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WAVE-BREAKING PHENOMENA AND GLOBAL SOLUTIONS FOR PERIODIC TWO-COMPONENT DULLIN-GOTTWALD-HOLM SYSTEMS

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Abstract. In this article we study the initial-value problem for the periodic two-component b-family system, including a special case, when $b = 2$, which is referred to as the two-component Dullin-Gottwald-Holm (DGH) system. We first show that the two-component b-family system can be derived from the theory of shallow-water waves moving over a linear shear flow. Then we establish several results of blow-up solutions corresponding to only wave breaking with certain initial profiles for the periodic two-component DGH system. Moreover, we determine the exact blow-up rate and lower bound of the lifespan for the system. Finally, we give a sufficient condition for the existence of the strong global solution to the periodic two-component DGH system.

1. Introduction

In recent years, Degasperis, Holm and Hone [22] (see also [33]) studied the following nonlinear b-family equation (up to a rescaling, shift and Galilean's transformation),

$$m_t - Au_x + um_x + bu_x m + \gamma u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where $m = u - \alpha^2 u_{xx}$. One can rewrite equation (1.1) in terms of $u(x,t)$ as follows:

$$u_t - \alpha^2 u_{xxt} - Au_x + (b+1)uu_x + \gamma u_{xxx} = \alpha^2 (bu_x u_{xx} + uu_{xxx}), \quad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

This equation can be regarded as a model of water waves by using asymptotic expansions directly in the Hamiltonian for Euler's equation in the shallow water regime [20, 33], where $u(t,x)$ stands for the horizontal velocity of the fluid, m is the momentum density, and A is a nonnegative parameter related to the critical shallow water speed. The real dimensionless constant b is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching, it is also the number of covariant dimensions associated with the momentum density m.

It is believed that the Korteweg-de Vries (KdV) equation ($\alpha = 0$ and $b = 2$), the Camassa-Holm (CH) equation ($b = 2$) [4, 26] (when $b = 2$ and $\gamma \neq 0$, it is also referred to as the Dullin-Gottwald-Holm (DGH) equation [4, 20]), and the

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Degasperis-Procesi (DP) equation ($b = 3$) [23] are the only three integrable equations in the b-family equation (1.2) [20, 21, 22, 23, 33, 34]. When $A = \gamma = 0$, (1.2) admits not only the peakon solutions for any b of the form $u(t,x) = ce^{-|x-ct|}$, $c \in \mathbb{R}$, but also multipeakon solutions [1, 22, 33] (see also [6] for the case of existence of infinite many peakons) defined by

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|},$$

where the canonical positions q_j and momenta p_j (with $j = 1, \dots, N$) satisfy the following system of ordinary differential equations with discontinuous right-hand side.

$$p_j' = (b-1) \sum_{k=1}^{N} p_j p_k sgn(q_j - q_k) e^{-|q_j - q_k|}$$

and

$$q_j' = \sum_{k=1}^{N} p_k e^{-|q_j - q_k|}.$$

If $\alpha = 0$ and $b = 2$, equation (1.2) becomes the well-known KdV equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. Its solitary waves are solitons. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global existence theory is now in hand [45]. It is observed that the KdV equation does not accommodate wave breaking (by wave breaking we understand that the wave profile remains bounded while its slope becomes unbounded in finite time [47]).

When $b = 2$ and $\gamma = 0$, equation (1.2) recovers the standard CH equation, modeling the unidirectional propagation of shallow water waves over a flat bottom [4, 13, 26]. The CH equation is also a model for the propagation of axially symmetric waves in the hyperelastic rods [19]. Its solitary waves are smooth if $A > 0$ and peaked in the limiting case $A = 0$ [4, 5, 6]. Recently, it was claimed in [38] that the CH equation might be relevant to the modeling of tsunami.

If $b = 3$ and $A = \gamma = 0$ in equation (1.2), then it recovers the DP equation. The DP equation can be also regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as the CH equation [13]. The formal integrability of the DP equation was obtained in [22] by constructing a Lax pair. It has a bi-Hamiltonian structure. The DP equation has not only peaked solitons and periodic peaked solitons, but also shock peakons [43] and the periodic shock waves [25].

The CH and DP equations have global strong solutions and also blow-up solutions in finite time, for instance, see [7, 9, 10, 14, 25, 40, 41, 42] and references therein, with a different class of initial profiles in the Sobolev spaces $H^s(\mathbb{R})$, $s > 3/2$. It is shown in [2] and [3] that solutions of the CH equation can be uniquely continued after breaking as either global conservative or global dissipative weak solutions. The advantage of the CH and DP equations in comparison with the KdV equation lies in the fact that the CH and DP equations have peaked solitons and models wave breaking. Wave breaking is one of the most intriguing long-standing problems of water wave theory [47]. The peaked solitons are the presence of solutions in the form of peaked solitary waves or "peakons" [4, 5, 6, 23]

$u(t,x) = ce^{-|x-ct|}, c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. The peakons replicate a feature that is characteristic for the waves of great height-waves of the largest amplitude that are exact solutions of the governing equations for water waves [8, 46, 11]. These peakons are shown to be stable [15, 16, 39].

The interest in the b-family equation inspired the search for various generalizations of this equation. The following two-component integrable Camassa-Holm system was first derived in [44] and can be viewed as a model in the context of shallow water theory [12, 35],

$$\begin{gathered} m_t - Au_x + um_x + 2u_x m + \rho\rho_x = 0, \\ m = u - u_{xx}, \\ \rho_t + (u\rho)_x = 0, \end{gathered} \tag{1.3}$$

where $\rho(t,x)$ is related to the free surface elevation from equilibrium(or scalar density), and the parameter A characterizes a linear underlying shear flow. Obviously, if $\rho = 0$, then (1.3) becomes the CH equation. Many recent works are devoted in studying system (1.3) (see, for instance, [12, 24, 27, 29, 30, 31, 32, 35, 49] and references therein).

In the presence of a linear shear flow and nonzero vorticity, we will follow Ivanov's approach [35] to derive the following two-component b-family system with any $b \neq -1$.

$$\begin{gathered} m_t - Au_x + um_x + bu_x m + \gamma u_{xxx} + \rho\rho_x = 0, \\ m = u - u_{xx}, \\ \rho_t + (u\rho)_x = 0. \end{gathered} \tag{1.4}$$

Note when $\rho = 0$, we recover the b-family equation (1.1). In terms of u and ρ, we obtain the equivalent form of system (1.4); that is,

$$\begin{gathered} u_t - u_{txx} - Au_x + (b+1)uu_x - bu_x u_{xx} - uu_{xxx} + \gamma u_{xxx} + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{gathered} \tag{1.5}$$

with the boundary assumptions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$.

Note that when $b = 2$, equation(1.5) is the two-component Camassa-Holm system, which has the bi-Hamiltonian structure and complete integrability via the inverse scattering transform method. It can be written as compatibility conditions of two linear systems (Lax pair) with a spectral parameter ξ, that is

$$\begin{gathered} \Psi_{xx} = \Big(-\xi^2\rho^2 + \xi\Big(m - \frac{A}{2} + \frac{\gamma}{2}\Big) + \frac{1}{4}\Big)\Psi, \\ \Psi_t = \Big(\frac{1}{2\xi} - u + \gamma\Big)\Psi_x + \frac{1}{2}u_x\Psi. \end{gathered}$$

Moreover, this system has the following two Hamiltonians

$$E(u,\rho) = \frac{1}{2}\int \big(u^2 + u_x^2 + (\rho-1)^2\big)\,dx$$

and

$$F(u,\rho) = \frac{1}{2}\int \big(u^3 + uu_x^2 - Au^2 - \gamma u_x^2 + 2u(\rho-1) + u(\rho-1)^2\big)\,dx.$$

The goal of this article is to study the initial-value problem for the periodic two-component b-family system, including a special case, $b = 2$, which is the two-component DGH system. We first derive the two-component b-family system from the shallow-water wave theory. Then we establish several results of blow-up solutions corresponding to only wave breaking with certain initial profiles for the periodic two-component DGH system. The difficulty to deal with blow-up solutions is that there is no uniform characteristics for this system. In this case, we make use of the different diffeomorphism of the trajectory q_2 defined in (4.4), which captures the maximum/minimum of u_x. Therefore the transport equation for ρ can coincide with the equation for u.

The rest of this paper is organized as follows. In Section 2, we follow the modeling approach in [35] to derive the two-component b-family system. Then applying Kato's semigroup theory, we establish the result of local well-posedness for the two component b-family system in Section 3. In Section 4, we analyze the wave-breaking phenomenon of the periodic two-component DGH system and give the precise blow-up scenarios and several wave-breaking data. In addition, we determine the blow-up rate and low bound of the lifespan. In the last section, we provide a sufficient condition for the existence of global solution.

Notation. Throughout this paper, we identity periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$.

2. Derivation of the model

Following Ivanov's approach in [35] , we consider the motion of an inviscid incompressible fluid with a constant density ϱ governed by the Euler equations

$$
\begin{gathered}
\vec{v}_t + (\vec{v}\cdot\nabla)\,\vec{v} = -\frac{1}{\varrho}\nabla P + \vec{g},\\
\nabla\cdot\vec{v} = 0,
\end{gathered}
$$

where $\vec{v}(t,x,y,z)$ is the velocity of the fluid, $P(t,x,y,z)$ is the pressure and $\vec{g} = (0,0,-g)$ is the gravity acceleration.

Using the shallow water approximation and non-dimensionalization, the above equations can be written as

$$
\begin{gathered}
u_t + \varepsilon(uu_x + wu_z) = -p_x,\\
\delta^2\left(w_t + \varepsilon(uw_x + ww_z)\right) = -p_z,\\
u_x + w_z = 0,\\
w = \eta_t + \varepsilon u\eta_x, \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta,\\
w = 0 \quad \text{on } z = 0,
\end{gathered}
$$

where $\vec{v} = (u,0,w)$ and $p(x,z,t)$ is the pressure variable measuring the deviation from the hydrostatic pressure distribution and $\eta(t,x)$ is the deviation from the mean level $z = h$ of the water surface. $\varepsilon = a/h$ and $\delta = h/\lambda$ are the two dimensionless parameters with a being the typical amplitude of the wave and λ being the typical wavelength of the wave.

In the presence of an underlying shear flow, the horizontal velocity of the flow becomes $u + \tilde{U}(z)$. We take the simplest case $\tilde{U}(z) = Az$ in which $A > 0$ is a constant. Notice that the Burns condition gives the shallow-water limit of the dispersion relation for the waves with vorticity , hence determines the speed of propagation of the linear waves. From Burns condition [17, 28] one has the following expression for the speed c of the traveling waves in linear approximation,

$$c = \frac{1}{2}\left(A \pm \sqrt{4+A^2}\right). \tag{2.1}$$

In the case of the constant vorticity $\omega = A$, we obtain the following equations for u_0 and η by ignoring the terms of $O(\varepsilon^2, \delta^4, \varepsilon\delta^2)$

$$\Big(u_0 - \frac{1}{2}\delta^2 u_{0,xx}\Big)_t + \varepsilon u_0 u_{0,x} + \eta_x - \frac{A}{3}\delta^2 u_{0,xxx} = 0, \tag{2.2}$$

$$\eta_t + A\eta_x + \Big((1+\varepsilon\eta)u_0 + \frac{A}{2}\varepsilon\eta^2\Big)_x - \frac{1}{6}\delta^2 u_{0,xxx} = 0, \tag{2.3}$$

where u_0 is the leading order approximation for u (see the details in [35]). Let both of the parameters ϵ and δ go to 0. Then by (2.2) and (2.3), we have the system of linear equations

$$\begin{gathered} u_{0,t} + \eta_x = 0,\\ \eta_t + A\eta_x + u_{0,x} = 0. \end{gathered}$$

This in turn implies that $\eta_{tt} + A\eta_{tx} - \eta_{xx} = 0$. Introducing a new variable

$$\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\mu u_{0,xx},$$

for some constants α, β and μ satisfying

$$\begin{gathered} \frac{\mu}{\alpha} = \frac{1}{6(c-A)},\\ \alpha = 1 + \frac{Ac}{2} + \frac{\beta}{\alpha}, \end{gathered}$$

then equations (2.2) and (2.3) become

$$\begin{gathered} m_t + Am_x - Au_{0,x} - \frac{1}{6c^2(c-A)}\delta^2 u_{0,xxx}\\ +\varepsilon\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)u_0 u_{0,x} + \frac{1}{2\varepsilon\alpha}(\rho^2)_x = 0,\\ \rho_t + A\rho_x + \alpha\varepsilon(\rho u_0)_x = 0, \end{gathered} \tag{2.4}$$

where $m = u_0 - \frac{1}{2}\delta^2 u_{0,xx}$. Since $b \neq -1$ and

$$(b+1)u_0 u_{0,x} = bmu_{0,x} + u_0 m_x + O(\delta^2),$$

equation (2.4) can be reformulated at the order of $O(\varepsilon, \delta^2)$ as

$$\begin{gathered} m_t + Am_x - Au_{0,x} - \frac{1}{6c^2(c-A)}\delta^2 u_{0,xxx}\\ + \frac{\varepsilon}{b+1}\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)(bmu_{0,x} + u_0 m_x) + \frac{1}{2\varepsilon\alpha}(\rho^2)_x = 0. \end{gathered}$$

Using the scaling $u_0 \to \frac{1}{\alpha\varepsilon}u_0,\ x \to \delta x$ and $\ t \to \delta t$, then (2.4) becomes

$$\begin{gathered} m_t + Am_x - Au_{0,x} - \frac{1}{6c^2(c-A)}u_{0,xxx} +\\ \frac{1}{(b+1)\alpha}\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)(bmu_{0,x} + u_0 m_x) + \frac{1}{2}(\rho^2)_x = 0,\\ m = u_0 - u_{0,xx}, \end{gathered}$$

$$\rho_t + A\rho_x + (\rho u_0)_x = 0.$$

Now if we choose

$$\frac{1}{(b+1)\alpha}\Big(1 - \frac{\alpha^2 + 2\beta}{\alpha}c^2\Big) = 1$$

and denote $\gamma = -\frac{1}{6c^2(c-A)}$, then we arrive at

$$\begin{gathered} m_t + Am_x - Au_{0,x} + bmu_{0,x} + u_0 m_x + \gamma u_{0,xxx} + \rho\rho_x = 0, \\ m = u_0 - u_{0,xx}, \\ \rho_t + A\rho_x + (\rho u_0)_x = 0. \end{gathered} \tag{2.5}$$

Thus the constants α, β, μ and c satisfy

$$\begin{gathered} \alpha = \frac{c^2(c^2+1)+1}{3c^2+b+1}, \quad \beta = \alpha^2 - \alpha\Big(1 + \frac{Ac}{2}\Big), \\ \mu = \frac{\alpha}{6(c-A)}, \quad c^2 - Ac - 1 = 0. \end{gathered}$$

With a further Galilean transformation $x \to x - ct$, $t \to t$, we can drop the terms $A\rho_x$ and Am_x in (2.5) and obtain the two-component b-family system (1.4) or (1.5).

3. Local well-posedness

In this section, we will apply Kato's semigroup theory to establish the local well-posedness for the following periodic initial-value problem to (1.5).

$$\begin{gathered} u_t + (u-\gamma)u_x = -\partial_x(1-\partial_x^2)^{-1}\Big(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2\Big), \\ t \ge 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, \quad t \ge 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), \quad t \ge 0,\ x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), \quad t \ge 0,\ x \in \mathbb{R}. \end{gathered} \tag{3.1}$$

For convenience, we present here Kato's theorem in a form suitable for our purpose. Consider the abstract quasilinear evolution equation

$$\begin{gathered} \frac{dv}{dt} + A(v)v = f(v) \quad t \ge 0, \\ v(0) = v_0. \end{gathered} \tag{3.2}$$

Let X and Y be two Hilbert spaces such that Y is continuously and densely embedded in X and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y,X)$ denote the space of all bounded linear operators from Y to X, particularly, it is denoted by $L(X)$ if $X = Y$. The linear operator A belongs to $G(X,1,\beta)$ where β is a real number, if $-A$ generates a C_0-semigroup such that $\|e^{-sA}\|_{L(X)} \le e^{\beta s}$. We make the following assumptions, where $\mu_i (1 = 1,2,3,4)$ are constants depending only on $\max\{\|y\|_Y,\ \|z\|_Y\}$:

(i) $A(y) \in L(Y,X)$ for $y \in Y$ with

$$\|\big(A(y) - A(z)\big)w\|_X \le \mu_1 \|y - z\|_X \|w\|_Y, \quad y,z,w \in Y$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.
(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|\big(B(y) - B(z)\big)w\|_X \le \mu_2 \|y - z\|_Y \|w\|_X, \quad y, z \in Y,\ w \in X.$$

(iii) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y, and satisfies

$$\|f(y) - f(z)\|_Y \le \mu_3 \|y - z\|_Y, \quad y, z \in Y$$

and

$$\|f(y) - f(z)\|_X \le \mu_4 \|y - z\|_X, \quad y, z \in Y.$$

Lemma 3.1 ([36]). *Assume conditions* (i), (ii) (iii) *hold. Given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to* (3.2) *such that*

$$v = v(\cdot, v_0) \in C\big([0, T); Y\big) \cap C^1\big([0, T); X\big).$$

Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is a continuous map from Y to $C\big([0, T); Y\big) \cap C^1\big([0, T); X\big)$.

We now provide the framework in which we shall reformulate problem (3.1).

Theorem 3.2. *Given an initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \ge 2$, there exists a maximal $T = T\left(\|(u_0, \rho_0)\|_{H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})}\right) > 0$ and a unique solution*

$$(u, \rho) \in C\left([0, T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})\right) \cap C^1\left([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})\right)$$

of system (3.1)*. Moreover, the solution (u, ρ) depends continuously on the initial value (u_0, ρ_0) and the maximal time of existence $T > 0$ is independent of s.*

The remaining of this section is devoted to the proof of Theorem 3.2. Let

$$U = \begin{pmatrix} u \\ \rho \end{pmatrix},$$

$$A(U) = \begin{pmatrix} (u - \gamma)\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix} \tag{3.3}$$

$$f(U) = \begin{pmatrix} -\partial_x (1 - \partial_x^2)^{-1} \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2\right) \\ -u_x \rho \end{pmatrix} \tag{3.4}$$

$Y = H^s \times H^{s-1}$, $X = H^{s-1} \times H^{s-2}$, $\Lambda = (1 - \partial_x^2)^{1/2}$ and

$$Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}.$$

Obviously, Q is an isomorphism of $H^s \times H^{s-1}$ onto $H^{s-1} \times H^{s-2}$. Thus, to derive Theorem 3.2, we only need to check that $A(U)$ and $f(U)$ satisfy the conditions (i)-(iii), and this can be formulated through several lemmas.

The following lemmas from [36] and [37] are useful in our proofs.

Lemma 3.3 ([36]). *Let r, t be two real numbers such that $-r < t \le r$. Then*

$$\|fg\|_{H^t} \le c\|f\|_{H^r} \|g\|_{H^t}, \quad if\ r > \frac{1}{2}$$

and

$$\|fg\|_{H^{r+t-\frac{1}{2}}} \le c\|f\|_{H^r} \|g\|_{H^t}, \quad if\ r < \frac{1}{2},$$

where c is a positive constant depending on r and t.

Lemma 3.4 ([37])**.** *Let $f \in H^r$ for some $r > \frac{3}{2}$. Then*

$$\|\Lambda^{-\bar{s}}[\Lambda^{\bar{s}+\bar{t}+1}, M_f]\Lambda^{-\bar{t}}\|_{L(L^2)} \leq c\|\partial_x f\|_{r-1}, \quad |\bar{s}|,\ |\bar{t}| \leq r-1,$$

where M_f is the operator of multiplication by f and c is a constant depending only on $\bar{s}$ and $\bar{t}$.

Lemma 3.5. *With $U \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})(s \geq 2)$, the operator $A(U)$ belongs to $G(H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}), 1, \beta)$.*

Proof. Taking the $H^{s-1} \times H^{s-2}$ inner product with $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ on both sides of the equation

$$\frac{dW}{dt} + A(U)W = 0$$

gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|W\|^2_{H^{s-1}\times H^{s-2}} \\
&= -\langle W, A(U)W\rangle_{(s-1)\times(s-2)} \\
&= -\Big\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} (u-\gamma)\partial_x w_1 \\ u\partial_x w_2 \end{pmatrix} \Big\rangle_{(s-1)\times(s-2)} \\
&= -\langle w_1, (u-\gamma)\partial_x w_1\rangle_{s-1} - \langle w_2, u\partial w_2\rangle_{s-2} \\
&= -\langle \Lambda^{s-1}w_1, \Lambda^{s-1}\big((u-\gamma)\partial_x w_1\big)\rangle - \langle \Lambda^{s-2}w_2, \Lambda^{s-2}\big(u\partial_x w_2\big)\rangle \\
&= -\langle \Lambda^{s-1}w_1, [\Lambda^{s-1}, u-\gamma]\partial_x w_1\rangle - \langle \Lambda^{s-1}w_1, (u-\gamma)\partial_x\Lambda^{s-1}w_1\rangle \\
&\quad - \langle \Lambda^{s-2}w_2, [\Lambda^{s-2}, u]\partial_x w_2\rangle - \langle \Lambda^{s-2}w_2, u\partial_x\Lambda^{s-2}w_2\rangle \\
&= -\langle \Lambda^{s-1}w_1, [\Lambda^{s-1}, u-\gamma]\partial_x w_1\rangle - \frac{1}{2}\langle \Lambda^{s-1}w_1, u_x\partial_x\Lambda^{s-1}w_1\rangle \\
&\quad - \langle \Lambda^{s-2}w_2, [\Lambda^{s-2}, u]\partial_x w_2\rangle - \frac{1}{2}\langle \Lambda^{s-2}w_2, \partial_x u\Lambda^{s-2}w_2\rangle \\
&\leq \|\Lambda^{s-1}w_1\|^2_{L^2}\|[\Lambda^{s-1}, u-\gamma]\Lambda^{2-s}\|_{L(L^2)} + \frac{1}{2}\|u_x\|_{L^\infty}\|\Lambda^{s-1}w_1\|_{L^2} \\
&\quad + \|\Lambda^{s-2}w_2\|^2_{L^2}\|[\Lambda^{s-2}, u]\Lambda^{3-s}\|_{L(L^2)} + \frac{1}{2}\|u_x\|_{L^\infty}\|\Lambda^{s-2}w_2\|_{L^2} \\
&\leq c\left(\|U\|_{H^s} + |\gamma|\right)\left(\|w_1\|^2_{H^{s-1}} + \|w_2\|^2_{H^{s-2}}\right) \\
&= c\left(\|U\|_{H^s} + |\gamma|\right)\|W\|^2_{H^{s-1}\times H^{s-2}},
\end{aligned}$$

where use has been made of Lemma 3.4 with $r = 0, \bar{t} = s-2$ and $\bar{s} = 0$, $\bar{t} = s-3$, respectively. By integrating both of sides in the above the estimate, it follows that $A(U) \in G\left(H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}), 1, c(\|u\|_{H^s} + \gamma)\right)$ □

Lemma 3.6. *The operator $A(U)$ defined by* (3.3) *belongs to $tL(H^s \times H^{s-1}, H^{s-1} \times H^{s-2})$. Moreover*

$$\begin{gathered}
\|\left(A(U) - A(V)\right)W\|_{H^{s-1}\times H^{s-2}} \leq \mu_1\|U - V\|_{H^s\times H^{s-1}}\|W\|_{H^s\times H^{s-1}}, \\
U, V, W \in H^s \times H^{s-1}.
\end{gathered} \tag{3.5}$$

Proof. In view of (3.3), we have

$$\left(A(U) - A(V)\right)W = \begin{pmatrix} (u-\gamma)\partial_x - (v_1-\gamma)\partial_x & 0 \\ 0 & u\partial_x - v_1\partial_x \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= \begin{pmatrix} (u - v_1)\partial_x w_1 \\ (u - v_1)\partial_x w_2 \end{pmatrix}.$$

Since H^{s-1} ($s \geq 2$) is a Banach algebra, taking $r = s-1$, $t = s-2$ in Lemma 3.3, we have

$$\begin{aligned}
&\|(A(U) - A(V))\, W\|_{H^{s-1}\times H^{s-2}} \\
&\leq \|(u - v_1)\partial_x w_1\|_{H^{s-1}} + \|(u - v_1)\partial_x w_2\|_{H^{s-2}} \\
&\leq c\|u - v_1\|_{H^{s-1}} \left(\|\partial_x w_1\|_{H^{s-1}} + \|\partial_x w_2\|_{H^{s-2}}\right) \\
&\leq c\|U - V\|_{H^{s-1}\times H^{s-2}} \|W\|_{H^{s-1}\times H^{s-2}}.
\end{aligned}$$

Taking $V = 0$ in (3.5), we deduce that $A(U) \in L\left(H^s \times H^{s-1}, H^{s-1} \times H^{s-2}\right)$. $\square$

Lemma 3.7 ([24])**.** *Let $B(U) = QA(U)Q^{-1} - A(U)$, for $U \in H^s \times H^{s-1}$ ($s \geq 2$). Then $B(U) \in L\left(H^{s-1} \times H^{s-2}\right)$ and*

$$\begin{gathered}
\|\,(B(U) - B(V))\, W\|_{H^{s-1}\times H^{s-2}} \leq \mu_2 \|U - V\|_{H^s\times H^{s-1}} \|W\|_{H^{s-1}\times H^{s-2}}, \\
U, V \in H^s \times H^{s-1},\ W \in H^{s-1} \times H^{s-2}.
\end{gathered}$$

Lemma 3.8 ([24])**.** *Let $U \in H^s \times H^{s-1}$ ($s \geq 2$). Then the operator $f(U)$ defined by (3.4) is bounded on bounded sets in $(H^{s-1} \times H^{s-2})$, and satisfies*

(a) $\|f(U) - f(V)\|_{H^s\times H^{s-1}} \leq \mu_3 \|U - V\|_{H^s\times H^{s-1}}$, $U, V \in H^s \times H^{s-1}$,
(b) $\|f(U) - f(V)\|_{H^{s-1}\times H^{s-2}} \leq \mu_4 \|U - V\|_{H^{s-1}\times H^{s-2}}$, $U, V \in H^s \times H^{s-1}$.

Proof of Theorem 3.2. The result follows from Lemmas 3.5–3.8. $\square$

4. Blow-up mechanism for $b = 2$

In this section, we investigate the problem of the wave-breaking phenomenon for the initial-value problem of the periodic two-component Dullin-Gottwald-Holm system which is a special case of (1.5) as $b = 2$.

4.1. Preliminaries.

The periodic two-component Dullin-Gottwald-Holm system can be written as

$$\begin{gathered}
u_t - u_{txx} - Au_x + \gamma u_{xxx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \rho\rho_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\
\rho_t + (u\rho)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\
u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
\rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}, \\
u(t, x + 1) = u(t, x), \quad t \geq 0,\ x \in \mathbb{R}, \\
\rho(t, x + 1) = \rho(t, x), \quad t \geq 0,\ x \in \mathbb{R}.
\end{gathered} \tag{4.1}$$

Let $G(x) = \frac{\cosh(x - [x] - 1/2)}{2\sinh(1/2)}, x \in \mathbb{S}$. Then $(1 - \partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$, $u = G * m$ and $m = u - u_{xx}$. Our system (4.1) can be written in the following "transport" type

$$
\begin{gathered}
u_t + (u-\gamma)u_x = -\partial_x G * \Big(u^2 + \frac{1}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2\Big), \quad t > 0, \quad x \in \mathbb{R}, \\
\rho_t + (u\rho)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\
u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\
\rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\
u(t,x+1) = u(t,x), \quad t \ge 0,\ x \in \mathbb{R}, \\
\rho(t,x+1) = \rho(t,x), \quad t \ge 0,\ x \in \mathbb{R}.
\end{gathered} \tag{4.2}
$$

To study the wave-breaking problem, we now briefly give the needed results without proof to pursue our goal. We consider the following two associated Lagrangian scales of the system (4.1)

$$
\begin{gathered}
\frac{\partial q_1}{\partial t} = u(t,q_1) - \gamma, \quad 0 < t < T, \\
q_1(0,x) = x, \quad x \in \mathbb{R},
\end{gathered} \tag{4.3}
$$

and

$$
\begin{gathered}
\frac{\partial q_2}{\partial t} = u(t,q_2), \quad 0 < t < T, \\
q_2(0,x) = x, \quad x \in \mathbb{R},
\end{gathered} \tag{4.4}
$$

where $u \in C^1([0,T), H^{s-1}(\mathbb{S}))$ is the first component of the solution (u,ρ) to (4.1).

Lemma 4.1 ([18, 12]). *Let (u,ρ) be the solution of system (4.1) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, and T the maximal time of existence. Then (4.3) has a unique solution $q_1 \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ and (4.4) has a unique solution $q_2 \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. These two solutions satisfy $q_i(t,x+1) = q_i(t,x)+1$, $i = 1,2$. Moreover, the maps $q_1(t,\cdot)$ and $q_2(t,\cdot)$ are increasing diffeomorphisms of $\mathbb{R}$ with*

$$
\begin{gathered}
q_{1x}(t,x) = \exp\Big(\int_0^t u_x(\tau, q_1(\tau,x))d\tau\Big) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}, \\
q_{2x}(t,x) = \exp\Big(\int_0^t u_x(\tau, q_2(\tau,x))d\tau\Big) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.
\end{gathered}
$$

The above lemmas indicate that $q_1(t,\cdot) : \mathbb{R} \to \mathbb{R}$ and $q_2(t,\cdot) : \mathbb{R} \to \mathbb{R}$ are diffeomorphisms of the line for each $t \in [0,T)$. Hence, the L^∞ norm of any function $v(t,\cdot) \in L^\infty(\mathbb{S})$ is preserved under the family of diffeomorphisms $q_1(t,\cdot)$ and $q_2(t,\cdot)$ with $t \in [0,T)$; that is,

$$
\|v(t,\cdot)\|_{L^\infty(\mathbb{S})} = \|v(t,q_1(t,\cdot))\|_{L^\infty(\mathbb{S})} = \|v(t,q_2(t,\cdot))\|_{L^\infty(\mathbb{S})}, \quad t \in [0,T). \tag{4.5}
$$

Similarly, we have

$$
\inf_{x\in\mathbb{S}} v(t,x) = \inf_{x\in\mathbb{S}} v(t,q_1(t,x)) = \inf_{x\in\mathbb{S}} v(t,q_2(t,x)), \quad t \in [0,T), \tag{4.6}
$$

$$
\sup_{x\in\mathbb{S}} v(t,x) = \sup_{x\in\mathbb{S}} v(t,q_1(t,x)) = \sup_{x\in\mathbb{S}} v(t,q_2(t,x)), \quad t \in [0,T). \tag{4.7}
$$

Lemma 4.2 ([24]). *Let (u,ρ) be the solution of (4.1) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, and T the maximal time of existence. Then we have*

$$
\rho(t,q_2(t,x))q_{2x}(t,x) = \rho_0(x), \quad (t,x) \in [0,T) \times \mathbb{S}. \tag{4.8}
$$

Moreover if there exists $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$, then $\rho(t,q_2(t,x_0)) = 0$ for all $t \in [0,T)$.

Lemma 4.3 ([9]). *Let $T > 0$ and $v \in C^1([0,T); H^2(\mathbb{R}))$. Then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}} (v_x(t,x)) = v_x(t,\xi(t)).$$

The function $m(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t,\xi(t)) \quad a.e. \quad on \quad (0,T).$$

We may use the following lemma derived in [31] to establish the blow-up criterion of solution to (4.1).

Lemma 4.4. *Let $0 < s < 1$. Suppose that $f_0 \in H^s(\mathbb{S}), g \in L^1([0,T]; H^s(\mathbb{S}))$, $v, v_x \in L^1([0,T]; L^\infty(\mathbb{S}))$ and that $f \in L^\infty([0,T]; H^s(\mathbb{S})) \bigcap C([0,T]; S'(\mathbb{S}))$ solves the one-dimensional linear transport equation*

$$f_t + v f_x = g,$$
$$f(0,x) = f_0(x).$$

Then $f \in C([0,T]; H^s(\mathbb{R}))$. More precisely, there exists a constant C depending only on s such that

$$\|f(t)\|_{H^s} \le \|f_0\|_{H^s} + C\Big(\int_0^t \|g(\tau)\|_{H^s} d\tau + \int_0^t \|f(\tau)\|_{H^s} V'(\tau) d\tau\Big).$$

Hence,

$$\|f(t)\|_{H^s} \le e^{CV(t)}\Big(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau\Big),$$

where $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty}) d\tau$.

The above lemma was proved using the Littlewood-Palay analysis for the transport equation and the Moser-type estimates. Using this result and performing the same argument as in [31], we can obtain the following blow-up criterion (up to a slight modification, the proof is omitted).

Theorem 4.5. *Let (u,ρ) be the solution of system (4.1) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, and T the maximal time of existence. Then*

$$T < \infty \Rightarrow \int_0^T \|u_x(\tau)\|_{L^\infty(\mathbb{S})} d\tau = \infty. \tag{4.9}$$

We now give several useful conservation laws of strong solutions to (4.1).

Lemma 4.6. *Let (u,ρ) be the solution of system (4.1) with initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, and T the maximal time of existence. Then for all $t \in [0,T)$, we have*

$$\int_{\mathbb{S}} u(t,x) dx = \int_{\mathbb{S}} u_0(x) dx,$$
$$\int_{\mathbb{S}} \rho(t,x) dx = \int_{\mathbb{S}} \rho_0(x) dx.$$

Proof. Integrating the first equation of (4.2) by parts, in view of the periodicity of u and G, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u dx = -\int_{\mathbb{S}} (u-\gamma) u_x dx - \int_{\mathbb{S}} \partial_x G * \Big(u^2 + \frac{1}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2\Big) dx = 0.$$

On the other hand, integrating the second equation of (4.2) by parts, in view of the periodicity of u and ρ, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}}\rho dx = -\int_{\mathbb{S}}(u\rho)_x dx = 0.$$

Therefore, the proof is complete. □

Lemma 4.7. *Let (u,ρ) be the solution of system (4.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s\geq 2$, and T the maximal time of existence. Then for all $t\in[0,T)$, we have*

$$\int_{\mathbb{S}}(u^2(t,x)+u_x^2(t,x)+\rho^2(t,x))dx = \int_{\mathbb{S}}(u_0^2(t,x)+u_{0x}^2(t,x)+\rho_0^2(t,x))dx.$$

Proof. Multiplying the first equation of (4.1) by $2u$ and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}}(u^2(t,x)+u_x^2(t,x))dx = \frac{d}{dt}\int_{\mathbb{S}}u_x(t,x)\rho^2(t,x)dx.$$

Multiplying the second equation of (4.1) by 2ρ and integrating by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}}\rho^2(t,x) = -\frac{d}{dt}\int_{\mathbb{S}}u_x(t,x)\rho^2(t,x)dx.$$

Adding the above two equalities, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}}(u^2(t,x)+u_x^2(t,x)+\rho^2(t,x))dx = 0.$$

This implies the desired result in this lemma. □

Lemma 4.8 ([48]). *For every $f\in H^1(\mathbb{S})$, we have*

$$\max_{x\in[0,1]} f^2(x) \leq \frac{e+1}{2(e-1)}\|f\|^2_{H^1(\mathbb{S})},$$

where the constant $\frac{e+1}{2(e-1)}$ is sharp.

By the conservation laws stated in Lemmas 4.6 and 4.7, we have the following corollary.

Corollary 4.9. *Let (u,ρ) be the solution of system (4.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s\geq 2$, and T the maximal time of existence. Then for all $t\in[0,T)$, we have*

$$\|u(t,\cdot)\|^2_{L^\infty(\mathbb{S})} \leq \frac{e+1}{2(e-1)}\|u(t,\cdot)\|^2_{H^1(\mathbb{S})} \leq \frac{e+1}{2(e-1)}\|(u_0,\rho_0)\|^2_{H^1(\mathbb{S})\times L^2(\mathbb{S})}.$$

Lemma 4.10 ([25]). *For all $f\in H^1(\mathbb{S})$, the following inequality holds*

$$G*(u^2+\frac{1}{2}u_x^2) \geq \kappa u^2(x),$$

with

$$\kappa = \frac{1}{2} + \frac{\arctan(\sinh(1/2))}{2\sinh(1/2)+2\arctan(\sinh(1/2))\sinh^2(1/2)} \approx 0.869.$$

Moreover, κ is the optimal constant obtained by the function

$$f_0 = \frac{1+\arctan(\sinh(x-[x]-1/2))\sinh(x-[x]-1/2)}{1+\arctan(\sinh(1/2))\sinh(1/2)}.$$

4.2. **Blow-up scenario.** Based on the above results, let us state the following theorem on the precise blow-up mechanism.

Theorem 4.11 (Wave-breaking criterion). *Let (u,ρ) be the solution of (4.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geq 2$, and T the maximal time of existence. Then the solution blows up in finite time if and only if*

$$\liminf_{t\to T_0^-}\{\inf_{x\in\mathbb{S}} u_x(t,x)\}=-\infty. \tag{4.10}$$

To prove this wave-breaking criterion, we use the following lemma to show that indeed u_x is uniformly bounded from above.

Lemma 4.12. *Let (u,ρ) be the solution of (4.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s\geq 2$, and T the maximal time of existence. Then*

$$\sup_{x\in\mathbb{S}} u_x(t,x)\leq\|u_{0,x}\|_{L^\infty}+\sqrt{\|\rho_0\|_{L^\infty}^2+C_1^2}. \tag{4.11}$$

The constants above are defined as follows.

$$C_0=\|(u_0,\rho_0)\|_{H^1\times L^2}^2, \tag{4.12}$$

$$C_1^2=\Big((1-\kappa)\frac{e+1}{e-1}+\frac{1}{2}\Big)C_0+\frac{(-1+\sinh 1)(\gamma-A)^2}{4\sinh^2(1/2)}, \tag{4.13}$$

$$C_2=\frac{5e+3}{4(e-1)}C_0+\frac{(-1+\sinh 1)(\gamma-A)^2}{8\sinh^2(1/2)}, \tag{4.14}$$

and κ is defined in Lemma 4.10.

Proof. The local well-posedness theorem and a density argument imply that it suffices to prove the desired estimates for $s\geq 3$. Thus, we take $s=3$ in the proof. Also, we assume that $u_0\not\equiv 0$. Otherwise, the results become trivial. Differentiating the first equation in (4.2) with respect to x. Using the identity $-\partial_x^2 G*f=f-G*f$, we obtain

$$u_{tx}+(u-\gamma)u_{xx}=-\frac{1}{2}u_x^2+u^2+\frac{1}{2}\rho^2-(\gamma-A)\partial_x^2G*u-G*\Big(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\Big). \tag{4.15}$$

Using Lemma 4.1 and the fact that

$$\sup_{x\in\mathbb{S}}\left(v_x(t,x)\right)=-\inf_{x\in\mathbb{S}}\left(-v_x(t,x)\right),$$

we can consider $\bar{m}(t)$ and $\eta(t)$ as follows,

$$\eta(t)\in\mathbb{S},\quad \bar{m}(t):=u_x(t,\eta(t))=\sup_{x\in\mathbb{S}}(u_x(t,x)),\quad t\in[0,T). \tag{4.16}$$

Hence,

$$u_{xx}(t,\eta(t))=0,\quad \text{a.e. } t\in[0,T).$$

For any $x\in\mathbb{S}$, take the trajectory $q_2(t,x)$ defined in (4.3). Then it follows from the second equation of (4.2) for the component ρ that

$$\frac{d\rho\left(t,q_2(t,x)\right)}{dt}=-u_x\left(t,q_2(t,x)\right)\rho\left(t,q_2(t,x)\right). \tag{4.17}$$

It is known that $q_2(t,\cdot):\mathbb{S}\to\mathbb{S}$ is a diffeomorphism for every $t\in[0,T)$. In view of Lemma 4.1, there exists $x_1(t)\in\mathbb{S}$ such that

$$q_2(t,x_1(t))=\eta(t),\quad t\in[0,T),$$

with $\eta(0) = x_1(0)$. Now define

$$\bar{\xi} = \rho(t, \eta(t)), \quad t \in [0, T).$$

Therefore, along the trajectory $q_2(t, x_1) = \eta(t)$, equations (4.15) and (4.17) become

$$\begin{gathered} \bar{m}'(t) = -\frac{1}{2}\bar{m}^2 + \frac{1}{2}\bar{\xi}^2 + f(t, \eta(t)), \\ \bar{\xi}'(t) = -\bar{\xi}\bar{m}, \end{gathered} \tag{4.18}$$

for $t \in [0, T)$, where "$'$" denotes the derivative with respect to t and $f(t, \eta(t))$ is

$$f = u^2 - (\gamma - A)\partial_x^2 G * u - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2).$$

We first derive the upper bound for f for later use in getting the wave-breaking results. Using Lemma 4.10 we have

$$f \le (1 - \kappa)u^2 - (\gamma - A)\partial_x G * \partial_x u, \tag{4.19}$$

for any $x \in \mathbb{S}$ and $t \in [0, T)$. Applying Young's inequality with $G = \frac{\cosh(x - [x] - 1/2)}{2\sinh(1/2)}$ leads to

$$\begin{aligned} |\gamma - A||\partial_x G * \partial_x u| &\le |\gamma - A|\|G_x\|_{L^2}\|u_x\|_{L^2} = |\gamma - A|\frac{\sqrt{\frac{1}{2}(-1 + \sinh 1)}}{2\sinh(1/2)}\|u_x\|_{L^2} \\ &\le \frac{(-1 + \sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|_{L^2}^2. \end{aligned} \tag{4.20}$$

Using Lemma 4.8, we obtain

$$u^2 \le \|u(t, \cdot)\|_{L^\infty(\mathbb{S})}^2 \le \frac{e + 1}{2(e - 1)}\|(u_0, \rho_0)\|_{H^1 \times L^2}^2. \tag{4.21}$$

Therefore, in view of (4.20), (4.21) and the conservation law in Lemma 4.7, we obtain the upper bound of f for any $x \in \mathbb{S}$ and $t \in [0, T)$,

$$\begin{aligned} f &\le (1 - \kappa)u^2 + |\gamma - A||\partial_x G * \partial_x u| \\ &\le (1 - \kappa)\frac{e + 1}{2(e - 1)}\|(u_0, \rho_0)\|_{H^1 \times L^2}^2 + \frac{(-1 + \sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)} + \frac{1}{4}\|u_x\|_{L^2}^2 \\ &\le \Big((1 - \kappa)\frac{e + 1}{2(e - 1)} + \frac{1}{4}\Big)\|(u_0, \rho_0)\|_{H^1 \times L^2}^2 + \frac{(-1 + \sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)} \\ &= \frac{1}{2}C_1^2. \end{aligned} \tag{4.22}$$

Attention is now turned to the lower bound of f. Similarly as before, we obtain

$$\begin{aligned} \big|G * \big(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\big)\big| &\le \|G\|_{L^\infty}\|u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\|_{L^1} \\ &\le \frac{\cosh(1/2)}{2\sinh(1/2)}\|(u_0, \rho_0)\|_{H^1 \times L^2}^2 \\ &= \frac{e + 1}{2(e - 1)}\|(u_0, \rho_0)\|_{H^1 \times L^2}^2. \end{aligned} \tag{4.23}$$

Using (4.20),(4.21) and (4.23), we have

$$\begin{aligned} -f &\le u^2 + |\gamma - A||\partial_x G * \partial_x u| + |G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)| \\ &\le \Big(\frac{e+1}{e-1} + \frac{1}{4}\Big)\|(u_0,\rho_0)\|^2_{H^1\times L^2} + \frac{(-1+\sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)} \\ &= \frac{5e+3}{4(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2} + \frac{(-1+\sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)}. \end{aligned} \tag{4.24}$$

Combining (4.22) and (4.24), we obtain

$$|f| \le \frac{5e+3}{4(e-1)}\|(u_0,\rho_0)\|^2_{H^1\times L^2} + \frac{(-1+\sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)} = C_2. \tag{4.25}$$

Since now $s \ge 3$, we have $u \in C_0^1(\mathbb{S})$. Therefore,

$$\inf_{x\in\mathbb{S}} u_x(t,x) \le 0, \quad \sup_{x\in\mathbb{S}} u_x(t,x) \ge 0, \quad t \in [0,T).$$

Hence, $\bar{m}(t) \ge 0$ for $t \in [0,T)$. From the second equation of (4.18), we obtain that

$$\bar{\xi}(t) = \bar{\xi}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}.$$

Hence,

$$|\rho(t,\eta(t))| = |\bar{\xi}(t)| \le |\bar{\xi}(0)| \le |\rho_0(x_1(0))| \le \|\rho_0\|_{L^\infty}.$$

Now define

$$P_1(t) = \bar{m}(t) - \|u_{0,x}\|_{L^\infty} - \sqrt{\|\rho_0\|^2_{L^\infty} + C_1^2}.$$

Note that $P_1(t)$ is a C^1- differentiable function in $[0,T)$ and satisfies

$$P_1(0) \le \bar{m}(0) - \|u_{0,x}\|_{L^\infty} \le 0.$$

We will show that

$$P_1(t) \le 0, \quad t \in [0,T). \tag{4.26}$$

If not, then suppose there is a $t_0 \in [0,T)$ such that $P_1(t_0) > 0$. Define

$$t_1 = \max\{t < t_0 : P_1(t) = 0\}.$$

Then $P_1(t_1) = 0$ and $P_1' \ge 0$, or equivalently,

$$\begin{gathered} \bar{m}(t_1) = \|u_{0,x}\|_{L^\infty} + \sqrt{\|\rho_0\|^2_{L^\infty} + C_1^2}, \\ \bar{m}'(t_1) \ge 0. \end{gathered}$$

On the other hand, we have

$$\begin{aligned} \bar{m}'(t_1) &= -\frac{1}{2}\bar{m}^2(t_1) + \frac{1}{2}\bar{\xi}^2(t_1) + f(t_1,\eta(t_1)) \\ &\le -\frac{1}{2}\Big(\|u_{0,x}\|_{L^\infty} + \sqrt{\|\rho_0\|^2_{L^\infty} + C_1^2}\Big)^2 + \frac{1}{2}\|\rho_0\|^2_{L^\infty} + \frac{C_1^2}{2} < 0, \end{aligned}$$

which is a contradiction. Therefore, $P_1(t) \le 0$, for $t \in [0,T)$, and we obtain (4.26). Therefore, the proof is complete. □

It is also found that if u_x is bounded from below, we may obtain the following estimates for $\|\rho\|_{L^\infty(\mathbb{S})}$.

Lemma 4.13. *Let (u,ρ) be the solution of (4.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s\geq 2$, and T the maximal time of existence. If there is an $M\geq 0$, such that*

$$\inf_{(t,x)\in[0,T)\times\mathbb{S}} u_x \geq -M, \tag{4.27}$$

then

$$\|\rho(t,\cdot)\|_{L^\infty(\mathbb{S})} \leq |\rho_0\|_{L^\infty(\mathbb{S})} e^{Mt}. \tag{4.28}$$

Proof. For any give $x\in\mathbb{S}$, we define

$$U(t)=u_x(t,q_2(t,x)),\quad \gamma(t)=\rho(t,q_2(t,x)),$$

with $q_2(t,x(t))=x$, for some $x(t)\in\mathbb{R}, t\in[0,T)$. Then the ρ equation of system (4.1) becomes

$$\gamma'=-\gamma U.$$

Thus,

$$\gamma(t)=\gamma(0)e^{-\int_0^t U(\tau)d\tau}.$$

From assumption (4.27), we see that

$$U(t)\geq -M,\quad t\in[0,T).$$

Hence,

$$|\rho(t,q_2(t,x(t))|=|\gamma(t)|\leq|\gamma(0)|e^{-\int_0^t U(\tau)d\tau}\leq\|\rho_0\|_{L^\infty}e^{Mt},$$

which together with (4.5) leads to (4.28). □

We are now in the position to prove Theorem 4.11.

Proof of Theorem 4.11. Assume that $T<\infty$ and (4.10) is not valid. Then there is some positive number $M>0$ such that

$$u_x(t,x)\geq -M,\quad \forall (t,x)\in[0,T)\times\mathbb{S}.$$

It is now inferred from Lemma 4.12 that $|u_x(t,x)|\leq C$, where

$$C=C(A,\gamma,M,\|(u_0,\rho_0)\|^2_{H^s\times H^{s-1}}).$$

Therefore, Theorem 4.5 in turn implies that the maximal existence time $T=\infty$, which contradicts the assumption that $T<\infty$. Conversely, the Sobolev embedding theorem $H^s(\mathbb{S})\hookrightarrow L^\infty(\mathbb{S})$ with $s>1/2$ implies that if (4.10) holds, the corresponding solution blows up in finite time. This completes the proof. □

Now, we give the following theorems with some initial conditions which guarantee wave breaking in finite time.

Theorem 4.14. *Let (u,ρ) be the solution of (4.1) with the initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s\geq 2$, and T the maximal time of existence. Assume that there is some $x_0\in\mathbb{S}$ such that*

$$\rho_0(x_0)=0,\quad u_{0,x}(x_0)=\inf_{x\in\mathbb{S}}u_{0,x}(x),$$

and

$$u_{0,x}(x_0)<-C_1, \tag{4.29}$$

where C_1 is defined as

$$C_1^2=\Big((1-\kappa)\frac{e+1}{e-1}+\frac{1}{2}\Big)\|(u_0,\rho_0)\|^2_{H^1\times L^2}+\frac{(-1+\sinh 1)(\gamma-A)^2}{4\sinh^2(1/2)}.$$

Then the corresponding solution to system (4.1) *blows up in the following sense: there exists a* T_1 *with*

$$0 < T_1 \le -\frac{2}{u_{0,x}(x_0) + \sqrt{-C_1 u_{0,x}(x_0)}} \tag{4.30}$$

such that

$$\liminf_{t\to T_0^-}\{\inf_{x\in\mathbb{S}} u_x(t,x)\} = -\infty.$$

Proof. Similar to the proof of Lemma 4.12, it suffices to consider $s \ge 3$. So in the following of this section $s = 3$ is taken for simplicity of notation.

we consider the functions $m(t)$ and $\xi(t) \in \mathbb{S}$ as in Lemma 4.12

$$m(t) := u_x(t,\xi(t)) = \inf_{x\in\mathbb{S}}(u_x(t,x)), \quad t \in [0,T).$$

Hence,

$$u_{xx}(t,\xi(t)) = 0, \quad \text{a.e.} \quad t \in [0,T). \tag{4.31}$$

Similar as before, we can choose $x_2(t) \in \mathbb{S}$ such that

$$q_2(t,x_2(t)) = \xi(t) \quad t \in [0,T).$$

Along the trajectory of $q_2(t,x)$, we have

$$\frac{d\rho(t,\xi(t))}{dt} = -\rho(t,\xi(t))u_x(t,\xi(t)).$$

It follows from the assumption of the theorem, that

$$m(0) = u_x(0,\xi(0)) = \inf_{x\in\mathbb{S}} u_{0,x}(x) = u_{0,x}(x_0).$$

Hence, we can choose $\xi(0) = x_0$ and then $\rho_0(\xi(0)) = \rho_0(x_0) = 0$. Thus, from (4.8) we obtain

$$\rho(t,\xi(t)) = 0, \quad t \in [0,T). \tag{4.32}$$

Differentiating the first equation in (4.2) with respect to x, evaluating the result at $x = \xi(t)$ and using (4.31) and (4.32), we deduce from (4.15) that

$$m'(t) = -\frac{1}{2}m^2(t) + f(t,\xi(t)). \tag{4.33}$$

Using the upper bound of f in (4.22), it is found that

$$m'(t) \le -\frac{1}{2}m^2(t) + \frac{1}{2}C_1^2, \quad \text{a.e. } t \in [0,T).$$

By assumption (4.29), $m(0) = u_{0,x}(x_0) < -C_1$, we deduce that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0,T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{C_1}{-u_{0,x}(x_0)}} \in \left(0, \frac{1}{2}\right). \tag{4.34}$$

Using that $m(t) < m(0) = u_{0,x}(x_0) < 0$, it follows that

$$m'(t) \le -\frac{1}{2}m^2(t) + \frac{1}{2}C_1^2 \le -\delta m^2(t), \quad \text{a.e. } t \in [0,T). \tag{4.35}$$

Integrating on both sides in (4.35), it is inferred that

$$m(t) \le \frac{u_{0,x}(x_0)}{1 + \delta u_{0,x}(x_0)t} \to -\infty \quad \textit{as} \quad t \to -\frac{1}{\delta u_{0,x}(x_0)}. \tag{4.36}$$

Hence,

$$T \le -\frac{1}{\delta u_{0,x}(x_0)}, \tag{4.37}$$

which proves (4.30). □

Corollary 4.15. *With the assumptions of Theorem 4.14, assume $s > 5/2$. There exists a T^* with $0 < T_1 \le T^*$, (T_1 is defined in (4.30)) such that*

(a) $\limsup_{t\to T^*}\{\sup_{x\in\mathbb{S}} \rho_x(t,x)\} = \infty$, *if* $\rho_{0,x}(x_0) > 0$,
(b) $\liminf_{t\to T^*}\{\inf_{x\in\mathbb{S}} \rho_x(t,x)\} = -\infty$, *if* $\rho_{0,x}(x_0) < 0$.

Proof. With the assumptions of Theorem 4.14, we have

$$\rho_0(x_0) = 0, \quad u_{0,x}(x_0) = \inf_{x\in\mathbb{S}} u_{0,x}(x),$$

and $u_{0,x}(x_0) < -C_1$. Evaluating ρ along the trajectory $q_2(t,x)$, we obtain

$$\frac{d\rho_x\left(t,q_2(t,x)\right)}{dt} = -u_{xx}\left(t,q_2(t,x)\right)\rho\left(t,q_2(t,x)\right) - 2u_x\left(t,q_2(t,x)\right)\rho_x\left(t,q_2(t,x)\right).$$

As in the proof of Theorem 4.14, we can choose $x_2(t) \in \mathbb{S}$ such that $q_2(t,x_2(t)) = \xi(t), t \in [0,T)$. Then we have

$$m(t) := u_x(t,\xi(t)) = \inf_{x\in\mathbb{S}}(u_x(t,x)), \quad t\in[0,T).$$

Hence, $u_{xx}(t,\xi(t)) = 0$, *a.e.* $t \in [0,T)$. This in turn implies

$$\frac{d\rho_x\left(t,\xi(t)\right)}{dt} = -2u_x\left(t,\xi(t)\right)\rho_x\left(t,\xi(t)\right),$$

and

$$\rho_x(t,\xi(t)) = \rho_{0,x}(x_0)e^{-2\int_0^t u_x(\tau,\xi(\tau))d\tau} = \rho_{0,x}(x_0)e^{-2\int_0^t \inf_{x\in\mathbb{S}} u_x(\tau,x)d\tau}.$$

Since $m(t)$ is strictly decreasing in $[0,T)$, by (4.36) we have

$$e^{-2\int_0^t \inf_{x\in\mathbb{S}} u_x(\tau,x)d\tau} \ge e^{-2\int_0^t \frac{u_{0,x}(x_0)}{1+\delta u_{0,x}(x_0)\tau}d\tau} \ge e^{-\frac{2}{\delta}\ln(1+\delta u_{0,x}(x_0)t)},$$

where δ is defined in (4.34). So

$$e^{-\frac{2}{\delta}\ln(1+\delta u_{0,x}(x_0)t)} \to +\infty,$$

if $t \to -\frac{1}{\delta u_{0,x}(x_0)}$. Therefore, it is inferred from (4.37) that there exists some T^* with $0 < T_1 \le T^*$ such that

$$\sup_{x\in\mathbb{S}} \rho_x(t,x) \ge \rho_x\left(t,\xi(t)\right) \to +\infty.$$

as $t \to T^*$. If $\rho_{0,x}(x_0) < 0$, the proof is similar to the above. This completes the proof of the corollary. □

Theorem 4.16. *Let (u,ρ) be the solution of (4.1) with the initial data $(u_0,\rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, and T the maximal time of existence. Also assume that $\int_{\mathbb{S}} \rho_0(x)dx = 0$, and $\|\rho_x(t,\cdot)\|_{L^\infty(\mathbb{S})} \le M$ (M is a positive constant). If there exists some $K_0 = K_0(C_0) > 0$ ($C_0 = \|(u_0,\rho_0)\|^2_{H^1\times L^2}$) such that*

$$\int_{\mathbb{S}} u_{0x}^3 dx < -K_0, \tag{4.38}$$

then the corresponding solution to (4.1) blows up in finite time.

Proof. Applying $u_x^2\partial_x$ to both sides of the first equation in (4.2) and integrating by parts with the fact that

$$-3\int_{\mathbb{S}} uu_x^2u_{xx}dx = \int_{\mathbb{S}} u_x^4dx.$$

We have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^3dx + \frac{1}{2}\int_{\mathbb{S}} u_x^4dx &= 3\int_{\mathbb{S}} u_x^2\Big(u^2 + (\gamma - A)u + \frac{1}{2}\rho^2\Big)dx \\ &\quad - 3\int_{\mathbb{S}} u_x^2G * \Big(u^2 + \frac{1}{2}u_x^2 + (\gamma - A)u + \frac{1}{2}\rho^2\Big)dx.\end{aligned} \tag{4.39}$$

Note that

$$\big|\int_{\mathbb{S}} u_x^3dx\big| \le \Big(\int_{\mathbb{S}} u_x^4dx\Big)^{1/2}\Big(\int_{\mathbb{S}} u_x^2dx\Big)^{1/2},$$

and $C_0 = \|(u_0,\rho_0)\|_{H^1\times L^2}^2$. Thus we have

$$\int_{\mathbb{S}} u_x^4dx \ge \frac{1}{C_0}\Big(\int_{\mathbb{S}} u_x^3dx\Big)^2. \tag{4.40}$$

Using Corollary 4.9, we obtain the estimate

$$\int_{\mathbb{S}} u_x^2u^2dx \le \|u\|_{L^\infty(\mathbb{S})}^2\int_{\mathbb{S}} u_x^2dx \le \frac{e+1}{2(e-1)}C_0^2. \tag{4.41}$$

By the assumption $\int_{\mathbb{S}}\rho_0(x)dx = 0$ and Lemma 4.2, we have

$$\int_{\mathbb{S}}\rho(t,x)dx = \int_{\mathbb{S}}\rho_0(x)dx = 0.$$

It then follows that for any $t \in [0,T)$, there exists $x_3(t) \in \mathbb{S}$ and $\rho(t,x_3(t)) = 0$. It is noted that

$$\rho(t,x) = \int_{x_3(t)}^{x(t)}\rho_x(t,s)ds, \quad x_3(t), x(t) \in \mathbb{S},$$

which implies that

$$|\rho(t,x)| \le \big|\int_{x_3(t)}^{x(t)}\rho_x(t,s)ds\big| \le M,$$

$$\int_{\mathbb{S}} u_x^2\rho^2dx \le M^2\int_{\mathbb{S}} u_x^2dx \le M^2C_0, \tag{4.42}$$

$$\big|\int_{\mathbb{S}} u_x^2udx\big| \le \|u\|_{L^\infty(\mathbb{S})}\int_{\mathbb{S}} u_x^2dx \le \Big(\frac{e+1}{2(e-1)}\Big)^{1/2}C_0^{3/2}, \tag{4.43}$$

and

$$\begin{aligned}&\int_{\mathbb{S}} u_x^2G * (\gamma - A)udx \\ &\ge -|\gamma - A|\|G\|_{L^\infty(\mathbb{S})}\|u\|_{L^\infty(\mathbb{S})}\int_{\mathbb{S}} u_x^2dx \\ &\ge -|\gamma - A|\frac{\cosh(1/2)}{2\sinh(1/2)}\Big(\frac{e+1}{2(e-1)}\Big)^{1/2}C_0^{3/2} = -|\gamma - A|\Big(\frac{e+1}{2(e-1)}\Big)^{3/2}C_0^{3/2}.\end{aligned} \tag{4.44}$$

In view of the above inequality (4.41), (4.42), (4.43) and (4.44), it follows from Lemma 4.10 that

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u_x^3dx &\le -\frac{1}{2C_0}\Big(\int_{\mathbb{S}} u_x^3dx\Big)^2 + 3\int_{\mathbb{S}} u_x^2\Big(u^2+\frac{1}{2}\rho^2\Big)dx + 3(\gamma-A)\int_{\mathbb{S}} u_x^2udx\\
&\quad -3\int_{\mathbb{S}} u_x^2G*\Big(u^2+\frac{1}{2}u_x^2\Big)+u_x^2G*(\gamma-A)u+u_x^2G*\Big(\frac{1}{2}\rho^2\Big)dx\\
&\le -\frac{1}{2C_0}\Big(\int_{\mathbb{S}} u_x^3dx\Big)^2 + 3\int_{\mathbb{S}} u_x^2\Big(u^2+\frac{1}{2}\rho^2\Big)dx + 3(\gamma-A)\int_{\mathbb{S}} u_x^2udx\\
&\quad -3\kappa\int_{\mathbb{S}} u_x^2u^2dx - 3\int_{\mathbb{S}} u_x^2G*(\gamma-A)udx\\
&= -\frac{1}{2C_0}\Big(\int_{\mathbb{S}} u_x^3dx\Big)^2 + 3(1-\kappa)\int_{\mathbb{S}} u_x^2u^2dx + \frac{3}{2}\int_{\mathbb{S}} u_x^2\rho^2dx\\
&\quad +3|\gamma-A|\int_{\mathbb{S}} u_x^2udx - 3\int_{\mathbb{S}} u_x^2G*(\gamma-A)udx\\
&\le -\frac{1}{2C_0}\Big(\int_{\mathbb{S}} u_x^3dx\Big)^2 + \frac{3(1-\kappa)(e+1)}{2(e-1)}C_0^2 + \frac{3}{2}M^2C_0\\
&\quad + \frac{3(3e-1)}{2(e-1)}|\gamma-A|\Big(\frac{e+1}{2(e-1)}\Big)^{1/2}C_0^{3/2}.
\end{aligned}
\tag{4.45}
$$

Set $h(t)=\int_{\mathbb{S}} u_x^3dx$, and

$$
K^2=\frac{3(1-\kappa)(e+1)}{2(e-1)}C_0^2+\frac{3}{2}M^2C_0+\frac{3(3e-1)}{2(e-1)}|\gamma-A|\left(\frac{e+1}{2(e-1)}\right)^{1/2}C_0^{3/2}.
$$

Note that if $h(0)<-\sqrt{2C_0}K$, then $h(t)<-\sqrt{2C_0}K$. Therefore, we can solve the above inequality (4.45) to obtain

$$
\frac{h(0)+\sqrt{2C_0}K}{h(0)-\sqrt{2C_0}K}e^{\sqrt{\frac{2}{C_0}}Kt}-1\le\frac{2\sqrt{2C_0}K}{h(0)-\sqrt{2C_0}K}\le 0.
$$

Due to the inequality

$$
0<\frac{h(0)+\sqrt{2C_0}K}{h(0)-\sqrt{2C_0}K}<1,
$$

then there exists T_1 satisfying

$$
0<T_1<\frac{1}{\sqrt{\frac{2}{C_0}}K}\ln\frac{h(0)+\sqrt{2C_0}K}{h(0)-\sqrt{2C_0}K},
$$

such that $\lim_{t\to T_1}\lim h(t)=-\infty$. This contradicts the assumption $u_x(t,x)>-M$. Let $K_0=\sqrt{2C_0}K$. As a result, we deduce that the solution blows up in finite time which is the desired result in the theorem. □

Next, we give a wave breaking result when the initial profile u_0 is odd and ρ_0 is even.

Theorem 4.17. *Let (u,ρ) be the solution of (4.1) with the initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S}), s\ge 2$, and T the maximal time of existence. Assume that $u_0\not\equiv 0$ is odd, ρ_0 is even, $u_{0,x}\le 0$ and $\rho_0(0)=0$. Assume $\gamma=A=0$. Then the corresponding solution to system (4.1) blows up in finite time.*

Proof. Similar to the proof of Lemma 4.12, it suffices to consider $s \geq 3$. Since u_0 is odd and ρ_0 is even, the corresponding solution $(u(t,x), \rho(t,x))$ satisfies that $u(t,x)$ is odd and $\rho(t,x)$ is even with respect to x for given $0 < t < T$. Hence, $u(t,0) = 0$ and $\rho_x(t,0) = 0$. Thanks to the transport equation of ρ in (4.1), we have

$$\begin{gathered} \rho_t(t,0) + \rho(t,0)u_x(t,0) = 0, \\ \rho(0,0) = 0. \end{gathered}$$

Thus, we obtain $\rho(t,0) = 0$. Evaluating (4.15) at $(t,0)$ and denoting $M(t) = u_x(t,0)$, we obtain

$$M'(t) + \frac{1}{2}M^2(t) = -G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)(t,0). \tag{4.46}$$

(a) If $u_{0,x} < 0$, then

$$M'(t) + \frac{1}{2}M^2(t) \leq 0. \tag{4.47}$$

Hence,

$$M(t) \leq M(0) = u_{0,x}(0) < 0, \quad \text{for} \quad \mathrm{t} \in [0, \mathrm{T}).$$

Integrating (4.47) on $[0,t]$, we obtain

$$-\frac{1}{M(t)} + \frac{1}{M(0)} \leq -\frac{1}{2}t.$$

Therefore,

$$u_x(t,0) = M(t) \leq \frac{2M(0)}{2 + M(0)t} \to -\infty, \quad t \to -\frac{2}{M(0)}, \tag{4.48}$$

which indicates that the maximal existence time $T \leq -(2/u_{0,x}(0))$.
(b) If $u_{0,x} = 0$, then

$$M'(t) \leq -G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)(t,0).$$

In view of $G * (\frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)(t,0) \geq 0$, we have

$$M'(t) \leq -G * u^2(t,0), t \in [0,T).$$

If there exists some $t' \in (0,T)$ such that

$$\int_{\mathbb{S}} G(x)u^2(t',x)dx = 0,$$

then we have $u(t',x) \equiv 0$. Using the uniqueness of strong solution guaranteed by Theorem 3.2, we obtain $u_0(x) = 0$. This contradicts the assumption $u_0 \not\equiv 0$. Thus, in view of the positivity of u^2 and G, we have $dM/dt(t) < 0$, $M(t)$ is strictly decreasing on [0,T). Then there exists some $t_0 \in (0,T)$ such that $M(t_0) < 0$. Solving inequality (4.47), we obtain

$$M'(t_0) \leq -\frac{1}{2}M^2(t_0) < 0.$$

Hence,

$$-\frac{1}{M(t)} + \frac{1}{M(t_0)} \leq -\frac{1}{2}(t - t_0), \quad t \in [t_0, T).$$

Consequently,

$$u_x(t,0) = M(t) \leq \frac{2M(t_0)}{2 + M(t_0)(t - t_0)} \to -\infty, \quad t \to t_0 - \frac{2}{M(t_0)}, \tag{4.49}$$

which indicates that the maximal existence time $0 < T \le t_0 - \frac{2}{M(t_0)}$. Therefore, the proof is complete. □

4.3. **Blow-up rate.** We now address the question of the blow-up rate of the slope to a breaking wave for system (4.1).

Theorem 4.18. *If $T < \infty$ is the blow-up time of the solution (u, ρ) to (4.1) with the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, satisfying the assumption of Theorem 4.14, then*

$$\lim_{t\to T^-} \Big(\Big(\inf_{x\in\mathbb{S}} u_x(t,x)\Big)(T-t)\Big) = -2. \tag{4.50}$$

Proof. We may again assume $s = 3$ to prove the theorem. In view of (4.33), we have

$$m'(t) = -\frac{1}{2}m^2(t) + f(t, \xi(t)).$$

Using (4.25), we deduce that 4.51

$$-\frac{1}{2}m^2(t) - C_2 \le m'(t) \le -\frac{1}{2}m^2(t) + C_2. \tag{4.51}$$

Choose $0 < \varepsilon < 1/2$. Since $m(t) \to -\infty$ as $t \to T^-$, we can find $t_0 \in (0, T)$ such that

$$m(t_0) < -\sqrt{2C_2 + \frac{C_2}{\varepsilon}}.$$

Since $m(t)$ is absolutely continuous on $[0, T)$. It is then inferred from (4.51) that $m(t)$ is strictly decreasing on $[t_0, T)$ and hence

$$m(t) < -\sqrt{2C_2 + \frac{C_2}{\varepsilon}} < -\sqrt{\frac{C_2}{\varepsilon}}, \qquad t \in [t_0, T).$$

This in turn implies that

$$\frac{1}{2} - \varepsilon < \frac{d}{dt}\Big(\frac{1}{m(t)}\Big) < \frac{1}{2} + \varepsilon, \quad a.e. \quad t \in [t_0, T).$$

Integrating the above relation on (t, T) with $t \in [t_0, T)$ and noticing that $m(t) \to -\infty$ as $t \to T^-$, we obtain

$$\left(\frac{1}{2} - \varepsilon\right)(T - t) < -\frac{1}{m(t)} < \left(\frac{1}{2} + \varepsilon\right)(T - t).$$

Since $\varepsilon \in (0, 1/2)$ is arbitrary, in view of the definition of $m(t)$, the above inequality implies (4.50). □

4.4. **Lower bound of the lifespan.** Our attention is now turned to a lower bound depending only on C_2 and $u_{0,x}(x_0) = \inf_{x\in\mathbb{S}} u_{0,x}(x)$ for the lifespan of the solution of system (4.1). We have the following result.

Theorem 4.19. *Assume $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, and $T_{\max} > 0$ is the lifespan of the corresponding solution to (4.1). Assume further there is some $x_0 \in S$ such that*

$$\rho_0(x_0) = 0, \quad u_{0,x}(x_0) = \inf_{x\in\mathbb{S}} u_{0,x}(x).$$

If $T_{\max} < \infty$, then the lifespan $T_{\max} > 0$ satisfies

$$\frac{T_{\max} \geq \overline{T} = \sqrt{\frac{2}{C_2}} \arctan \Big(-\sqrt{2C_2}}{\inf_{x\in\mathbb{S}} u_{0,x}(x) \Big)} \tag{4.52}$$

where

$$C_2 = \frac{5e+3}{4(e-1)} \|(u_0, \rho_0)\|^2_{H^1 \times L^2} + \frac{(-1+\sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)}$$

is defined in(4.14).

Proof. Let us first assume that the initial data $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 3$. In view of (4.33), we have

$$m'(t) = -\frac{1}{2} m^2(t) + f(t, q_2(t, x_2)) \geq -\frac{1}{2} m^2(t) - C_2.$$

Integrating this inequality, we obtain

$$\arctan \frac{m(t)}{\sqrt{2C_2}} \geq \arctan \frac{m(0)}{\sqrt{2C_2}} - \sqrt{\frac{C_2}{2}} t, \qquad \forall t < \min(T_{\max}, \overline{T}).$$

This in turn implies that

$$m(t) \geq \frac{\sqrt{2C_2} m(0) - 2C_2 \tan \big(\sqrt{\frac{C_2}{2}} t \big)}{\sqrt{2C_2} + m(0) \tan (\sqrt{\frac{C_2}{2}} t)}.$$

Due to (4.10), there appears the result (4.52) from the above inequality.

If $s \in [2, 3)$, it is easy to see the lifespan $T^s_{\max}$as a function of s for the initial data $u_{0,x}(x_0) = \inf_{x\in\mathbb{S}} u_{0,x}(x)$ with $s \geq 2$ is nonincreasing. So $T^s_{\max} \geq T^r_{\max}$ for $2 \leq s \leq r$. This ensures the validity of lower bound of the lifespan $T^s_{\max}$ in (4.52) for all $s \geq 2$. □

5. Existence of global solution

In this section, we provide a sufficient condition for the existence of a global solution of system (4.1).

Theorem 5.1. *Assume the initial data $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$, $s \geq 2$. If*

$$\inf_{x\in\mathbb{S}} \rho_0(x) > 0, \tag{5.1}$$

then the corresponding solution (u, ρ) to the initial-value problem of system (4.1), *as given by Theorem 3.2, exists globally in time.*

Proof. As before we prove this theorem for $s \geq 3$. By Theorem 4.5, to obtain global existence, it suffices to control $|u_x(t, x)|$. We will achieve this by proving the following key results.

$$\big| \inf_{x\in(\mathbb{S})} u_x(t, x) \big|, \quad \big| \sup_{x\in(\mathbb{S})} u_x(t, x) \big| \leq C_4 e^{C_3 t}, \tag{5.2}$$

where

$$C_3 = 1 + \frac{5e+3}{4(e-1)} \|(u_0, \rho_0)\|^2_{H^1 \times L^2} + \frac{(-1+\sinh 1)(\gamma - A)^2}{8\sinh^2(1/2)},$$

$$C_4 = \frac{1}{\inf_{x\in(\mathbb{S}} \rho_0(x)} \left(1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty} \right).$$

We first estimate $|\inf_{x\in(\mathbb{S})} u_x(t,x)|$. Recall that $m(t)$, $\xi(t)$ and $x_2(t)$ are defined by

$$m(t) := u_x(t,\xi(t)) = \inf_{x\in\mathbb{S}}(u_x(t,x)), \quad t\in[0,T),$$
$$u_{xx}(t,\xi(t)) = 0, \quad \text{a.e. } t\in[0,T).$$

We can choose $x_2(t)\in\mathbb{R}$, such that $q_2(t,x_2(t)) = \xi(t)$. Let $\zeta(t) = \rho\big(t,\xi(t)\big)$. Evaluating (4.16) along the trajectory $q_2(t,x)$ at $\xi(t)$ leads to

$$m'(t) = -\frac{1}{2}m^2(t) + \frac{1}{2}\zeta^2(t) + f(t,\xi(t)), \tag{5.3}$$
$$\zeta'(t) = -\zeta m, \quad t\in[0,T). \tag{5.4}$$

In view of (5.1), it follows (5.4) that $\zeta(t)$ and $\zeta(0)$ are all positive. We define the following Lyapunov function, which is due to Constantin and Ivanov [12]

$$w(t) = \zeta(t) + \frac{1}{\zeta(t)}(1+m^2(t)). \tag{5.5}$$

It is always positive in $[0,T)$ since $\zeta(t)$ and $\zeta(0)$ are all positive. Differentiating and using (5.3) and (5.4), we obtain

$$\begin{aligned} w'(t) &= \zeta'(t) - \frac{1}{\zeta^2(t)}(1+m^2(t))\zeta'(t) + \frac{2}{\zeta(t)}m'(t)m(t)\\ &= \frac{2m(t)}{\zeta(t)}\left(\frac{1}{2} + f(t,\xi(t))\right)\\ &\le \frac{1}{\zeta(t)}(1+m^2(t))\left(\frac{1}{2} + |f(t,\xi(t)))|\right)\\ &\le C_3 w(t). \end{aligned} \tag{5.6}$$

Solving (5.6) and recalling the definitions of C_3 and C_4, we infer that

$$\begin{aligned} w(t) &\le w(0)e^{C_3 t} = \frac{1}{\zeta(0)}\left(\zeta^2(0) + 1 + m^2(0)\right)e^{C_3 t}\\ &\le \frac{1}{\zeta(0)}\left(1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty}\right)e^{C_3 t}\\ &= C_4 e^{C_3 t}. \end{aligned} \tag{5.7}$$

It is easy to see that $\zeta(t)\le w(t)$ and $|m(t)|\le w(t)$. Therefore, for $t\in[0,T)$,

$$\big|\inf_{x\in(\mathbb{S})} u_x(t,x)\big| = |m(t)| \le w(t) \le C_4 e^{C_3 t}.$$

To estimate $|\sup_{x\in(\mathbb{S})} u_x(t,x)|$, recalling $\bar{m}(t)$, $\eta(t)$ and $x_1(t)$ as defined in Lemma 4.12, let $\bar\zeta(t) = \rho(t,\eta(t))$. For $t\in[0,T)$, we obtain

$$\bar m'(t) = -\frac{1}{2}\bar m^2(t) + \frac{1}{2}\bar\zeta^2(t) + f(t,\eta(t)),$$
$$\bar\zeta'(t) = -\bar\zeta\bar m.$$

Define

$$\bar w(t) = \bar\zeta(t) + \frac{1}{\bar\zeta(t)}(1+\bar m^2(t)).$$

Similar to (5.6) and (5.7), we have

$$\bar w(t) \le C_3\bar w(t) \quad\text{and}\quad \bar w(t)\le C_4 e^{C_3 t}.$$

Therefore,

$$\big|\sup_{x\in(\mathbb{S})} u_x(t,x)\big| = |\bar{m}(t)| \le \bar{w}(t) \le C_4 e^{C_3 t}, \quad t\in[0,T).$$

Therefore, the proof is complete. □

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