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EXISTENCE OF POSITIVE SOLUTIONS FOR EVEN-ORDER m-POINT BOUNDARY-VALUE PROBLEMS ON TIME SCALES

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ABSTRACT. In this article, we consider a nonlinear even-order m-point boundary-value problems on time scales. We establish the criteria for the existence of at least one, two and three positive solutions for higher order nonlinear m-point boundary-value problems on time scales by using the four functionals fixed point theorem, Avery-Henderson fixed point theorem and the five functionals fixed point theorem, respectively.

1. INTRODUCTION

Higher order multi-point boundary value problems on time scales have attracted the attention of many researchers in recent years; see for example [1, 2, 3, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein.

In this article, we are concerned with the existence of single and multiple positive solutions to the following nonlinear higher order m-point boundary value problem (BVP) on time scales:

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = f(t, y(t)), \quad t \in [t_1, t_m] \subset \mathbb{T},\ n \in \mathbb{N} \\ y^{\Delta^{2i+1}}(t_m) = 0, \quad \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = \sum_{k=2}^{m-1} y^{\Delta^{2i+1}}(t_k),\end{gathered} \tag{1.1}$$

where $\alpha > 0$ and $\beta > 0$ are given constants, $t_1 < t_2 < \ldots < t_{m-1} < t_m$, $m \geq 3$ and $0 \leq i \leq n-1$. We assume that $f : [t_1, t_m] \times [0, \infty) \to [0, \infty)$ is continuous.

Throughout this article we assume $\mathbb{T}$ is any time scale and $[t_1, t_m]$ is a subset of $\mathbb{T}$ such that $[t_1, t_m] = \{t \in \mathbb{T} : t_1 \leq t \leq t_m\}$. Some basic definitions and theorems on time scales can be found in the books [7, 8], which are excellent references for calculus of time scales.

In this article, existence results of at least one positive solution of (1.1) are first established as a result of the four functional fixed point theorem. Second, we apply the Avery-Henderson fixed-point theorem to prove the existence of at least two positive solutions to (1.1). Finally, we use the five functional fixed-point theorem to show that the existence of at least three positive solutions to (1.1). The results

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are even new for the difference equations and differential equations as well as for dynamic equations on general time scales.

2. Preliminaries

To state the main results of this paper, we will need the following lemmas.

Lemma 2.1. *If $\alpha \neq 0$, then Green's function for the boundary value problem*

$$
\begin{gathered}
-y^{\Delta^2}(t) = 0, \quad t \in [t_1, t_m], \\
y^{\Delta}(t_m) = 0, \quad \alpha y(t_1) - \beta y^{\Delta}(t_1) = \sum_{k=2}^{m-1} y^{\Delta}(t_k), \quad m \geq 3
\end{gathered}
$$

is given by

$$
G(t,s) = \begin{cases} H_1(t,s), & t_1 \leq s \leq \sigma(s) \leq t_2, \\ H_2(t,s), & t_2 \leq s \leq \sigma(s) \leq t_3, \\ \dots \\ H_{m-2}(t,s), & t_{m-2} \leq s \leq \sigma(s) \leq t_{m-1}, \\ H_{m-1}(t,s), & t_{m-1} \leq s \leq \sigma(s) \leq t_m, \end{cases} \tag{2.1}
$$

where

$$
H_j(t,s) = \begin{cases} t + \frac{\beta+j-1}{\alpha} - t_1, & t \leq s, \\ s + \frac{\beta+j-1}{\alpha} - t_1, & s \leq t, \end{cases}
$$

for all $j = 1, 2, \dots, m-1$.

Proof. A direct calculation gives that if $h \in C[t_1, t_m]$, then the boundary-value problem

$$
\begin{gathered}
-y^{\Delta^2}(t) = h(t), \quad t \in [t_1, t_m], \\
y^{\Delta}(t_m) = 0, \quad \alpha y(t_1) - \beta y^{\Delta}(t_1) = \sum_{k=2}^{m-1} y^{\Delta}(t_k), \quad m \geq 3
\end{gathered}
$$

has the unique solution

$$
\begin{aligned}
y(t) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s)\Delta s + \frac{1}{\alpha} \sum_{k=2}^{m-1} \int_{t_k}^{t_m} h(s)\Delta s + \int_t^{t_m} (t-s)h(s)\Delta s \\
&= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s)\Delta s - \sum_{j=2}^{m-1} \frac{j-1}{\alpha} \int_{t_j}^{t_{j+1}} h(s)\Delta s + \int_t^{t_m} (t-s)h(s)\Delta s.
\end{aligned}
$$

Hence, we obtain (2.1). $\square$

Lemma 2.2. *If $\alpha > 0$ and $\beta > 0$, then the Green's function $G(t,s)$ in (2.1) satisfies the inequality*

$$
G(t,s) \geq \frac{t - t_1}{t_m - t_1} G(t_m, s)
$$

for $(t,s) \in [t_1, t_m] \times [t_1, t_m]$.

Proof. (i) Let $s \in [t_1, t_m]$ and $t \le s$. Then we obtain

$$\frac{G(t,s)}{G(t_m,s)} = \frac{t + \frac{\beta+j-1}{\alpha} - t_1}{t_m + \frac{\beta+j-1}{\alpha} - t_1} > \frac{t-t_1}{t_m-t_1}.$$

(ii) For $s \in [t_1, t_m]$ and $s \le t$, we have

$$\frac{G(t,s)}{G(t_m,s)} = 1 \ge \frac{t-t_1}{t_m-t_1}.$$

□

Lemma 2.3. *If $\alpha > 0$ and $\beta > 0$, then the Green's function $G(t,s)$ in (2.1) satisfies*

$$0 < G(t,s) \le G(s,s)$$

for $(t,s) \in [t_1, t_m] \times [t_1, t_m]$.

Proof. Since $\alpha > 0$ and $\beta > 0$, $H_j(t,s) > 0$ for all $j = 1, 2, \dots, m-1$. Then we obtain $G(t,s) > 0$ from (2.1).

Now, we will show that $G(t,s) \le G(s,s)$. (i) Let $s \in [t_1, t_m]$ and $t \le s$. Since $G(t,s)$ is nondecreasing in t, $G(t,s) \le G(s,s)$.

(ii) For $s \in [t_1, t_m]$ and $s \le t$, it is clear that $G(t,s) = G(s,s)$. □

Lemma 2.4. *If $\alpha > 0$, $\beta > 0$ and $s \in [t_1, t_m]$, then the Green's function $G(t,s)$ in (2.1) satisfies*

$$\min_{t \in [t_{m-1}, t_m]} G(t,s) \ge K \|G(.,s)\|,$$

where

$$K = \frac{\beta + \alpha(t_{m-1} - t_1)}{\beta + m - 2 + \alpha(t_m - t_1)} \tag{2.2}$$

and $\|x\| = \max_{t \in [t_1, t_m]} |x(t)|$.

Proof. Since the Green's function $G(t,s)$ in (2.1) is nondecreasing in t, We have $\min_{t \in [t_{m-1}, t_m]} G(t,s) = G(t_{m-1}, s)$ In addition, it is obvious that $\|G(.,s)\| = G(s,s)$ for $s \in [t_1, t_m]$ by Lemma 2.3. Then we have

$$G(t_{m-1}, s) \ge K G(s,s)$$

from the branches of the Green's function $G(t,s)$. □

If we let $G_1(t,s) := G(t,s)$ for G as in (2.1), then we can recursively define

$$G_j(t,s) = \int_{t_1}^{t_m} G_{j-1}(t,r) G(r,s) \Delta r$$

for $2 \le j \le n$ and $G_n(t,s)$ is Green's function for the homogeneous problem

$$(-1)^n y^{\Delta^{2n}}(t) = 0, \quad t \in [t_1, t_m],$$

$$y^{\Delta^{2i+1}}(t_m) = 0, \quad \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = \sum_{k=2}^{m-1} y^{\Delta^{2i+1}}(t_k),$$

where $m \ge 3$ and $0 \le i \le n-1$.

Lemma 2.5. *Let $\alpha > 0$, $\beta > 0$. The Green's function $G_n(t,s)$ satisfies the following inequalities*

$$0 \le G_n(t,s) \le L^{n-1}\|G(.,s)\|, \quad (t,s) \in [t_1,t_m] \times [t_1,t_m],$$
$$G_n(t,s) \ge K^n M^{n-1}\|G(.,s)\|, \quad (t,s) \in [t_{m-1},t_m] \times [t_1,t_m]$$

where K is given in (2.2)*, and*

$$L = \int_{t_1}^{t_m} \|G(.,s)\|\Delta s > 0, \tag{2.3}$$

$$M = \int_{t_{m-1}}^{t_m} \|G(.,s)\|\Delta s > 0. \tag{2.4}$$

The proof of the above lemma is done using induction on n and Lemma 2.4.

Let $\mathcal{B}$ denote the Banach space $C[t_1,t_m]$ with the norm $\|y\| = \max_{t\in[t_1,t_m]} |y(t)|$. Define the cone $P \subset \mathcal{B}$ by

$$P = \{y \in \mathcal{B} : y(t) \ge 0, \min_{t\in[t_{m-1},t_m]} y(t) \ge \frac{K^n M^{n-1}}{L^{n-1}}\|y\|\} \tag{2.5}$$

where K, L, M are given in (2.2), (2.3), (2.4), respectively.

Note that (1.1) is equivalent to the nonlinear integral equation

$$y(t) = \int_{t_1}^{t_m} G_n(t,s) f(s, y(s))\Delta s. \tag{2.6}$$

We can define the operator $A : P \to \mathcal{B}$ by

$$Ay(t) = \int_{t_1}^{t_m} G_n(t,s) f(s, y(s))\Delta s, \tag{2.7}$$

where $y \in P$. Therefore, solving (2.6) in P is equivalent to finding fixed points of the operator A.

It is clear that $AP \subset P$ and $A : P \to P$ is a completely continuous operator by a standard application of the Arzela-Ascoli theorem.

Now we state the fixed point theorems which will be applied to prove main theorems. We are now in a position to present the four functionals fixed point theorem. Let φ and Ψ be nonnegative continuous concave functionals on the cone P, and let η and θ be nonnegative continuous convex functionals on the cone P. Then for positive numbers r, τ, μ and R, define the sets

$$Q(\varphi,\eta,r,R) = \{x \in P : r \le \varphi(x), \eta(x) \le R\},$$
$$U(\Psi,\tau) = \{x \in Q(\varphi,\eta,r,R) : \tau \le \Psi(x)\},$$
$$V(\theta,\mu) = \{x \in Q(\varphi,\eta,r,R) : \theta(x) \le \mu\}.$$

The following theorem can be found in [6].

Theorem 2.6 (Four Functionals Fixed Point Theorem)**.** *Suppose P is a cone in a real Banach space E, φ and Ψ are nonnegative continuous concave functionals on P, η and θ are nonnegative continuous convex functionals on P, and there exist nonnegative positive numbers r, τ, μ and R, such that $A : Q(\varphi,\eta,r,R) \to P$ is a completely continuous operator, and $Q(\varphi,\eta,r,R)$ is a bounded set. If*

(i) $\{x \in U(\Psi,\tau) : \eta(x) < R\} \cap \{x \in V(\theta,\mu) : r < \varphi(x)\} \ne \emptyset$
(ii) *$\varphi(Ax) \ge r$, for all $x \in Q(\varphi,\eta,r,R)$, with $\varphi(x) = r$ and $\mu < \theta(Ax)$,*
(iii) *$\varphi(Ax) \ge r$, for all $x \in V(\theta,\mu)$, with $\varphi(x) = r$,*

(iv) $\eta(Ax) \le R$, *for all* $x \in Q(\varphi,\eta,r,R)$, *with* $\eta(x) = R$ *and* $\Psi(Ax) < \tau$,
(v) $\eta(Ax) \le R$, *for all* $x \in U(\Psi,\tau)$, *with* $\eta(x) = R$,

then A *has a fixed point* x *in* $Q(\varphi,\eta,r,R)$.

Theorem 2.7 (Avery-Henderson Fixed Point Theorem [5]). *Let* P *be a cone in a real Banach space* E. *Set*

$$P(\phi,r) = \{u \in P : \phi(u) < r\}.$$

Assume there exist positive numbers r *and* M, *nonnegative increasing continuous functionals* η, ϕ *on* P, *and a nonnegative continuous functional* θ *on* P *with* $\theta(0) = 0$ *such that*

$$\phi(u) \le \theta(u) \le \eta(u) \quad \text{and} \quad \|u\| \le M\phi(u)$$

for all $u \in \overline{P(\phi,r)}$. *Suppose that there exist positive numbers* $p < q < r$ *such that*

$$\theta(\lambda u) \le \lambda\theta(u), \quad \text{for } 0 \le \lambda \le 1 \text{ and } u \in \partial P(\theta,q).$$

If $A : \overline{P(\phi,r)} \to P$ *is a completely continuous operator satisfying*

(i) $\phi(Au) > r$ *for all* $u \in \partial P(\phi,r)$,
(ii) $\theta(Au) < q$ *for all* $u \in \partial P(\theta,q)$,
(iii) $P(\eta,p) \ne \emptyset$ *and* $\eta(Au) > p$ *for all* $u \in \partial P(\eta,p)$,

then A *has at least two fixed points* u_1 *and* u_2 *such that*

$$p < \eta(u_1) \text{ with } \theta(u_1) < q \quad \text{and} \quad q < \theta(u_2) \text{ with } \phi(u_2) < r.$$

Now, we will present the five functionals fixed point theorem. Let φ,η,θ be nonnegative continuous convex functionals on the cone P, and γ,Ψ nonnegative continuous concave functionals on the cone P. For nonnegative numbers h,a,b,d and c, define the following convex sets:

$$\begin{gathered}
P(\varphi,c) = \{x \in P : \varphi(x) < c\},\\
P(\varphi,\gamma,a,c) = \{x \in P : a \le \gamma(x), \varphi(x) \le c\},\\
Q(\varphi,\eta,d,c) = \{x \in P : \eta(x) \le d, \varphi(x) \le c\},\\
P(\varphi,\theta,\gamma,a,b,c) = \{x \in P : a \le \gamma(x), \theta(x) \le b, \varphi(x) \le c\},\\
Q(\varphi,\eta,\Psi,h,d,c) = \{x \in P : h \le \Psi(x), \eta(x) \le d, \varphi(x) \le c\}.
\end{gathered} \tag{2.8}$$

The following theorem can be found in [4].

Theorem 2.8 (Five Functionals Fixed Point Theorem). *Let* P *be a cone in a real Banach space* E. *Suppose that there exist nonnegative numbers* c *and* M, *nonnegative continuous concave functionals* γ *and* Ψ *on* P, *and nonnegative continuous convex functionals* φ,η *and* θ *on* P, *with*

$$\gamma(x) \le \eta(x), \|x\| \le M\varphi(x), \forall x \in \overline{P(\varphi,c)}.$$

Suppose that $A : \overline{P(\varphi,c)} \to \overline{P(\varphi,c)}$ *is a completely continuous and there exist nonnegative numbers* h,a,k,b, *with* $0 < a < b$ *such that*

(i) $\{x \in P(\varphi,\theta,\gamma,b,k,c) : \gamma(x) > b\} \ne \emptyset$ *and* $\gamma(Ax) > b$ *for* $x \in P(\varphi,\theta,\gamma,b,k,c)$,
(ii) $\{x \in Q(\varphi,\eta,\Psi,h,a,c) : \eta(x) < a\} \ne \emptyset$ *and* $\eta(Ax) < a$ *for* $x \in Q(\varphi,\eta,\Psi,h,a,c)$,
(iii) $\gamma(Ax) > b$, *for* $x \in P(\varphi,\gamma,b,c)$, *with* $\theta(Ax) > k$,
(iv) $\eta(Ax) < a$, *for* $x \in Q(\varphi,\eta,a,c)$, *with* $\Psi(Ax) < h$,

then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, c)}$ such that

$$\eta(x_1) < a, \quad \gamma(x_2) > b, \quad \eta(x_3) > a \quad \text{with } \gamma(x_3) < b.$$

3. Main Results

Now, we will give the sufficient conditions to have at least one positive solution for (1.1). The four functionals fixed point theorem will be used to prove the next theorem.

Theorem 3.1. *Let $\alpha > 0$ and $\beta > 0$. Suppose that there exist constants r, R, μ, τ with $0 < r < \tau \le \mu < R, r = \frac{K^n M^{n-1}}{L^{n-1}}\mu$ and $R = \frac{\tau L^{n-1}}{K^n M^{n-1}}$. If the function f satisfies the following conditions:*

(i) *$f(t, y) \ge \frac{r}{K^n M^n}$ for all $(t, y) \in [t_{m-1}, t_m] \times [r, \mu]$,*
(ii) *$f(t, y) \le \frac{R}{L^n}$ for all $(t, y) \in [t_1, t_m] \times [0, R]$,*

then (1.1) has at least one positive solution y such that $r \le y(t) \le R$ for $t \in [t_1, t_m]$.

Proof. Define the maps

$$\varphi(y) = \Psi(y) = \min_{t \in [t_{m-1}, t_m]} y(t),$$
$$\theta(y) = \max_{t \in [t_{m-1}, t_m]} y(t),$$
$$\eta(y) = \max_{t \in [t_1, t_m]} y(t).$$

Then φ and Ψ are nonnegative continuous concave functionals on P, and η and θ are nonnegative continuous convex functionals on P. Since

$$\|y\| = \max_{t \in [t_1, t_m]} |y(t)| = \eta(y) \le R$$

for all $y \in Q(\varphi, \eta, r, R)$, $Q(\varphi, \eta, r, R)$ is a bounded set. Note that the operator $A : Q(\varphi, \eta, r, R) \to P$ is completely continuous by a standard application of the Arzela-Ascoli theorem.

Now, we verify that the remaining conditions of Theorem 2.6. We obtain

$$\Psi(\frac{\mu}{2}) = \mu \ge \tau, \quad \eta(\frac{\mu}{2}) = \mu < R,$$
$$\theta(\frac{\mu}{2}) = \mu, \quad \varphi(\frac{\mu}{2}) = \mu > r.$$

Then, we have $\frac{\mu}{2} \in \{y \in U(\Psi, \tau) : \eta(y) < R\} \cap \{y \in V(\theta, \mu) : \varphi(y) > r\}$, which means that (i) in Theorem 2.6 is fulfilled.

Now, we shall verify that condition (ii) of Theorem 2.6 is satisfied. By Lemma 2.5, we obtain

$$\begin{aligned}\theta(Ay) &= \int_{t_1}^{t_m} G_n(t_m, s) f(s, y(s)) \Delta s \\ &\le L^{n-1} \int_{t_1}^{t_m} \|G(., s)\| f(s, y(s)) \Delta s.\end{aligned}$$

Since $\theta(Ay) > \mu$, we find that

$$\int_{t_1}^{t_m} \|G(., s)\| f(s, y(s)) \Delta s > \frac{\mu}{L^{n-1}}. \tag{3.1}$$

Then, we obtain

$$\begin{aligned}\varphi(Ay) &= \int_{t_1}^{t_m} G_n(t_{m-1}, s) f(s, y(s))\Delta s\\ &\geq K^n M^{n-1} \int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s > r,\end{aligned}$$

using Lemma 2.5 and (3.1).

Now, we shall show that condition (iii) of Theorem 2.6 holds. Since $\varphi(y) = r$ and $y \in V(\theta, \mu)$, we find that $r \leq y(t) \leq \mu$ for $t \in [t_{m-1}, t_m]$. By Lemma 2.5 and the hypothesis (i), we have

$$\begin{aligned}\varphi(Ay) &= \int_{t_1}^{t_m} G_n(t_{m-1}, s) f(s, y(s))\Delta s\\ &\geq K^n M^{n-1} \int_{t_{m-1}}^{t_m} \|G(., s)\| f(s, y(s))\Delta s \geq r.\end{aligned}$$

Now, we shall verify that condition (iv) of Theorem 2.6 is fulfilled. We get

$$\begin{aligned}\Psi(Ay) &= \int_{t_1}^{t_m} G_n(t_{m-1}, s) f(s, y(s))\Delta s\\ &\geq K^n M^{n-1} \int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s\end{aligned}$$

using Lemma 2.5. Since $\Psi(Ay) < \tau$,

$$\int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s < \frac{\tau}{K^n M^{n-1}}. \tag{3.2}$$

Then, by Lemma 2.5 and (3.2) we obtain

$$\begin{aligned}\eta(Ay) &= \int_{t_1}^{t_m} G_n(t_m, s) f(s, y(s))\Delta s\\ &\leq L^{n-1} \int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s < R.\end{aligned}$$

Finally, we shall show that condition (v) of Theorem 2.6 is satisfied. Since $\eta(y) = R$, we find $0 \leq y(t) \leq R$ for $t \in [t_1, t_m]$. Using Lemma 2.5 and the hypothesis (ii), we have

$$\begin{aligned}\eta(Ay) &= \int_{t_1}^{t_m} G_n(t_m, s) f(s, y(s))\Delta s\\ &\leq L^{n-1} \int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s \leq R.\end{aligned}$$

Hence, by Theorem 2.6, the (1.1) has at least one positive solution y such that $r \leq y(t) \leq R$ for $t \in [t_1, t_m]$. This completes the proof. □

Now we will use the Avery-Henderson fixed point theorem to prove the next theorem.

Theorem 3.2. *Assume $\alpha > 0$, $\beta > 0$. Suppose there exist numbers $0 < p < q < r$ such that the function f satisfies the following conditions:*

(i) $f(t, y) > \frac{r}{K^n M^n}$ *for* $(t, y) \in [t_{m-1}, t_m] \times [r, \frac{rL^{n-1}}{K^n M^{n-1}}]$;

(ii) $f(t,y) < \frac{q}{L^n}$ *for* $(t,y) \in [t_1, t_m] \times [0, \frac{qL^{n-1}}{K^n M^{n-1}}]$;

(iii) $f(t,y) > \frac{p}{K^n M^n}$ *for* $t \in [t_{m-1}, t_m] \times [\frac{K^n M^{n-1}}{L^{n-1}} p, p]$,

where K, L, M, *are defined in* (2.2), (2.3), (2.4), *respectively. Then* (1.1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$p < \max_{t \in [t_1, t_m]} y_1(t) \quad \textit{with} \quad \max_{t \in [t_{m-1}, t_m]} y_1(t) < q$$

$$q < \max_{t \in [t_{m-1}, t_m]} y_2(t) \quad \textit{with} \quad \min_{t \in [t_{m-1}, t_m]} y_2(t) < r.$$

Proof. Define the cone P as in (2.5). From Lemma 2.5, $AP \subset P$ and A is completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by

$$\phi(y) := \min_{t \in [t_{m-1}, t_m]} y(t), \quad \theta(y) := \max_{t \in [t_{m-1}, t_m]} y(t), \quad \eta(y) := \max_{t \in [t_1, t_m]} y(t).$$

For each $y \in P$, we have $\phi(y) \le \theta(y) \le \eta(y)$, and from (2.5)

$$\|y\| \le \frac{L^{n-1}}{K^n M^{n-1}} \phi(y).$$

Moreover, $\theta(0) = 0$ and for all $y \in P$, $\lambda \in [0,1]$ we obtain $\theta(\lambda y) = \lambda \theta(y)$.

We now verify that the remaining conditions of Theorem 2.7 hold.

Claim 1: If $y \in \partial P(\phi, r)$, then $\phi(Ay) > r$: Since $y \in \partial P(\phi, r)$ and $\|y\| \le \frac{L^{n-1}}{K^n M^{n-1}} \phi(y)$, we have $r \le y(t) \le \frac{rL^{n-1}}{K^n M^{n-1}}$ for $t \in [t_{m-1}, t_m]$. Then, by hypothesis (i) and Lemma 2.5 we find that

$$\begin{aligned} \phi(Ay) &= \int_{t_1}^{t_m} G_n(t_{m-1}, s) f(s, y(s)) \Delta s \\ &\ge K^n M^{n-1} \int_{t_{m-1}}^{t_m} \|G(., s)\| f(s, y(s)) \Delta s > r. \end{aligned}$$

Claim 2: If $y \in \partial P(\theta, q)$, then $\theta(Ay) < q$: Since $y \in \partial P(\theta, q)$ and $\|y\| \le \frac{L^{n-1}}{K^n M^{n-1}} \phi(y)$, $0 \le y(t) \le \frac{qL^{n-1}}{K^n M^{n-1}}$ for $t \in [t_1, t_m]$. Thus, using hypothesis (ii) and Lemma 2.5, we obtain

$$\begin{aligned} \theta(Ay) &= \int_{t_1}^{t_m} G_n(t_m, s) f(s, y(s)) \Delta s \\ &\le L^{n-1} \int_{t_1}^{t_m} \|G(., s)\| f(s, y(s)) \Delta s < q. \end{aligned}$$

Claim 3: $P(\eta, p) \ne \emptyset$ and $\eta(Ay) > p$ for all $y \in \partial P(\eta, p)$: Since $\frac{p}{2} \in P$ and $p > 0$, $\frac{p}{2} \in P(\eta, p)$. If $y \in \partial P(\eta, p)$ and $\eta(y) \ge \frac{K^n M^{n-1}}{L^{n-1}} \|y\|$, we obtain $\frac{K^n M^{n-1}}{L^{n-1}} p \le y(t) \le \|y\| = p$ for $t \in [t_{m-1}, t_m]$. Hence, by hypothesis (iii) and Lemma 2.5 we have

$$\begin{aligned} \eta(Ay) &= \int_{t_1}^{t_m} G_n(t_m, s) f(s, y(s)) \Delta s \\ &\ge K^n M^{n-1} \int_{t_{m-1}}^{t_m} \|G(., s)\| f(s, y(s)) \Delta s > p. \end{aligned}$$

Since the conditions of Theorem 2.7 are satisfied, BVP (1.1) has at least two positive solutions y_1 and y_2 such that

$$p < \max_{t \in [t_1, t_m]} y_1(t) \quad \text{with} \quad \max_{t \in [t_{m-1}, t_m]} y_1(t) < q$$

$$q < \max_{t\in[t_{m-1},t_m]} y_2(t) \quad \text{with} \quad \min_{t\in[t_{m-1},t_m]} y_2(t) < r. \qquad \square$$

Now, we will apply the five functionals fixed point theorem to investigate the existence of at least three positive solutions for (1.1).

Theorem 3.3. *Let $\alpha > 0$ and $\beta > 0$. Suppose that there exist constants a, b, c with $0 < a < b < \frac{bL^{n-1}}{K^n M^{n-1}} < c$ such that the function f satisfies the following conditions:*

- (i) *$f(t, y) \le \frac{c}{L^n}$ for $(t, y) \in [t_1, t_m] \times [0, c]$,*
- (ii) *$f(t, y) > \frac{b}{K^n M^n}$ for $(t, y) \in [t_{m-1}, t_m] \times [b, \frac{bL^{n-1}}{K^n M^{n-1}}]$,*
- (iii) *$f(t, y) < \frac{a}{L^n}$ for $(t, y) \in [t_1, t_m] \times [0, a]$,*

where K, L, M are as defined in (2.2), (2.3), (2.4), respectively. Then (1.1) has at least three positive solutions y_1, y_2 and y_3 such that

$$\max_{t\in[t_1,t_m]} y_1(t) < a < \max_{t\in[t_1,t_m]} y_3(t),$$
$$\min_{t\in[t_{m-1},t_m]} y_3(t) < b < \min_{t\in[t_{m-1},t_m]} y_2(t).$$

Proof. Define the cone P as in (2.5) and define these maps

$$\gamma(y) = \Psi(y) = \min_{t\in[t_{m-1},t_m]} y(t),$$
$$\theta(y) = \max_{t\in[t_{m-1},t_m]} y(t),$$
$$\varphi(y) = \eta(y) = \max_{t\in[t_1,t_m]} y(t).$$

Then γ and Ψ are nonnegative continuous concave functionals on P, and φ, η and θ are nonnegative continuous convex functionals on P. Let $P(\varphi, c)$, $P(\varphi, \gamma, a, c)$, $Q(\varphi, \eta, d, c)$, $P(\varphi, \theta, \gamma, a, b, c)$ and $Q(\varphi, \eta, \Psi, h, d, c)$ be defined by (2.8). It is clear that

$$\gamma(y) \le \eta(y), \quad \|y\| = \varphi(y), \quad \forall y \in \overline{P(\varphi, c)}.$$

If $y \in \overline{P(\varphi, c)}$, then we have $y(t) \in [0, c]$ for all $t \in [t_1, t_m]$. By Lemma 2.5 and the hypothesis (i), we obtain

$$\begin{aligned}\varphi(Ay) &= \int_{t_1}^{t_m} G_n(t_m, s) f(s, y(s))\Delta s \\ &\le L^{n-1} \int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s \le c.\end{aligned}$$

This proves that $A : \overline{P(\varphi, c)} \to \overline{P(\varphi, c)}$.

Now we verify that the remaining conditions of Theorem 2.8. Let $y_1 = \frac{b+\epsilon_1}{2}$ such that $0 < \epsilon_1 < (\frac{L^{n-1}}{K^n M^{n-1}} - 1)b$. Since

$$\gamma(y_1) = b + \epsilon_1 > b,$$
$$\theta(y_1) = b + \epsilon_1 < \frac{bL^{n-1}}{K^n M^{n-1}},$$
$$\varphi(y_1) = b + \epsilon_1 < \frac{bL^{n-1}}{K^n M^{n-1}} < c,$$

we obtain

$$\{y \in P(\varphi, \theta, \gamma, b, \frac{bL^{n-1}}{K^n M^{n-1}}, c) : \gamma(y) > b\} \ne \emptyset.$$

If $y \in P(\varphi,\theta,\gamma,b,\frac{bL^{n-1}}{K^nM^{n-1}},c)$, then we have $b \le y(t) \le \frac{bL^{n-1}}{K^nM^{n-1}}$ for all $t \in [t_{m-1},t_m]$. By using Lemma 2.5 and the hypothesis (ii), we obtain

$$\begin{aligned}\gamma(Ay) &= \int_{t_1}^{t_m} G_n(t_{m-1},s)f(s,y(s))\Delta s\\ &\ge K^nM^{n-1}\int_{t_{m-1}}^{t_m}\|G(.,s)\|f(s,y(s))\Delta s > b.\end{aligned}$$

Thus, the condition (i) of Theorem 2.8 holds.

Let $y_2 = \frac{a-\epsilon_2}{2}$ such that $0 < \epsilon_2 < (1-\frac{K^nM^{n-1}}{L^{n-1}})a$. Since

$$\begin{gathered}\eta(y_2) = a-\epsilon_2 < a,\\ \Psi(y_2) = a-\epsilon_2 > \frac{K^nM^{n-1}}{L^{n-1}}a,\\ \varphi(y_2) = a-\epsilon_2 < c,\end{gathered}$$

we find that

$$\{y \in Q(\varphi,\eta,\Psi,\frac{K^nM^{n-1}}{L^{n-1}}a,a,c) : \eta(y) < a\} \ne \emptyset.$$

If $y \in Q(\varphi,\eta,\Psi,\frac{K^nM^{n-1}}{L^{n-1}}a,a,c)$, then we obtain $0 \le y(t) \le a$, for $t \in [t_1,t_m]$. Hence,

$$\begin{aligned}\eta(Ay) &= \int_{t_1}^{t_m} G_n(t_m,s)f(s,y(s))\Delta s\\ &\le L^{n-1}\int_{t_1}^{t_m}\|G(.,s)\|f(s,y(s))\Delta s < a\end{aligned}$$

by Lemma 2.5 and the hypothesis *(iii)*. It follows that condition (ii) of Theorem 2.8 is fulfilled.

Now, we shall show that the condition (iii) of Theorem 2.8 is satisfied. We have

$$\begin{aligned}\theta(Ay) &= \int_{t_1}^{t_m} G_n(t_m,s)f(s,y(s))\Delta s\\ &\le L^{n-1}\int_{t_1}^{t_m}\|G(.,s)\|f(s,y(s))\Delta s\end{aligned}$$

using Lemma 2.5. Since $\theta(Ay) > \frac{bL^{n-1}}{K^nM^{n-1}}$, we obtain

$$\int_{t_1}^{t_m}\|G(.,s)\|f(s,y(s))\Delta s > \frac{b}{K^nM^{n-1}}. \tag{3.3}$$

Then, by Lemma 2.5 and (3.3) we find that

$$\begin{aligned}\gamma(Ay) &= \int_{t_1}^{t_m} G_n(t_{m-1},s)f(s,y(s))\Delta s\\ &\ge K^nM^{n-1}\int_{t_1}^{t_m}\|G(.,s)\|f(s,y(s))\Delta s > b.\end{aligned}$$

Finally, we shall verify that the condition (iv) of Theorem 2.8 holds. By Lemma 2.5, we obtain

$$\Psi(Ay) = \int_{t_1}^{t_m} G_n(t_{m-1},s)f(s,y(s))\Delta s$$

$$\geq K^n M^{n-1} \int_{t_1}^{t_m} \|G(.,s)\| f(s,y(s))\Delta s.$$

Since $\Psi(Ay) < \frac{K^n M^{n-1}}{L^{n-1}} a$, we have

$$\int_{t_1}^{t_m} \|G(.,s)\| f(s,y(s))\Delta s < \frac{a}{L^{n-1}}. \tag{3.4}$$

Then, we find that

$$\begin{aligned}\eta(Ay) &= \int_{t_1}^{t_m} G_n(t_m,s) f(s,y(s))\Delta s \\ &\leq L^{n-1} \int_{t_1}^{t_m} \|G(.,s)\| f(s,y(s))\Delta s < a.\end{aligned}$$

using Lemma 2.5 and (3.4).

Since the conditions of Theorem 2.8 are satisfied, (1.1) has at least three positive solutions $y_1, y_2, y_3 \in \overline{P(\varphi, c)}$ such that

$$\begin{gathered}\max_{t\in[t_1,t_m]} y_1(t) < a < \max_{t\in[t_1,t_m]} y_3(t), \\ \min_{t\in[t_{m-1},t_m]} y_3(t) < b < \min_{t\in[t_{m-1},t_m]} y_2(t).\end{gathered}$$

This completes the proof. □

Example 3.4. Let $\mathbb{T} = \{(1/5)^n : n \in \mathbb{N}\} \cup \{0\} \cup [3,5]$. We consider the boundary value problem

$$\begin{gathered}-y^{\Delta^4}(t) = \frac{2013y^2}{y^2+2013}, \quad t \in [\frac{1}{5}, 5] \subset \mathbb{T} \\ y^{\Delta}(5) = 0, \quad \frac{1}{2} y(\frac{1}{5}) - 2y^{\Delta}(\frac{1}{5}) = y^{\Delta}(3) + y^{\Delta}(4), \\ y^{\Delta^3}(5) = 0, \quad \frac{1}{2} y^{\Delta^2}(\frac{1}{5}) - 2y^{\Delta^3}(\frac{1}{5}) = y^{\Delta^3}(3) + y^{\Delta^3}(4).\end{gathered} \tag{3.5}$$

If we take $r = 84500$, $\tau = 187083$, $\mu = 633177.9584$ and $R = 1401856$, then all the conditions in Theorem 3.1 are satisfied. Thus, BVP (3.5) has at least one positive solution y such that $84500 \leq y(t) \leq 1401856$ for $t \in [\frac{1}{5}, 5]$.

If we take $p = 2\times 10^{-5}$, $q = 26 \times 10^{-6}$ and $r = 24 \times 10^{-3}$, then all the conditions in Theorem 3.2 are fulfilled. Hence, BVP (3.5) has at least two positive solutions y_1 and y_2 satisfying

$$\begin{gathered}2\times 10^{-5} < \max_{t\in[\frac{1}{5},5]} y_1(t) \quad \text{with} \quad \max_{t\in[4,5]} y_1(t) < 26\times 10^{-6} \\ 26\times 10^{-6} < \max_{t\in[4,5]} y_2(t) \quad \text{with} \quad \min_{t\in[4,5]} y_2(t) < 24\times 10^{-3}.\end{gathered}$$

If we take $a = 15.10^{-4}$, $b = 1$ and $c = 1320000$, then all the conditions in Theorem 3.3 hold. Therefore, BVP (3.5) has at least three positive solutions y_1, y_2 and y_3 such that

$$\begin{gathered}\max_{t\in[\frac{1}{5},5]} y_1(t) < 15\times 10^{-4} < \max_{t\in[\frac{1}{5},5]} y_3(t), \\ \min_{t\in[4,5]} y_3(t) < 1 < \min_{t\in[4,5]} y_2(t).\end{gathered}$$

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