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INTERVAL OSCILLATION CRITERIA FOR SECOND-ORDER FORCED DELAY DIFFERENTIAL EQUATIONS UNDER IMPULSE EFFECTS

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Abstract. We establish some oscillation criteria for a forced second-order differential equation with impulses. These results extend some well-known results for forced second-order impulsive differential equations with delay.

1. Introduction

In this article, we consider the second-order impulsive delay differential equation

$$\begin{gathered}(p(t)x'(t))' + q(t)x(t-\tau) + \sum_{i=1}^{n} q_i(t)\Phi_{\alpha_i}(x(t-\tau)) = e(t), \quad t \ge t_0,\ t \ne t_k,\\ x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k), \quad k = 1, 2, \dots,\end{gathered} \tag{1.1}$$

where

$$x(t_k^-) := \lim_{t\to t_k^-} x(t), \quad x(t_k^+) := \lim_{t\to t_k^+} x(t),$$

$$x'(t_k^-) := \lim_{h\to 0^-} \frac{x(t_k+h) - x(t_k)}{h}, \quad x'(t_k^+) := \lim_{h\to 0^+} \frac{x(t_k+h) - x(t_k)}{h},$$

$\Phi_\alpha(s) := |s|^{\alpha-1}s$, $0 \le t_0 < t_1 < \dots < t_k < \dots$, $\lim_{k\to\infty} t_k = \infty$, the exponents of the nonlinearities satisfy

$$\alpha_1 > \alpha_2 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0,$$

the functions p, q, q_i, e are piecewise left continuous at each t_k, more precisely, they belong to the set

$$\begin{gathered}PLC[t_0,\infty) := \big\{h : [t_0,\infty) \to \mathbb{R}|,\ h \text{ is continuous on each interval } (t_k, t_{k+1}),\\ h(t_k^\pm) \text{ exists, and } h(t_k) = h(t_k^-) \text{ for all } k \in \mathbb{N}\big\},\end{gathered}$$

and $p > 0$ is a nondecreasing function.

A function $x \in PLC[t_0,\infty)$ is said to be a solution of (1.1) if $x(t)$ satisfies (1.1), and $x(t)$ and $x'(t)$ are left continuous at every t_k, $k \in \mathbb{N}$.

In the past few decades, there has been a great deal of work on the oscillatory behavior of the solutions of second order differential equations, see [2, 13] and the

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references cited therein. Impulsive differential equations are an effective tool for the simulation of processes and phenomena observed in control theory, population dynamics, economics, etc. Research in this direction was initiated by Gopalsamy and Zhang in [6]. Since then there has been an increasing interest in finding the oscillation criteria for such equations, see [1, 3, 4, 8, 9, 10, 11] their references.

Liu and Xu [8], obtained several oscillation theorems for the equation

$$\begin{gathered} (r(t)x'(t))' + p(t)|x(t)|^{\alpha-1}x(t) = q(t), \quad t \ge t_0,\ t \ne t_k, \\ x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k), \quad x(t_0^+) = x_0, \quad x'(t_0^+) = x_0', \end{gathered} \tag{1.2}$$

which is a special case of (1.1).

More recently, Guvenilir [7] established interval criteria for the oscillation of second-order functional differential equations with oscillatory potentials for the equation

$$(k(t)x'(t))' + p(t)x(g(t)) + q(t)|x(g(t))|^{\gamma-1}x(g(t)) = e(t), \quad t \ge 0. \tag{1.3}$$

We note that when $g(t) = t - \tau$, this equation is included in (1.1).

In this article, some new sufficient conditions for the oscillation of solutions of (1.1) are presented, and illustrated by an example. It should be noted that the derivation in this work adopts new estimates which are not a routine extension of the existing techniques used for the non-delay case.

As is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise the solution is said to be non-oscillatory.

2. Main Results

We will assume the following three conditions throughout this article.

- (H1) $\tau \ge 0$, $b_k, a_k > 0$, $t_{k+1} - t_k > \tau$, $k = 1, 2, \dots$; $\alpha_1 > \alpha_2 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0$, $(n > m \ge 1)$.
- (H2) $p, q, q_i, e \in PLC[t_0, \infty)$.
- (H3) For any $T \ge 0$, there exist intervals $[c_1, d_1]$ and $[c_2, d_2]$ contained in $[T, \infty)$ such that $c_1 < d_1 \le d_1 + \tau \le c_2 < d_2$, $c_j, d_j \notin \{t_k\}$, $j = 1, 2$, $k = 1, 2, \dots$ and

$$\begin{gathered} q(t) \ge 0, \quad q_i(t) \ge 0 \quad \text{for } t \in [c_1 - \tau, d_1] \cup [c_2 - \tau, d_2], \quad i = 1, 2, \dots, n; \\ e(t) \le 0 \quad \text{for } t \in [c_1 - \tau,\, d_1]; \\ e(t) \ge 0 \quad \text{for } t \in [c_2 - \tau,\, d_2]. \end{gathered}$$

Denote

$$\begin{gathered} I(s) := \max\{i : t_0 < t_i < s\}; \quad \beta_j := \max\{p(t) : t \in [c_j, d_j]\}, \quad j = 1, 2; \\ \Omega := \{\omega \in C^1[c_j, d_j] : \omega(t) \not\equiv 0,\ \omega(c_j) = \omega(d_j) = 0,\ j = 1, 2\}; \\ \Gamma := \big\{G \in C^1[c_j, d_j] : G \ge 0,\ G \not\equiv 0,\ G(c_j) = G(d_j) = 0, \\ G'(t) = 2g(t)\sqrt{G(t)},\ j = 1, 2\big\}. \end{gathered}$$

Before giving the main results, we introduce the following Lemma.

Lemma 2.1 ([12]). *For any n-tuple $(\alpha_1, \dots, \alpha_n)$ satisfying $\alpha_1 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0$, there exists an n-tuple $(\eta_1, \dots, \eta_n)$ satisfying either (a)*

$$\sum_{i=1}^n \alpha_i \eta_i = 1, \quad \sum_{i=1}^n \eta_i < 1, \quad 0 < \eta_i < 1,$$

or (b)

$$\sum_{i=1}^{n}\alpha_i\eta_i=1,\quad \sum_{i=1}^{n}\eta_i=1,\quad 0<\eta_i<1.$$

Theorem 2.2. *Assume that conditions* (H1)–(H3) *hold, and that there exists* $\omega\in\Omega$ *such that*

$$\begin{aligned}
&\int_{c_j}^{d_j} p(t)\omega'^2(t)dt-\int_{c_j}^{t_{I(c_j)+1}} Q(t)Q^j_{I(c_j)}(t)\omega^2(t)dt\\
&-\sum_{k=I(c_j)+2}^{I(d_j)}\int_{t_{k-1}}^{t_k} Q(t)Q^j_k(t)\omega^2(t)dt-\int_{t_{I(d_j)}}^{d_j} Q(t)Q^j_{I(d_j)}(t)\omega^2(t)dt\\
&\le L(\omega,c_j,d_j),
\end{aligned}\tag{2.1}$$

where $L(\omega,c_j,d_j):=0$ *for* $I(c_j)=I(d_j)$, *and*

$$\begin{aligned}
&L(\omega,c_j,d_j)\\
&:=\beta_j\big\{\omega^2(t_{I(c_j)+1})\frac{a_{I(c_j)+1}-b_{I(c_j)+1}}{a_{I(c_j)+1}(t_{I(c_j)+1}-c_j)}+\sum_{k=I(c_j)+2}^{I(d_j)}\omega^2(t_k)\frac{a_k-b_k}{a_k(t_k-t_{k-1})}\big\}
\end{aligned}$$

for $I(c_j)<I(d_j)$, $j=1,2,$

$$\begin{gathered}
Q(t):=q(t)+\eta_0^{-\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}q_i^{\eta_i}(t)|e(t)|^{\eta_0},\\
Q^j_k(t):=\begin{cases}\frac{t-t_k}{a_k\tau+b_k(t-t_k)}, & t\in(t_k,t_k+\tau),\\ \frac{t-t_k-\tau}{t-t_k}, & t\in[t_k+\tau,t_{k+1}],\end{cases}\\
k=I(c_j),I(c_j)+1,\dots,I(d_j),
\end{gathered}$$

and $\eta_1,\eta_2,\dots,\eta_n$ *are positive constants satisfying part (a) of Lemma 2.1. Then all solutions of* (1.1) *are oscillatory.*

Proof. Suppose that $x(t)$ is a non-oscillatory solution of (1.1). By re-defining t_0 if necessary, without loss of generality, we may assume that $x(t-\tau)>0$ for all $t\ge t_0>0$. Define the Riccati transformation

$$v(t):=\frac{p(t)x'(t)}{x(t)}.$$

It follows from (1.1) that $v(t)$ satisfies

$$v'(t)=-q(t)\frac{x(t-\tau)}{x(t)}-\sum_{i=1}^{n}q_i(t)|x(t-\tau)|^{\alpha_i-1}\frac{x(t-\tau)}{x(t)}+\frac{e(t)}{x(t)}-\frac{v^2(t)}{p(t)},\tag{2.2}$$

for all $t\ne t_k,t\ge t_0$, and $v(t_k^+)=\frac{b_k}{a_k}v(t_k)$ for all $k\in\mathbb{N}$.

From the assumptions, we can choose $c_1,d_1\ge t_0$ such that $q(t)\ge 0$ and $q_i(t)\ge 0$ for $t\in[c_1-\tau,d_1]$, $i=1,2,\dots,n$, and $e(t)\le 0$ for $t\in[c_1-\tau,d_1]$. By Lemma 2.1, there exist $\eta_i>0$, $i=1,\dots,n$, such that $\sum_{i=1}^n\alpha_i\eta_i=1$ and $\sum_{i=1}^n\eta_i<1$. Define $\eta_0:=1-\sum_{i=1}^n\eta_i$ and let

$$u_0:=\eta_0^{-1}\Big|\frac{e(t)x(t-\tau)}{x(t)}\Big|x^{-1}(t-\tau),$$

$$u_i := \eta_i^{-1} q_i(t) \frac{x(t-\tau)}{x(t)}\, x^{\alpha_i - 1}(t-\tau), \quad i = 1, 2, \dots, n\,.$$

Then by the arithmetic-geometric mean inequality (see [5]), we have

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}$$

and so

$$\begin{aligned} v'(t) \leq -\eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \frac{x^{\eta_i}(t-\tau)}{x^{\eta_i}(t)} x^{(\alpha_i - 1)\eta_i}(t-\tau) |e(t)|^{\eta_0} \\ \times \frac{x^{\eta_0}(t-\tau)}{x^{\eta_0}(t)} x^{-\eta_0}(t-\tau) - q(t)\frac{x(t-\tau)}{x(t)} - \frac{v^2(t)}{p(t)}, \quad t \neq t_k. \end{aligned} \tag{2.3}$$

Since

$$\prod_{i=0}^{n} \frac{x^{\eta_i}(t-\tau)}{x^{\eta_i}(t)} = \frac{x^{\eta_0+\eta_1+\dots+\eta_n}(t-\tau)}{x^{\eta_0+\eta_1+\dots+\eta_n}(t)} = \frac{x(t-\tau)}{x(t)}\,,$$

$$\prod_{i=1}^{n} x^{(\alpha_i-1)\eta_i}(t-\tau) x^{-\eta_0}(t-\tau) = 1\,,$$

we obtain

$$\begin{aligned} v'(t) &\leq -q(t)\frac{x(t-\tau)}{x(t)} - \eta_0^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i} q_i^{\eta_i}(t) \frac{x(t-\tau)}{x(t)} |e(t)|^{\eta_0} - \frac{v^2(t)}{p(t)} \\ &= -Q(t)\frac{x(t-\tau)}{x(t)} - \frac{v^2(t)}{p(t)}, \quad t \neq t_k. \end{aligned} \tag{2.4}$$

Multiply both sides of (2.4) by $\omega^2(t)$, with w as prescribed in the hypothesis of the theorem. Then integrate from c_1 to d_1; using integration by parts on the left side, we have

$$\begin{aligned} &\sum_{k=I(c_1)+1}^{I(d_1)} \omega^2(t_k)[v(t_k) - v(t_k^+)] \\ &\leq 2\int_{c_1}^{d_1} \omega(t)\omega'(t)v(t)dt - \int_{c_1}^{d_1} \omega^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt - \int_{c_1}^{d_1} \frac{v^2(t)\omega^2(t)}{p(t)}dt \\ &= 2\int_{c_1}^{d_1} \omega(t)\omega'(t)v(t)dt - \int_{c_1}^{t_{I(c_1)+1}} \omega^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt \\ &\quad - \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} \omega^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt \\ &\quad - \int_{t_{I(d_1)}}^{d_1} \omega^2(t)Q(t)\frac{x(t-\tau)}{x(t)}dt - \int_{c_1}^{d_1} \frac{v^2(t)\omega^2(t)}{p(t)}dt. \end{aligned} \tag{2.5}$$

To estimate $\frac{x(t-\tau)}{x(t)}$, we first consider the situation where $I(c_1) < I(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{I(c_1)+1},\ t_{I(c_1)+2}, \dots,\ t_{I(d_1)}$.

Case (1). $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$.

(i) If $t \in [t_k + \tau, t_{k+1}]$, then $t - \tau \in [t_k, t_{k+1} - \tau]$. Since $t_{k+1} - t_k > \tau$, there are no impulsive moments in $(t - \tau, t)$. As in the proof of [1, Lemma 2.4], we have

$$x(t) > x(t) - x(t_k^+) = x'(\xi)(t - t_k), \quad \xi \in (t_k,\ t).$$

Since the function $p(t)x'(t)$ is nonincreasing,

$$x(t) > x'(\xi)(t - t_k) > \frac{p(t)x'(t)}{p(\xi)}(t - t_k).$$

From the fact that $p(t)$ is nondecreasing, we have

$$\frac{p(t)x'(t)}{x(t)} < \frac{p(\xi)}{t - t_k} < \frac{p(t)}{t - t_k}.$$

We obtain $\frac{x'(t)}{x(t)} < \frac{1}{t-t_k}$. Upon integrating from $t-\tau$ to t, we obtain $\frac{x(t-\tau)}{x(t)} > \frac{t-t_k-\tau}{t-t_k}$.

(ii) If $t \in (t_k, t_k + \tau)$, then $t - \tau \in (t_k - \tau, t_k)$, and there is an impulsive moment t_k in $(t - \tau, t)$. Similar to (i), we obtain $\frac{x'(s)}{x(s)} < \frac{1}{s-t_k+\tau}$ for $s \in (t_k - \tau, t_k]$. Upon integrating from $t-\tau$ to t_k, we obtain $\frac{x(t-\tau)}{x(t_k)} > \frac{t-t_k}{\tau}$. Since $x(t)-x(t_k^+) < x'(t_k^+)(t-t_k)$, we have

$$\frac{x(t)}{x(t_k^+)} < 1 + \frac{x'(t_k^+)}{x(t_k^+)}(t - t_k) = 1 + \frac{b_k x'(t_k)}{a_k x(t_k)}(t - t_k)\,.$$

Using $\frac{x'(t_k)}{x(t_k)} < \frac{1}{\tau}$ and $x(t_k^+) = a_k x(t_k)$, this implies

$$\frac{x(t_k)}{x(t)} > \frac{\tau}{a_k\tau + b_k(t - t_k)}\,.$$

Therefore,

$$\frac{x(t - \tau)}{x(t)} > \frac{t - t_k}{a_k\tau + b_k(t - t_k)}.$$

Case (2). $t \in [c_1, t_{I(c_1)+1}]$. We consider three sub-cases.

(i) If $t_{I(c_1)} > c_1 - \tau$, $t \in [t_{I(c_1)} + \tau, t_{I(c_1)+1}]$, then there are no impulsive moments in $(t - \tau, t)$. Making a similar analysis of case 1(i), we obtain $\frac{x(t-\tau)}{x(t)} > \frac{t-\tau-t_{I(c_1)}}{t-t_{I(c_1)}}$.

(ii) If $t_{I(c_1)} > c_1 - \tau$, $t \in [c_1, t_{I(c_1)} + \tau)$, then $t - \tau \in [c_1 - \tau,\ t_{I(c_1)})$ and there is an impulsive moment $t_{I(c_1)}$ in $(t - \tau,\ t)$. Similar to case 1(ii), we have

$$\frac{x(t - \tau)}{x(t)} > \frac{t - t_{I(c_1)}}{a_{I(c_1)}\tau + b_{I(c_1)}(t - t_{I(c_1)})}\,.$$

(iii) If $t_{I(c_1)} < c_1 - \tau$, then there are no impulsive moments in $(t - \tau, t)$. So

$$\frac{x(t - \tau)}{x(t)} > \frac{t - \tau - t_{I(c_1)}}{t - t_{I(c_1)}}\,.$$

Case (3). $t \in (t_{I(d_1)},\ d_1]$. There are three sub-cases to consider:

(i) If $t_{I(d_1)} + \tau < d_1, t \in [t_{I(d_1)} + \tau, d_1]$, then there are no impulsive moments in $(t - \tau, t)$. Similar to case 2(i), we have

$$\frac{x(t - \tau)}{x(t)} > \frac{t - \tau - t_{I(d_1)}}{t - t_{I(d_1)}}\,.$$

(ii) If $t_{I(d_1)}+\tau<d_1, t\in[t_{I(d_1)}, t_{I(d_1)}+\tau)$, then there is an impulsive moment $t_{I(d_1)}$. Similar to case 2(ii), we obtain

$$\frac{x(t-\tau)}{x(t)}>\frac{t-t_{I(d_1)}}{a_{I(d_1)}\tau+b_{I(d_1)}(t-t_{I(d_1)})}\,.$$

(iii) If $t_{I(d_1)}+\tau\ge d_1$, then there is an impulsive moment $t_{I(d_1)}$ in $(t-\tau,\,t)$. Similar to case 3(ii), we obtain

$$\frac{x(t-\tau)}{x(t)}>\frac{t-t_{I(d_1)}}{a_{I(d_1)}\tau+b_{I(d_1)}(t-t_{I(d_1)})}\,.$$

Combining all these cases, we have

$$\frac{x(t-\tau)}{x(t)}>\begin{cases}Q^1_{I(c_1)}(t), & \text{for } t\in[c_1,t_{I(c_1)+1}],\\ Q^1_k(t), & \text{for } t\in(t_k,t_{k+1}],\ k=I(c_1)+1,\dots,I(d_1)-1,\\ Q^1_{I(d_1)}(t), & \text{for } t\in(t_{I(d_1)},d_1].\end{cases}$$

Hence by (2.5), we have

$$\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)}\omega^2(t_k)[v(t_k)-v(t_k^+)]\\
&\le 2\int_{c_1}^{d_1}\omega(t)\omega'(t)v(t)dt-\int_{c_1}^{t_{I(c_1)+1}}\omega^2(t)Q(t)Q^1_{I(c_1)}(t)dt\\
&\quad-\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}}\omega^2(t)Q(t)Q^1_k(t)dt-\int_{t_{I(d_1)}}^{d_1}\omega^2(t)Q(t)Q^1_{I(d_1)}(t)dt\\
&\quad-\int_{c_1}^{d_1}\frac{v^2(t)\omega^2(t)}{p(t)}dt\\
&=-\int_{c_1}^{t_{I(c_1)+1}}\frac{1}{p(t)}[p(t)\omega'(t)-v(t)\omega(t)]^2dt\\
&\quad-\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}}\frac{1}{p(t)}[p(t)\omega'(t)-v(t)\omega(t)]^2dt\\
&\quad-\int_{t_{I(d_1)}}^{d_1}\frac{1}{p(t)}[p(t)\omega'(t)-v(t)\omega(t)]^2dt\\
&\quad+\int_{c_1}^{t_{I(c_1)+1}}[p(t)\omega'^2(t)-Q(t)Q^1_{I(c_1)}(t)\omega^2(t)]dt\\
&\quad+\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}}[p(t)\omega'^2(t)-Q(t)Q^1_k(t)\omega^2(t)]dt\\
&\quad+\int_{t_{I(d_1)}}^{d_1}[p(t)\omega'^2(t)-Q(t)Q^1_{I(d_1)}(t)\omega^2(t)]dt\,.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} \omega^2(t_k)[v(t_k)-v(t_k^+)] \\
&< \int_{c_1}^{t_{I(c_1)+1}} [p(t)\omega'^2(t)-Q(t)Q^1_{I(c_1)}(t)\omega^2(t)]dt \\
&\quad + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} [p(t)\omega'^2(t)-Q(t)Q^1_k(t)\omega^2(t)]dt \\
&\quad + \int_{t_{I(d_1)}}^{d_1} [p(t)\omega'^2(t)-Q(t)Q^1_{I(d_1)}(t)\omega^2(t)]dt\,,
\end{aligned}
\tag{2.6}
$$

for if not, we must have $p(t)\omega'(t) = v(t)\omega(t)$ or $x(t)\omega'(t) = x'(t)\omega(t)$ on $[c_1, d_1]$. Upon integrating, $x(t)$ will be a multiple of $\omega(t)$, which contradicts the facts that ω vanishes at c_1 and d_1 while $x(t)$ does not.

On the other hand, since $\big(p(t)x'(t)\big)' < 0$ for all $t \in (c_1,\, t_{I(c_1)+1}]$, $p(t)x'(t)$ is nonincreasing in $(c_1,\, t_{I(c_1)+1}]$. Thus

$$
x(t) > x(t)-x(c_1) = x'(\xi)(t-c_1) \ge \frac{p(t)x'(t)}{p(\xi)}(t-c_1), \quad \text{for some } \xi \in (c_1,\, t)\ ,
$$

and hence $\frac{p(t)x'(t)}{x(t)} < \frac{p(\xi)}{t-c_1}$. Letting $t \to t^-_{I(c_1)+1}$, we have

$$
v(t_{I(c_1)+1}) \le \frac{\beta_1}{t_{I(c_1)+1}-c_1}\,. \tag{2.7}
$$

Making a similar analysis on $(t_{k-1}, t_k]$, $k = I(c_1)+2, \dots, I(d_1)$, it is not difficult to see that

$$
v(t_k) \le \frac{\beta_1}{t_k - t_{k-1}}\,. \tag{2.8}
$$

Here we must point out that (2.7) and (2.8) play a key role in our method for estimating $v(t_j)$, which is different from the usual techniques for the case without impulses. From (2.7) and (2.8), and noting that $a_k \le b_k$, we have

$$
\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} \frac{a_k-b_k}{a_k}\omega^2(t_k)v(t_k) \\
&\ge \beta_1\Big[\sum_{k=I(c_1)+2}^{I(d_1)} \frac{a_k-b_k}{a_k(t_k-t_{k-1})}\omega^2(t_k) + \frac{a_{I(c_1)+1}-b_{I(c_1)+1}}{a_{I(c_1)+1}(t_{I(c_1)+1}-c_1)}\omega^2(t_{I(c_1)+1})\Big] \\
&= L(\omega, c_1, d_1)\,.
\end{aligned}
$$

Since

$$
\sum_{k=I(c_1)+1}^{I(d_1)} \omega^2(t_k)[v(t_k)-v(t_k^+)] = \sum_{k=I(c_1)+1}^{I(d_1)} \frac{a_k-b_k}{a_k}\omega^2(t_k)v(t_k)\,,
$$

by (2.6), we have

$$
L(\omega, c_1, d_1) < \int_{c_1}^{t_{I(c_1)+1}} [p(t)\omega'^2(t)-Q(t)Q^1_{I(c_1)}(t)\omega^2(t)]dt
$$

$$
\begin{aligned}
&+\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}}[p(t)\omega'^2(t)-Q(t)Q_k^1(t)\omega^2(t)]dt\\
&+\int_{t_{I(d_1)}}^{d_1}[p(t)\omega'^2(t)-Q(t)Q_{I(d_1)}^1(t)\omega^2(t)]dt\,,
\end{aligned}
$$

which contradicts (2.1).

If $I(c_1)=I(d_1)$, then $L(\omega,c_1,d_1)=0$, and there are no impulsive moments in $[c_1,d_1]$. Similar to the proof of (2.6), we obtain

$$
\int_{c_1}^{d_1}[p(t)\omega'^2(t)-Q(t)Q_{I(c_1)}(t)\omega^2(t)]dt>0\,. \tag{2.9}
$$

This again contradicts our assumption. Finally, if $x(t)$ is eventually negative, we can consider $[c_2,d_2]$ and reach a similar contradiction. The proof of Theorem 2.2 is complete. □

Theorem 2.3. *Assume conditions* (H1)–(H3) *hold,* $a_k\le b_k$ *and there exists a* $G\in\Gamma$ *such that*

$$
\begin{aligned}
&\int_{c_j}^{d_j}p(t)g^2(t)dt-\int_{c_j}^{t_{I(c_j)+1}}Q(t)G(t)Q_{I(c_j)}^j(t)dt\\
&-\sum_{k=I(c_j)+1}^{I(d_j)-1}\int_{t_k}^{t_{k+1}}Q(t)G(t)Q_k^j(t)dt-\int_{t_{I(d_j)}}^{d_j}Q(t)G(t)Q_{I(d_j)}^j(t)dt\\
&\le R(G,c_j,d_j)\,,
\end{aligned}
\tag{2.10}
$$

where $R(G,c_j,d_j):=0$ *for* $I(c_j)=I(d_j)$, $j=1,2$, *and*

$$
\begin{aligned}
&R(G,c_j,d_j)\\
&:=\frac{a_{I(c_j)+1}-b_{I(c_j)+1}}{a_{I(c_j)+1}(t_{I(c_j)+1}-c_1)}G(t_{I(c_j)+1})\beta_j+\sum_{k=I(c_j)+2}^{I(d_j)}\frac{a_k-b_k}{a_k}\frac{\beta_j}{t_k-t_{k-1}}G(t_k)
\end{aligned}
$$

for $I(c_j)<I(d_j)$, *then all solutions of* (1.1) *are oscillatory.*

Proof. Similar to the proof of Theorem 2.2, suppose $x(t-\tau)>0$ for $t\ge t_0$. If $I(c_1)<I(d_1)$, multiplying $G(t)$ throughout (2.4) and integrating over $[c_1,d_1]$, we obtain

$$
\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)}G(t_k)\frac{a_k-b_k}{a_k}v(t_k)\\
&\le-\int_{c_1}^{d_1}Q(t)G(t)\frac{x(t-\tau)}{x(t)}dt-\int_{c_1}^{d_1}\frac{v^2(t)G(t)}{p(t)}dt+2\int_{c_1}^{d_1}v(t)g(t)\sqrt{G(t)}dt\\
&<-\int_{c_1}^{d_1}\Big(\sqrt{\frac{G(t)}{p(t)}}v(t)-\sqrt{p(t)}g(t)\Big)^2dt+\int_{c_1}^{d_1}p(t)g^2(t)dt\\
&\quad-\int_{c_1}^{t_{I(c_1)+1}}Q(t)G(t)Q_{I(c_1)}^1(t)dt-\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}}Q(t)G(t)Q_k^1(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{t_{I(d_1)}}^{d_1} Q(t)G(t)Q^1_{I(d_1)}(t)dt \\
&\le \int_{c_1}^{d_1} p(t)g^2(t)dt - \int_{c_1}^{t_{I(c_1)+1}} Q(t)G(t)Q^1_{I(c_1)}(t)dt \\
&- \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} Q(t)G(t)Q^1_k(t)dt - \int_{t_{I(d_1)}}^{d_1} Q(t)G(t)Q^1_{I(d_1)}(t)dt\,.
\end{aligned}
$$

On the other hand, from the proof of Theorem 2.2, we have

$$
v(t_{I(c_1)+1}) \le \frac{\beta_1}{t_{I(c_1)+1} - c_1}, \quad v(t_k) \le \frac{\beta_1}{t_k - t_{k-1}}, \tag{2.11}
$$

for $k = I(c_1) + 2, \dots, I(d_1)$. So

$$
\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} \frac{a_k - b_k}{a_k} G(t_k)v(t_k) \\
&\ge \frac{a_{I(c_1)+1} - b_{I(c_1)+1}}{a_{I(c_1)+1}(t_{I(c_1)+1} - c_1)} G(t_{I(c_1)+1})\beta_1 + \sum_{k=I(c_1)+2}^{I(d_1)} \frac{a_k - b_k}{a_k} \frac{\beta_1}{t_k - t_{k-1}} G(t_k) \\
&= R(G, c_1, d_1)\,.
\end{aligned}
$$

This contradicts (2.10). If $I(c_1) = I(d_1)$, the proof is similar to that of Theorem 2.2, and so it is omitted here. The proof of Theorem 2.3 is complete. □

Next, let $D = \{(t,s) : t_0 \le s \le t\}$. A function $H \in C(D, \mathbb{R})$ is said to belong to the class $\mathfrak{H}$ if

- (A1) $H(t,t) = 0$, $H(t,s) > 0$ for $t > s$; and
- (A2) H has partial derivatives $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on D such that

$$
\frac{\partial H}{\partial t} = 2h_1(t,s)\sqrt{H(t,s)}, \quad \frac{\partial H}{\partial s} = -2h_2(t,s)\sqrt{H(t,s)}.
$$

Similar to [8, Theorem 2.3], we have the following Theorem.

Theorem 2.4. *Assume the conditions* (H1)–(H3) *hold. Suppose that there are* $\delta_j \in (c_j, d_j)$, $j = 1, 2$, *and* $H \in \mathfrak{H}$ *such that*

$$
\begin{aligned}
&\frac{1}{H(d_j,\delta_j)}\Big[\int_{\delta_j}^{d_j} Q(s)Q_j(s)H(d_j,s)ds - \int_{\delta_j}^{d_j} p(s)h_2^2(d_j,s)ds\Big] \\
&+ \frac{1}{H(\delta_j,c_j)}\Big[\int_{c_j}^{\delta_j} Q(s)Q_j(s)H(s,c_j)ds - \int_{c_j}^{\delta_j} p(s)h_1^2(s,c_j)ds\Big] \\
&> P(H, c_j, d_j)\,,
\end{aligned} \tag{2.12}
$$

where $P(H, c_j, d_j) := 0$ *for* $I(c_j) = I(d_j)$*, and*

$$
\begin{aligned}
P(H, c_j, d_j) := \frac{\beta_j}{H(d_j, \delta_j)} \Big(& H(d_j, t_{I(\delta_j)+1}) \frac{b_{I(\delta_j)+1} - a_{I(\delta_j)+1}}{a_{I(\delta_j)+1}(t_{I(\delta_j)+1} - \delta_j)} \\
& + \sum_{i=I(\delta_j)+2}^{I(d_j)} H(d_j, t_i) \frac{b_i - a_i}{a_i(t_i - t_{i-1})} \Big) \\
& + \frac{\beta_j}{H(\delta_j, c_j)} \Big(H(t_{I(c_j)+1}, c_j) \frac{b_{I(c_j)+1} - a_{I(c_j)+1}}{a_{I(c_j)+1}(t_{I(c_j)+1} - c_j)} \\
& + \sum_{i=I(c_j)+2}^{I(\delta_j)} H(t_i, c_j) \frac{b_i - a_i}{a_i(t_i - t_{i-1})} \Big)
\end{aligned} \tag{2.13}
$$

for $I(c_j) < I(d_j)$*,* $j = 1, 2$*. Then all solutions of* (1.1) *are oscillatory.*

Example 2.5. *Consider the impulsive differential equation*

$$
\begin{gathered}
x''(t) + m\cos(t/2)x(t - \frac{\pi}{8}) + 8\cos(t/2)|x(t - \frac{\pi}{8})|^{\frac{3}{2}} x(t - \frac{\pi}{8}) \\
+ \cos^3 \frac{t}{2} |x(t - \frac{\pi}{8})|^{-\frac{1}{2}} x(t - \frac{\pi}{8}) = \sin \frac{t}{2}, \quad t \neq 2k\pi \pm \frac{\pi}{4}, \\
x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = a_k x'(t_k), \quad t_k = 2k\pi \pm \frac{\pi}{4}.
\end{gathered} \tag{2.14}
$$

In this equation, $\tau = \pi/8$, $t_{k+1} - t_k \geq \pi/2 > \pi/8$, $\alpha_1 = 5/2$, $\alpha_2 = 1/2$, and m is a positive constant. For any $T > 0$, we can choose k large enough such that $T < c_1 = 4k\pi - \frac{\pi}{2} < d_1 = 4k\pi$ and $c_2 = 4k\pi + \frac{\pi}{8} < d_2 = 4k\pi + \frac{\pi}{2}$ satisfy (H3), then there is an impulsive moment $t_k = 4k\pi - \frac{\pi}{4}$ in $[c_1, d_1]$ and an impulsive moment $t_{k+1} = 4k\pi + \frac{\pi}{4}$ in $[c_2, d_2]$. Let $\omega(t) = \sin(8t) \in \Omega_\omega(c_j, d_j)$, $j = 1, 2$, we have

$$
\int_{c_1}^{d_1} (\omega'(t))^2 dt = 32 \int_{c_1}^{d_1} (\cos 16t + 1) dt = 16\pi, \tag{2.15}
$$

$t_{I(c_1)} = 4k\pi - \frac{7}{4}\pi$, $t_{I(d_1)} = 4k\pi - \frac{\pi}{4}$. Choose $\eta_0 = \eta_1 = \eta_2 = 1/3$. Then

$$
\begin{aligned}
Q(t) &= m\cos(t/2) + \big[(\frac{1}{3})^{-1/3}\big]^3 \big(8\cos(t/2)\big)^{1/3} \cos(t/2)\ \big|\sin \frac{t}{2}\big|^{1/3} \\
&= \cos(\frac{t}{2})(m - 3\sin^{1/3} t) \\
&\geq m\cos(t/2).
\end{aligned} \tag{2.16}
$$

Hence

$$
\begin{aligned}
& \int_{4k\pi - \frac{\pi}{2}}^{4k\pi - \frac{\pi}{4}} Q(t) \frac{t - \frac{\pi}{8} - t_{I(c_1)}}{t - t_{I(c_1)}} \sin^2(8t) dt \\
& + \int_{4k\pi - \frac{\pi}{4}}^{4k\pi - \frac{\pi}{8}} Q(t) \frac{t - t_{I(d_1)}}{a_{I(d_1)}(t + \frac{\pi}{8} - t_{I(d_1)})} \sin^2(8t) dt \\
& + \int_{4k\pi - \frac{\pi}{8}}^{4k\pi} Q(t) \frac{t - \frac{\pi}{8} - t_{I(d_1)}}{t - t_{I(d_1)}} \sin^2(8t) dt \\
& > \frac{9}{10} m \int_{4k\pi - \frac{\pi}{2}}^{4k\pi - \frac{\pi}{4}} \cos(t/2) \sin^2(8t) dt > 16\pi
\end{aligned} \tag{2.17}
$$

for m large enough. On the other hand, note that $a_k = b_k > 0$, so that $L(\omega, c_j, d_j) = 0$. It follows from Theorem 2.2 that all the solutions of (2.14) are oscillatory.

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