) ensures that the points $(u_\pm, 0)$ can be connected by doing an analysis of the phase portrait of the saddle point $(u_-, 0)$. We also estimate of the domain attraction of the node $(u_+, 0)$. This technique is different the one in [1, Theorem 5.1], it does not use of the existence of a kinetic function.

In the case of traveling waves our manuscript can be used to establish existence and uniqueness of traveling waves when the flux-function associated is $f(u) = u^2/2$

and when the diffusive-dispersive coefficients are positive constants. Moreover, we can adapt our work to establish existence and uniqueness of traveling waves for the equation

$$u_t + f(u)_x = au_{xx} + bu_{xxt} \tag{1.7}$$

where $a > 0$ and $b \neq 0$, see in Appendix II.

An outline of this article is as follows. In Section 2, we recall the concept of traveling wave solution connecting the states $u\pm$ together with the concept of week solution. We close the section with the method of linearization for differential equations. In section 3, we begin by recalling well-known concepts and results. Also, the stability of equilibrium point of the associated differential equation is established. After declaring an invariance theorem we establish a result about traveling waves, which is essential to the existence of trajectories connecting the states $u\pm$. In section 4, an estimate of domain of attraction is provided. In Section 5, the analysis of the phase portrait close to the saddle point shows that there are exactly two semi-orbits of system. We establish that only one semi-orbit enters the attraction domain of the attracting equilibrium point $(u_+, 0)$, and it converges to $(u_-, 0)$ as $y \to -\infty$. This gives the desired saddle-attractor connection.

2. Traveling waves: a weak solution

We consider partial differential equation

$$u_t + f(u)_x = (a(u)u_x)_x + (b(u)u_x)_{xx} \tag{2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the flux function, the functions $a = a(u) > 0$, $b = b(u) > 0$ are $C^1(\mathbb{R})$ and $C^2(\mathbb{R})$ respectively. We seek the existence of traveling wave solution $u(x,t) = s(x-ct)$, for some constant speed $c \in \mathbb{R}$, satisfying the following conditions at infinity:

$$\lim_{y\to\pm\infty} s^{(j)}(y) = 0, \quad j = 1, 2 \quad \text{and} \quad \lim_{y\to\pm\infty} s(y) = u_\pm, \quad u_- \neq u_+. \tag{2.2}$$

Note that from (2.1) the function $s = s(y)$ satisfies the ordinary differential equation

$$(f(s(y)) - cs(y))' = (a(s(y))s'(y))' + (b(s(y))s'(y))'' \tag{2.3}$$

where

$$(\cdot)' = \frac{d}{dy}(\cdot), \quad y \in \mathbb{R}.$$

Integrating (2.3) on $]-\infty, y[$ and using the conditions at infinity (2.2) we have

$$\begin{aligned} &-c(s(y) - u_-) + f(s(y)) - f(u_-) \\ &= a(s(y))s'(y) - \lim_{y\to-\infty} a(s(y))s'(y) + (b(s(y))s'(y))' - \lim_{y\to-\infty}(b(s(y))s'(y))' \end{aligned} \tag{2.4}$$

Define the function $h(s) = -c(s - u_-) + f(s) - f(u_-)$ for $s \in \mathbb{R}$. Since

$$\lim_{y\to-\infty} a(s(y))s'(y) = \lim_{y\to-\infty} b'(s(y))(s'(y))^2 = 0,$$

we can re-write (2.4) in the form

$$h(s(y)) = a(s(y))s'(y) + (b(s(y))s'(y))', \quad y \in \mathbb{R}. \tag{2.5}$$

We obtain the following lemma.

Lemma 2.1. *Let $u(x,t) = s(x-ct)$, $c \in \mathbb{R}$, be a* traveling wave *solution of* (2.1) *satisfying* (2.2). *Then, in equation* (2.5), *letting $y \to +\infty$ we obtain*

$$-c(u_+ - u_-) + f(u_+) - f(u_-) = 0; \tag{2.6}$$

i.e., the triple (u_-, u_+, c) satisfies the Rankine-Hugoniot relation.

Remark 2.2. In agreement with Lemma 2.1, it will be assumed that

$$c = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Setting $w(y) = b(s(y))s'(y)$ in (2.5) we have the second-order system

$$\begin{aligned} s'(y) &= \frac{w(y)}{b(s(y))}, \\ w'(y) &= h(s(y)) - \frac{a(s(y))}{b(s(y))}w(y). \end{aligned} \tag{2.7}$$

Let F be vector field

$$F(s,w) = \left(\frac{w}{b(s)}, h(s) - \frac{a(s)}{b(s)}w\right),$$

a point in the (s,w)-plane is an equilibrium point of (2.7) if and only if it has of the form $(s,0)$ with $h(s) = 0$.

Proposition 2.3. *A point (s,w) is an equilibrium point of the* (2.7) *if and only if $w = 0$ and the triple (s, u_-, c) satisfies the Rankine-Hugoniot relation for the associate conservation law $u_t + f(u)_x = 0$.*

Remark 2.4. Denote by Γ the set of equilibrium points of (2.7) and let $(u_i, 0) \in \Gamma$. Then

$$h(s) = -c(s - u_i) + f(s) - f(u_i).$$

Geometrically, Γ is the intersection of the straight line connecting $(u_\pm, 0)$ and the graph of f.

Definition 2.5 (Weak Solution)**.** A discontinuous function of the form

$$u(x,t) = \begin{cases} u_-, & x - ct \le 0 \\ u_+, & x - ct > 0 \end{cases}$$

is said to be a weak solution of the conservation law $u_t + f(u)_x = 0$ if it satisfies the Rankine-Hugoniot relation

$$-c(u_+ - u_-) + f(u_+) - f(u_-) = 0.$$

We know that weak solutions are not unique. To choose the only physically relevant solution we use the Oleinik entropy criterion, that requires

$$\frac{f(u_+) - f(u_-)}{u_+ - u_-} < \frac{f(u) - f(u_-)}{u - u_-}, \quad \forall u \in (u_+,\, u_-), \tag{2.8}$$

or equivalently,

$$\frac{f(u) - f(u_+)}{u - u_+} < \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \forall u \in (u_+,\, u_-). \tag{2.9}$$

In view of proposition 2.3, assuming that $(s, 0)$ is a equilibrium point, the function

$$u(x,t) = \begin{cases} u_-, & x - ct \le 0 \\ s, & x - ct > 0 \end{cases} \tag{2.10}$$

is a weak solution of the conservation law $u_t + f(u)_x = 0$. Conversely, if $u(x,t)$ is as in (2.10) and also a weak solution then the points $(s,0)$ e $(u_-,0)$ are equilibrium points of the F.

Another fact that follows from the existence of weak solution physically relevant is that $h(s) < 0$ in (u_+, u_-), we assume without loss of generality that $u_+ < u_-$. We can also conclude that $f'(u_+) \le c \le f'(u_-)$.

2.1. **Linearization.** The Jacobian matrix is

$$DF(s,w) = \begin{pmatrix} -w\left(\frac{b'(s)}{b^2(s)}\right) & \frac{1}{b(s)} \\ (f'(s)-c) - w\left(\frac{a'(s)b(s)-a(s)b'(s)}{b^2(s)}\right) & -\frac{a(s)}{b(s)} \end{pmatrix}$$

We choose $(u_i, 0)$ an arbitrary equilibrium point and obtain

$$DF(u_i,0) = \begin{pmatrix} 0 & \frac{1}{b(u_i)} \\ f'(s)-c & -\frac{a(u_i)}{b(u_i)} \end{pmatrix}.$$

Therefore, the characteristic equation of $DF(u_i, 0)$ is

$$|DF(u_i,0) - \lambda Id| = \lambda^2 + \frac{a(u_i)}{b(u_i)}\lambda - \frac{1}{b(u_i)}(f'(u_i) - c) = 0,$$

which admits two roots:

$$\begin{gathered} \lambda_1 = -\frac{a(u_i)}{2b(u_i)} - \sqrt{\frac{(a(u_i))^2}{4(b(u_i))^2} + \frac{f'(u_i)-c}{b(u_i)}}, \\ \lambda_2 = -\frac{a(u_i)}{2b(u_i)} + \sqrt{\frac{(a(u_i))^2}{4(b(u_i))^2} + \frac{f'(u_i)-c}{b(u_i)}} \end{gathered} \tag{2.11}$$

where $i = \pm$. Next, we recall some concepts and theorems that will be useful for the classification of the equilibrium points of field F.

3. Definitions and statement of results

Consider the nonlinear system

$$X' = G(X), \tag{3.1}$$

where $G : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and D is a neighborhood of the $X = X_0$.

Definition 3.1. The equilibrium point $X = X_0$ of (3.1) is

(1) stable, if for each $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that $\|X(t) - X_0\| < \epsilon$ for all $t \ge 0$ whenever $\|X(0) - X_0\| < \delta$;
(2) unstable, if not stable;
(3) asymptotically stable, if it is stable and δ can be chosen such that

$$\lim_{t\to+\infty} \|X(t) - X_0\| = 0$$

whenever $\|X(0) - X_0\| < \delta$.

Theorem 3.2 ([8])**.** *Let $X = X_0$ be an equilibrium point for the nonlinear system (3.1) and let $A = DG(X_0)$. Then*

1. *If $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues λ_i of A then X_0 is asymptotically stable relative to the nonlinear system;*
2. *If $\mathrm{Re}(\lambda_i) > 0$ for one or more of the eigenvalues λ_i then X_0 is unstable relative to the nonlinear system, where $i = 1, 2$.*

3.1. **Nonlinear classification system.** From Theorem 3.2 and (2.11), we have the following classification:

1. If $f'(u_i) - c > 0$ then $(u_i, 0)$ is a saddle point (of the linearized system). Thus, the point $(u_i, 0)$ is unstable.
2. If $f'(u_i) - c < 0$ and since $a(u_i) > 0$ then $(u_i, 0)$ is asymptotically stable.

In the remainder of this article is devoted to the case when $f'(u_+) < c < f'(u_-)$ and thus shall be ensured the existence and uniqueness of traveling wave connecting the states $u_\pm$. Now follow with some more definitions and results of the theory of differential equations.

Definition 3.3 (Invariant set)**.** (1) A set $M \subset D$ is said to be *invariant* set with respect to (3.1) if $X(0) \in M$ then $X(t) \in M$ for all $t \in \mathbb{R}$.

(2) A set $M \subset D$ is said to be *positively invariant* set (negatively invariant set) with respect to $X' = G(X)$, if $X(0) \in M$ then $X(t) \in M$ for all $t \geq 0$ ($t \leq 0$).

Definition 3.4. A trajectory $X(t)$ of (3.1) approaches a set $M \subset D$ as $t \to +\infty$, if for every $\epsilon > 0$ there is $T > 0$ such that

$$\mathrm{dist}(X(t), M) \doteq \inf_{p \in M} \|X(t) - p\| < \epsilon, \quad \forall t > T.$$

Theorem 3.5 (LaSalle's invariance principle [8])**.** *Let $V : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that*

$$\dot{V}(s, w) \doteq \nabla V(s, w) \cdot G(s, w) \leq 0$$

for all $X = (s, w) \in \Omega$, with Ω compact positively invariant in D. Let E be the set of all points in Ω where $\dot{V}(s, w) = 0$. Let M be the largest invariant set in E. Then every trajectory of $X' = G(X)$, starting Ω; i.e., $X(0) \in \Omega$, approaches of M as $t \to +\infty$.

Now turning our attention to system (2.7), we consider the function

$$V(s, w) = -\int_{u_+}^{s} h(x)b(x)\,dx + \frac{w^2}{2}, \quad (s, w) \in \mathbb{R}^2. \tag{3.2}$$

Note that

$$\dot{V}(s, w) = \frac{w^2}{b(s)}(-a(s)) \leq 0 \quad \text{and} \quad \dot{V}(s, w) = 0 \Leftrightarrow w = 0.$$

Proposition 3.6. *Suppose that there is a compact $\Omega \subset \mathbb{R}^2$ positively invariant with respect to system (2.7) with vector field*

$$F(s, w) = \Big(\frac{w}{b(s)}, h(s) - \frac{a(s)}{b(s)} w\Big).$$

Then every solution of this system starting in Ω approaches the set $\Gamma \cap \Omega$.

Proof. With the quantities (3.2) at hand, we can rewrite this context $E = \{(s, w) \in \Omega; w = 0\}$. Using Theorem 3.5 every solution starting in Ω approaches the set M the largest invariant set in E. It is easy to see that $\Gamma \cap \Omega$ is invariant, here Γ is defined in Remark 2.4, so $\Gamma \cap \Omega \subset M$. Let us show that $M \subset \Gamma \cap \Omega$. Indeed, given $(u, 0) \in M$, let $(s(y), w(y))$ be the solution of the system with initial data $(s(0), w(0)) = (u, 0)$. We have that $(s(y), w(y)) \in M$ for all $y \in \mathbb{R}$, since it is M invariant, so $w(y) = 0$, $y \in \mathbb{R}$. On the other hand, follow of the system that

$$s'(y) = \frac{w(y)}{b(s(y))} = 0,$$

which implies $s(y) = u$. Thus, $(u, 0)$ is an equilibrium point of the system (2.7); i.e., $(u, 0) \in \Gamma \cap \Omega$. □

4. Traveling wave solution: domain of attraction

The goal of this section is to determine a compact Ω positively invariant in $\mathbb{R}^2$ and estimate the domain of attraction of the equilibrium point $(u_+, 0)$. Initially we give the definition of the domain of attraction.

Definition 4.1. Let $X = X_0$ be an equilibrium point asymptotically stable of the system $X' = G(X)$. Denote by $\phi(t, \overline{X})$ the solution starting at $\overline{X}$ in $t = 0$. Then, the *domain attraction* corresponding to X_0 is the set from $\overline{X}$ such that

$$\lim_{t\to+\infty} \|\phi(t, \overline{X}) - X_0\| = 0.$$

4.1. **Estimation of the domain attraction.** Let $m = \min_{(s,w)\in\partial(D\cup R)} V(s, w)$, where the set D and R are given by

$$D = \Big\{(s, w) \in \mathbb{R}^2 : (s - u_+)^2 + \frac{w^2}{\gamma^2} \le (u_+ - q)^2,\ u_+ \le s \le q\Big\},$$

$$R = \Big\{(s, w) \in \mathbb{R}^2 : (s - u_+)^2 + \frac{(u_+ - p)^2}{(\gamma|u_+ - q|)^2} w^2 \le (u_+ - p)^2,\ p \le s \le u_+\Big\}$$

(see Figure 1), with

$$\gamma^2 > (\mathrm{Lip}_{[p,u_-]} f + |c|) \max_{[p,u_-]} b(s) \tag{4.1}$$

and

$$\int_p^{u_-} h(x)b(x)\,dx > 0 \quad \text{and} \quad h > 0 \text{ in } (p, u_+). \tag{4.2}$$

where p was chosen satisfying the inequality $2u_+ - u_- \le p < u_+$.

The condition (4.2) makes sense since we have $f'(u_+) < c$. Indeed, from $f'(u_+) < c$, there is a $t > 0$ such that

$$\frac{f(u) - f(u_+)}{u - u_+} < c, \quad \forall u \in (u_+ - t, u_+).$$

Hence, $h(u) > 0$ for all $u \in (u_+ - t, u_+)$ thus

$$\int_u^{u_+} h(x)b(x)\,dx > 0, \quad \forall u \in (u_+ - t, u_+).$$

Choose $p \in (u_+ - t, u_+)$. Being $I(v) = \int_p^v h(x)b(x)\,dx > 0$ continuous positive in $(p, u_+]$ there is some $u_+ < q \le u_-$ such that $I(q) > 0$, i.e., $\int_p^q h(x)b(x)\,dx > 0$. In this work we assume that holds $q = u_-$.

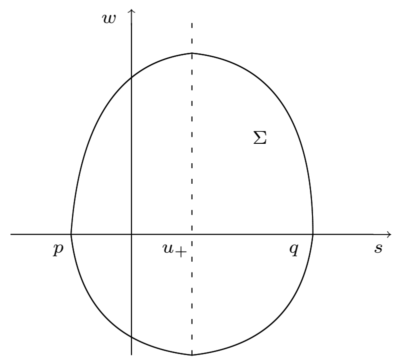

FIGURE 1. Region containing the domain of attraction

Lemma 4.2. *Let $\Sigma = D \cup R$ be defined as above and $\partial\Sigma$ denote its boundary. Let γ be given by (4.1). Then we have*

$$0 < m = \min_{(s,w)\in\partial\Sigma} V(s,w) = V(u_-,0) = \int_{u_+}^{u_-} -h(x)b(x)\,dx.$$

Proof. Let $E = \partial D \cap \partial\Sigma$. Then

$$E = \{(s,w) \in \mathbb{R}^2 : (s-u_+)^2 + \frac{w^2}{\gamma^2} = (u_+ - u_-)^2,\ u_+ \le s \le u_-\}$$

and we have

$$w^2 = \gamma^2((u_+ - u_-)^2 - (s-u_+)^2),$$

we replace w in $V(\cdot,\cdot)$ in (3.2) we have

$$\min_{(s,w)\in E} V(s,w) = \min_{s\in[u_+,u_-]} \Big[-\int_{u_+}^{s} h(x)b(x)\,dx + \frac{\gamma^2}{2}((u_+ - u_-)^2 - (s-u_+)^2)\Big].$$

Define $g(s) = -\int_{u_+}^{s} h(x)b(x)\,dx + \frac{\gamma^2}{2}((u_+ - u_-)^2 - (s-u_+)^2)$, for $s \in [u_+,u_-]$. It follows that

$$\begin{aligned} g'(s) &= -h(s)b(s) - \gamma^2(s-u_+) \\ &= -(s-u_+)\Big(\gamma^2 + b(s)\Big(\frac{f(s)-f(u_+)}{s-u_+} - c\Big)\Big) < 0, \quad s \in (u_+,u_-) \end{aligned}$$

where the inequality above comes from (4.1), because, since $-(s-u_+) < 0$ we can deduce that $\Big(\gamma^2 + b(s)\big(\frac{f(s)-f(u_+)}{s-u_+} - c\big)\Big) > 0$, in fact

$$\begin{aligned} cb(s) - b(s)\Big(\frac{f(s)-f(u_+)}{s-u_+}\Big) &\le |c|b(s) + b(s)\Big|\frac{f(s)-f(u_+)}{s-u_+}\Big| \\ &\le (\mathrm{Lip}_{[p,u_-]} f + |c|) \max_{[p,u_-]} b(s) < \gamma^2. \end{aligned} \tag{4.3}$$

Therefore, the function g is strictly decreasing in $[u_+,u_-]$ and realizes its minimum value in $s = u_-$. Thus,

$$\min_{(s,w)\in E} V(s,w) = V(u_-,0).$$

On the other hand, on $F = \partial R \cap \partial(D \cup R)$

$$F = \Big\{(s,w) \in \mathbb{R}^2 : (s-u_+)^2 + \frac{(u_+-p)^2}{(\gamma|u_+-u_-|)^2} w^2 = (u_+-p)^2,\ p \le s \le u_+\Big\}$$

one has

$$w^2 = \gamma^2\Big((u_+-u_-)^2 - (s-u_+)^2 \frac{(u_+-u_-)^2}{(u_+-p)^2}\Big),$$

we replace w in $V(\cdot,\cdot)$ in (3.2) we have

$$\begin{aligned}\min_{(s,w)\in F} V(s,w) = \min_{s\in[p,u_+]} \Big[&- \int_{u_+}^{s} h(x)b(x)\,dx \\ &+ \frac{\gamma^2}{2}\Big((u_+-u_-)^2 - (s-u_+)^2 \frac{(u_+-u_-)^2}{(u_+-p)^2}\Big)\Big].\end{aligned}$$

Setting

$$G(s) = -\int_{u_+}^{s} h(x)b(x)\,dx + \frac{\gamma^2}{2}\left((u_+-u_-)^2 - (s-u_+)^2 \frac{(u_+-u_-)^2}{(u_+-p)^2}\right),$$

for $s \in [p, u_+]$, we have

$$\begin{aligned}G'(s) &= -h(s)b(s) - \gamma^2(s-u_+)\frac{(u_+-u_-)^2}{(u_+-p)^2} \\ &= -(s-u_+)\Big(\gamma^2 \frac{(u_+-u_-)^2}{(u_+-p)^2} + b(s)\Big(\frac{f(s)-f(u_+)}{s-u_+} - c\Big)\Big), \quad s \in (p, u_+).\end{aligned}$$

Note that

$$\frac{(u_+-u_-)^2}{(u_+-p)^2} > 1 \quad \text{and} \quad -(s-u_+) > 0$$

so

$$\gamma^2 \frac{(u_+-u_-)^2}{(u_+-p)^2} + b(s)\Big(\frac{f(s)-f(u_+)}{s-u_+} - c\Big) > \gamma^2 + b(s)\Big(\frac{f(s)-f(u_+)}{s-u_+} - c\Big).$$

But, we show that $\gamma^2 + b(s)\big(\frac{f(s)-f(u_+)}{s-u_+} - c\big) > 0$ similarly as in (4.3). Therefore G is strictly decreasing in $[p, u_+]$, and

$$\min_{(s,w)\in F} V(s,w) = V(p,0).$$

Now let us compare the values $V(u_-,0)$ and $V(p,0)$. Clearly from (4.2) it follows that

$$V(p,0) = \int_p^{u_+} h(x)b(x)\,dx = \int_p^{u_-} h(x)b(x)\,dx - \int_{u_+}^{u_-} h(x)b(x)\,dx > V(u_-,0).$$

So the lemma is proved. □

We are now able to build a compact positively invariant. Consider $l \in (0, m)$, where m is given in Lemma 4.2, and define a set

$$\Omega_l = \{(s,w) \in D \cup R : V(s,w) \le l\}.$$

Assertion 4.3. $\Omega_l \subset \operatorname{int}(D \cup R)$.

Proof. To prove this, we suppose to the contrary, then there exists $(s_0, w_0) \in \Omega_l \cap \partial(D \cup R)$, therefore $V(s_0, w_0) \ge m > l$ producing a contradiction. □

Assertion 4.4. *Let Ω_l be o set above. Then, it is compact positively invariant.*

For proof of the assertion 4.4 we need of Lemma 4.5, whose proof can be found at [8].

Lemma 4.5. *Suppose that there exists a compact set $W \subset \mathbb{R}^2$ such that every local solution of $X' = G(X)$, $y > 0$, $X(0) = X_0 = (s(0), w(0)) \in W$, lies entirely in W. Then, there is a unique solution passing through X_0 defined in $(0, \infty)$.*

Proof of Assertion 4.4. We clearly have that Ω_l is compact, since $\Omega_l = (D \cup R) \cap V^{-1}((-\infty, l])$. Let $(s(y), w(y))$ be a solution for system starting in Ω_l, i.e, $(s(0), w(0)) \in \Omega_l$. We saw earlier that $\dot{V}(s, w) \le 0$, then

$$\frac{d}{dy}(V(s(y), w(y))) \le 0$$

so the function $V(s(y), w(y))$ is decreasing for $y \in J = (0, \omega_+)$, we denote by J the maximal interval associated to the maximal solution $(s(y), w(y))$. Thus,

$$V(s(y), w(y)) \le V(s(0), w(0)) \le l(< m).$$

Consequently,

$$(s(y), w(y)) \in \Omega_l \quad \text{for} \quad y \in J,$$

provided that $(s(y), w(y)) \in \Sigma$ for all $y \in J$, since $m = \min_{(s,w)\in\partial\Sigma} V(s, w)$. It then follows from Lemma 4.5 that $\omega_+ = \infty$. So the assertion 4.4 follows. □

We have proven $(u_+, 0) \in \Omega_l$ for all $l \in (0, m)$ and is the only equilibrium point of the system (2.7) in Ω_l, once Ω_l lies entirely in the interior of $D \cup R$ and the function h is positive in (p, u_+) and negative in (u_+, u_-).

Therefore, according to Proposition 3.6 every solution starting in Ω_l tends toward $(u_+, 0)$ when $y \to +\infty$. Furthermore, sets Ω_l are an approximation of the domain of attraction of the point $(u_+, 0)$ which is the subject of the following lemma.

Lemma 4.6. *The domain of attraction of the equilibrium point $(u_+, 0)$ contains the set*

$$W = \{(s, w) \in D \cup R : V(s, w) < V(u_-, 0)\}.$$

Moreover, the line segment $[u_+, u_-) \times \{0\}$ is contained in W.

Proof. It is sufficient to prove that $W = \cup_{0<l<m}\Omega_l$. It is easy to see that $\Omega_l \subset W$. It remains to check that $W \subset \cup_{0<l<m}\Omega_l$. For this, let (l_n) be a sequence in $(0, m)$ such that $l_n \to m$ as $n \to \infty$. As $V(u_-, 0) - V(s, w) > 0$ for $(s, w) \in W$ follows from the definition of limit that

$$l_n > \int_{u_+}^{s} -h(x)b(x)\,dx + \frac{w^2}{2}$$

for some $n = n(V(u_-, 0) - V(s, w))$. Thus, $(s, w) \in \Omega_{l_n}$ and the identity $W = \cup_{0<l<m}\Omega_l$ holds. For $u_+ < u < u_-$ we have

$$V(u, 0) = \int_{u_+}^{u} -h(x)b(x)\,dx < \int_{u_+}^{u_-} -h(x)b(x)\,dx = V(u_-, 0) = m$$

and then $(u, 0) \in \omega$. □

5. Existence of semi-orbits

In this section we recall the basic results and concepts. The reader is referred to [11] and [5] for more details.

Definition 5.1. Let X_0 a equilibrium point of a (s,w)-planar C^r vector field $G = (G_1, G_2)$. We say that

$$DG(X_0) = \begin{pmatrix} \frac{\partial G_1}{\partial s}(X_0) & \frac{\partial G_1}{\partial w}(X_0) \\ \frac{\partial G_2}{\partial s}(X_0) & \frac{\partial G_2}{\partial w}(X_0) \end{pmatrix}$$

is the *linear part* of the vector field G at the equilibrium point X_0. The equilibrium point X_0 is called *hyperbolic* if the two eigenvalues de $DG(X_0)$ have real part different from 0.

Theorem 5.2 (The Stable Manifold Theorem [5])**.** . *Assume $A = DG(X_0)$ has eigenvalues λ_1, λ_2 with $\lambda_1 < 0 < \lambda_2$. Then there are two orbits of $X' = G(X)$ that approach X_0 as $y \to +\infty$, along a smooth curve tangent at X_0 to the eigenvectors for λ_1 and two orbits that approach X_0 as $y \to -\infty$, along a smooth curve tangent at X_0 to the eigenvectors of λ_2.*

5.1. **Semi-orbits.** We now return to the system (2.7). We recall that the equilibrium point $(u_-, 0)$ is a saddle point, so it is *hyperbolic*. It follows from Theorem 5.2 that there are two orbits of (2.7), that approach $(u_-, 0)$ as $y \to -\infty$, along a smooth curve tangent at $(u_-, 0)$ to the eigenvectors of λ_2, given in (2.11).

More specifically, according to the analysis of phase portrait close to the equilibrium point $(u-, 0)$ [see Appendix I] there are exactly two semi-orbits of system (2.7) that converge to $(u_-, 0)$ as $y \to -\infty$, one orbit approaches the $(u-, 0)$ from region $S = \{(s,w) \in \mathbb{R}^2 : s < u_-, w < 0\}$, while the other approaches from region $T = \{(s,w) \in \mathbb{R}^2 : s > u_-, w > 0\}$ (see Figure 2).

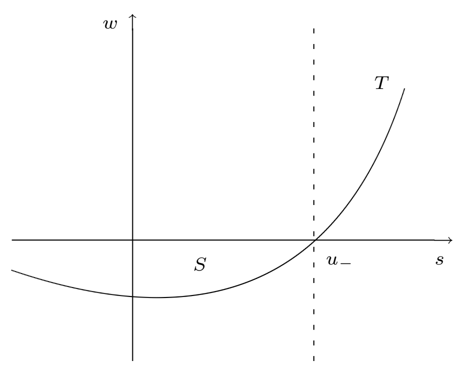

Figure 2. The two semi-orbits connection u_-

Proposition 5.3. *Let $(s(y), w(y))$ be a trajectory of system (2.7) globally defined entering the saddle point $(u_-, 0)$ from the region T as $y \to -\infty$, then such trajectory cannot tend to $(u_+, 0)$ as $y \to +\infty$.*

Proof. Multiply (2.5) by $b(s(y))s'(y)$ and integrate on $(-\infty, z)$,

$$\int_{-\infty}^{z} h(s(y)b(s(y))s'(y)\,dy = \int_{-\infty}^{z} a(s(y)b(s(y))(s'(y))^2\,dy$$

$$+\frac{1}{2}\int_{-\infty}^{z}[(b(s(y))s'(y))^2]'\,dy$$

equivalently,

$$\int_{u_-}^{s(z)} h(x)b(x)\,dx = \int_{-\infty}^{z} a(s(y)b(s(y))(s'(y))^2\,dy + \frac{1}{2}[b(s(z))s'(z)]^2. \tag{5.1}$$

Suppose now that $(s(y),w(y)) \to (u_+,0)$ as $y \to \infty$, then there is some z_0 such that $s(z_0) = u_-$. Choosing $z = z_0$ and replacing it in (5.1), we have

$$0 = \int_{-\infty}^{z_0} a(s(y)b(s(y))(s'(y))^2\,dy + \frac{1}{2}[b(u_-)s'(z_0)]^2.$$

As each term is non-negative we must have

$$\int_{-\infty}^{z_0} a(s(y)b(s(y))(s'(y))^2\,dy = 0$$

and by continuity it follows that $s'(y) = 0$ for $y \in (-\infty, z_0]$. Then $s(y) = u_-$ for $y \in (-\infty, z_0]$. Thus, $(s(z_0), w(z_0)) = (u_-, 0)$ and by uniqueness of solutions $(s(y), w(y)) = (u_-, 0)$ for every $y \in \mathbb{R}$. Therefore, $(s(y), w(y)) \to (u_-, 0)$ as $y \to \infty$, which contradicts our assumption since $u_- \neq u_+$. $\square$

We conclude that the unique semi-orbit that can tend to $(u_+, 0)$ as $y \to \infty$ is the one that enters the region S. In the next section we will prove that this semi-orbit is the traveling wave desired.

6. Existence of traveling wave solutions

Let $(s(z), w(z))$ be a solution (2.7) starting in $D \cup R$ and defined at least for values of y sufficiently negative such that

$$(s(y), w(y)) \to (u_-, 0), \quad y \to -\infty$$

and that $s(y) < u_-$, $w(y) < 0$ for values of y sufficiently negative, that corresponds to semi-orbit that approach the $(u_-, 0)$ from region S. We can assure that $(s(y), w(y))$ belong $D \cup R$ for y sufficiently negative, this follows from convergence mentioned above .

Let us multiply the second equation of (2.7) by $w(z) = b(s(z))s'(z)$ and integrate from $-\infty$ to y

$$\int_{-\infty}^{y} w(z)w'(z)\,dz = \int_{-\infty}^{y} w(z)h(s(z))\,dz - \int_{-\infty}^{y} \frac{a(s(z))}{b(s(z))}(w(z))^2\,dz. \tag{6.1}$$

Note that the second term on the right of (6.1) is non-negative. We prove that

$$I = \int_{-\infty}^{y} \frac{a(s(z))}{b(s(z))}(w(z))^2\,dz > 0.$$

Suppose, by contradiction, that $I = 0$ then, by continuity, $w(z) = 0$ for every $z \in (-\infty, y]$, so $s'(z) = 0$, which means precisely that $s(z)$ is constant for $z \in (-\infty, y]$. However, we have $s(z) \to u_-$ as $z \to -\infty$ it then follows that $s(z) = u_-$ on $(-\infty, y]$ and thus $(s(y), w(y)) = (u_-, 0)$ which contradicts the uniqueness of solutions. From this fact it follows that

$$\frac{(w(y))^2}{2} < \int_{-\infty}^{y} b(s(z))s'(z)h(s(z))\,dz = \int_{s(y)}^{u_-} -b(x)h(x)\,dx\,. \tag{6.2}$$

We can write

$$\int_{s(y)}^{u_-} -b(x)h(x)\,dx = \int_{u_+}^{u_-} -b(x)h(x)\,dx - \int_{u_+}^{s(y)} -b(x)h(x)\,dx$$

so rewriting (6.2) we have

$$\int_{u_+}^{s(y)} -b(x)h(x)\,dx + \frac{w^2(y)}{2} < \int_{u_+}^{u_-} -b(x)h(x)\,dx.$$

Thus,

$$V(s(y), w(y)) < V(u_-, 0).$$

Consequently, we have $(s(y), w(y)) \in W$, and

$$(s(z), w(z)) \to (u_+, 0) \quad \text{as } z \to \infty.$$

Using the fact that $(s(y), w(y)) \in W$, then $(s(y), w(y)) \in \Omega_l$, for some $l \in (0, m)$. We formulate the Cauchy problem

$$\begin{gathered} X'(t) = F(X(t)) \\ X(y) = (s(y), w(y)). \end{gathered} \tag{6.3}$$

Thus, $(s(t), w(t))$ is a solution and $(s(t), w(t)) \in \Omega_l$, for $t \geq y$, recalling once Ω_l is invariant positively. Moreover,

$$(s(t), w(t)) \to (u_+, 0) \quad \text{as } t \to \infty\,,$$

recalling once again the fact $(s(y), w(y))$ belong to domain of attraction of the equilibrium point $(u_+, 0)$.

Therefore, we prove the existence of a single orbit (up to translation of the independent variable) connecting the states $(u_\pm, 0)$ with flux function $f : \mathbb{R} \to \mathbb{R}$ de class C^1. We then checked our main result.

Theorem 6.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Suppose there is a weak solution connecting the states $u_\pm$ with constant speed c, given by* (2.6)*. Further, assume* (2.8) *and $f'(u_+) < c < f'(u_-)$. Then there is (up to translation of the independent variable y) a unique solution $s(y)$ of* (2.3) *satisfying* (2.2)*.*

7. Appendix I

Here we will carry out an analysis of the phase portrait close to the equilibrium point $(u-, 0)$. Let us eliminate the possibilities for the behavior of semi-orbits near the saddle point.

Case 1: If $w(y) > 0$ and $s(y) < u_-$ for $y \in (-\infty, 0] = I$, then of (2.7) we have $s'(y) > 0$ in I, so $s(y)$ is increscent. Consequently, $\lim_{y\to-\infty} s(y) \neq u_-$. Therefore $(s(y), w(y)) \nrightarrow (u_-, 0)$ as $y \to -\infty$ and the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s < u_-, w > 0\}$.

Case 2: If $w(y) < 0$ and $s(y) > u_-$ for $y \in (-\infty, 0] = I$, then of (2.7) we have $s'(y) < 0$ in I, so $s(y)$ is decreasing in I. Consequently, $\lim_{y\to-\infty} s(y) \neq u_-$. Therefore, the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s > u_-, w < 0\}$.

Case 3: If $w(y) = 0$ and $s(y) > u_-$ in I. It follows from (2.7) that $s'(y) = 0$ in I, so $s(y)$ is constant. If $\lim_{y\to-\infty} s(y) = u_-$, we have $s(y) = u_-$, contradiction. Therefore the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s > u_-, w = 0\}$.

Case 4: If $w(y) = 0$ and $s(y) < u_-$ in I, is entirely analogous to the previous item. Therefore, the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s < u_-, w = 0\}$.

Case 5: If $s(y) = u_-$ for $y \in (-\infty, 0] = I$, is similar. Therefore the semi-orbits do not tend to $(u_-, 0)$ the region $S = \{(s, w) \in \mathbb{R}^2 : s = u_-, w = 0\}$.

Case-possible: Theorem 5.2 ensures that there are two semi-orbit along a smooth curve tangent in $(u_-, 0)$ to the eigenvectors of λ_2 (up to translation). Therefore, we have a semi-orbit tending to the point $(u_-, 0)$ of the region $S = \{(s, w) \in \mathbb{R}^2 : s < u_-, w < 0\}$, while the other approaches from region $T = \{(s, w) \in \mathbb{R}^2 : s > u_-, w > 0\}$.

8. Appendix II

We consider an partial differential equations

$$u_t + uu_x = au_{xx} + bu_{xxx} \tag{8.1}$$

where $a > 0, b > 0$. We seek existence of traveling waves solution $u(x, t) = s(x - ct)$, for some constant speed $c \in \mathbb{R}$, satisfying the following conditions at infinity:

$$\lim_{y\to\pm\infty} s^{(j)}(y) = 0,\ j = 1, 2 \quad \text{and} \quad \lim_{y\to\pm\infty} s(y) = u_\pm,\ u_- \neq u_+. \tag{8.2}$$

Following [3], the following problem is equivalent to (8.1)-(8.2). Now consider

$$u_t + uu_x = au_{xx} + bu_{xxx} \tag{8.3}$$

where $a > 0, b > 0$, and the new conditions at infinity are

$$\lim_{y\to\pm\infty} s^{(j)}(y) = 0,\ j = 1, 2, \quad \lim_{y\to-\infty} s(y) = 2\eta > 0 \quad \text{and} \quad \lim_{y\to\infty} s(y) = 0. \tag{8.4}$$

Now we must determine p as in (4.2); i.e, obtain p such that

$$\int_p^{2\eta} bh(x)\,dx > 0 \tag{8.5}$$

with $-2\eta < p < 0$ and $h(x) = x^2 - 2\eta x$. As we have $b > 0$ just take

$$\int_p^{2\eta} h(x)\,dx = 1/3(p - 2\eta)(-p^2 + p\eta + 2\eta^2) > 0. \tag{8.6}$$

So choosing $p \in (-2\eta, -\eta)$ we have

$$\int_p^{2\eta} bh(x)\,dx > 0. \tag{8.7}$$

Therefore we are able to apply our work and we have existence and uniqueness of traveling waves when the flux-function associated is given by $f(u) = u^2/2$.

For the BBM-Burgers equation

$$u_t + f(u)_x = au_{xx} + bu_{xxt} \tag{8.8}$$

where $a > 0$ and $b \in \mathbb{R}$, we establish the classification of equilibrium points as in section 3.1 keeping $c > 0$ and the sign of b was studied separately to establish the desired connection. When $c < 0$ proceed in the same manner. For $b = 0$ we refer to [10] for existence.

For (8.8) the associated system becomes

$$
\begin{aligned}
s'(y) &= \frac{w(y)}{-bc} \\
w'(y) &= h(s(y)) - \frac{a}{-bc} w(y)
\end{aligned} \tag{8.9}
$$

and $V(s,w) = -\int_{u_+}^{s} -bch(x)\,dx + \frac{w^2}{2}$.

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