Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 46, pp. 1–42.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A HYPERBOLIC-PARABOLIC SYSTEM ARISING IN PULSE COMBUSTION: EXISTENCE OF SOLUTIONS FOR THE LINEARIZED PROBLEM

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Abstract. A mixed hyperbolic-parabolic system is derived for a lumped parameter continuum model of pulse combustion. For a regularized version of the initial-boundary value problem for an associated linear system, with time-dependent boundary conditions, Galerkin approximations are used to establish the existence of a suitable class of unique solutions. Standard parabolic theory is then employed to established higher regularity for the solutions of the regularized problem. Finally, a priori estimates are derived which allow for letting the artificial viscosity, in the regularized system, approach zero so as to obtain the existence of a unique solution for the original mixed hyperbolic-parabolic problem.

1. Introduction

Pulse combustion is a process in which pressure, velocity, and temperature, vary periodically with time; it was first observed by Rayleigh [52]. A basic pulse combustor consists of a set of intake valves for air and fuel, a combustion chamber, and a tailpipe from which the combustion products are expelled as a consequence of the oscillating pressure field in the chamber. As air and gas enter the chamber, combustion raises the temperature and pressure; when the pressure rises above atmospheric pressure, the valves begin to close, the air-fuel input is reduced or stopped and the combustion products begin flowing through the tailpipe, leading to a decrease in chamber pressure. Once the chamber pressure falls below atmospheric pressure the valves open to admit another fuel-air charge. The gaseous products in the tailpipe execute oscillations which are superimposed on a mean flow and which produce a periodic jet that issues from the open end of the tailpipe; it is this periodic jet which drives the resulting impingement heat transfer process (see figures 1, 2 and 3).

A variety of models have been constructed to simulate the behavior of pulse combustors, e.g., [2, 3, 6, 16, 18, 19, 22, 24, 26, 27, 28, 36, 37, 38, 47, 51, 60, 62] and some work has been done to examine factors influencing combustion in the

2000 *Mathematics Subject Classification.* 35M33, 35B65, 80A25.
Key words and phrases. Pulse combustion; linear hyperbolic-parabolic system;
Galerkin approximation; global solution.

Submitted July 10, 2012. Published February 8, 2013.
This article is based on the Ph.D. dissertation of the first author at Northern Illinois University.

combustion chamber and the tailpipe flow field in [20, 21, 39, 54]. For analysis of the associated problem of acoustic oscillations in resonance tubes one may consult [23, 32, 40, 41, 44, 56, 61].

A description of the complex processes occurring in a typical pulse combustor may be found in [3, 26], e.g., when the chamber pressure is equal to atmospheric pressure turbulence is present within a layer separating the fresh air-fuel mixture and the residual gas from the preceding cycle; this layer contains ignition nuclei and is broken up into vortices which are carried into the fresh mixture thus igniting it and producing a flame which transits the length of the combustion chamber. Analytical models of pulse combustor operation which take into account the full range of physical processes present are not feasible. Most pulse combustion models attack the problem by writing down a set of conservation laws for the ongoing combustion process and presenting results, in graphical form, based on numerical analysis of the governing system; such an approach is not illuminating if the goal is the production of a combined model which would allow for making qualitative (as well as quantitative) predictions of the effect of varying pulse combustor physical and geometrical design characteristics. With the exception of the work in [26], none of the literature has attempted to deal with the influence of value design and operation on pulse combustor performance including the important issue of the qualitative behavior of the jet which issues from the combustor tailpipe; a notable exception is [2] where, however, the model (in its original form) assumes an instantaneous opening and closing of the valves that is unrealistic and has the effect of inducing a discontinuity in the mathematical model. Shortcomings in the model presented in [2] have been addressed in [8]. Recently results were obtained by applying the technique of averaging, to the dynamical system generated by the lumped parameter pulse combustor model discussed in [8] and were presented in [9].

The lumped parameter model of pulse combustion, which is found in [2], while not dealing with all the chemical kinetics processes involved, incorporates a realistic valve dynamics submodel and is capable of producing closed-form approximations for pressure and temperature variations in the chamber and velocity oscillations in the tailpipe. The work in [2] begins with a statement of energy balance and assumes that there are two uniform regions in the combustion chamber, a 'cool' zone consisting of the reactants and a hot zone containing the combustion products; these regions are separated by a moving flame front. The model assumes a spatially uniform pressure $p(t)$, ignores friction in the tailpipe, does not account for heat loss from the chamber, and also assumes that the combustion products in the tailpipe are incompressible. Balance of energy in [2] yields the equation

$$\Big(\frac{c_v V_B}{R}\Big)\frac{dp}{dt} = h_R\dot{m}_R + \frac{\Delta H}{1+r}\dot{m}_B - h_0\bar{\rho}Av(t) \tag{1.1}$$

where c_v/R is the (approximate) constant ratio of specific heat (at constant volume) to the gas constant for the air and fuel, V_B is the chamber volume, A is the cross-sectional area of the cylindrical tailpipe, r is the air-fuel (mass) ratio, ΔH is the heat of combustion per unit mass of fuel, h_R is the enthalpy, per unit mass, of the reactant mixture, h_0 is the enthalpy, per unit mass of the combustion products, $v(t)$ is the velocity of the combustion products in the tailpipe, $\bar{\rho}$ is the average density of the combustion products in the tailpipe, and $\dot{m}_R, \dot{m}_B$ are, respectively, the mass flow rate of the reactants and the mass burning rate of the reactant mixture in the chamber. Coupled to (1.1), as a consequence of the continuity equation, is the balance of momentum equation

$$\bar{\rho} L \frac{dv}{dt} = p(t) - p_a \equiv \tilde{p}(t) \tag{1.2}$$

In (1.2), L is the length of the tailpipe while p_a is atmospheric pressure at the open end of the tailpipe. To (1.1) and (1.2) we must append constitutive equations relating the reactant mass flow rate $\dot{m}_R$ and the mass burning rate $\dot{m}_B$ to the chamber pressure p. In [2] it was assumed that the valves are, at any time t, either fully-open, or fully-closed, depending on whether or not there exists a pressure induced driving force for flow into the chamber; these inflows of air and fuel (gas), with respective mass flow rates $\dot{m}_a$ and $\dot{m}_g$, were described by the orifice flow equations

$$\dot{m}_a = \begin{cases} \sqrt{2\rho_a} C_{Da} A_a \sqrt{p_a - p}, & p < p_a \\ 0, & p \geq p_a, \end{cases} \tag{1.3a}$$

$$\dot{m}_g = \begin{cases} \sqrt{2\rho_g} C_{Dg} A_g \sqrt{p_g - p}, & p < p_g \\ 0, & p \geq p_g, \end{cases} \tag{1.3b}$$

where C_{D_a}, C_{Dg} are the discharge coefficients of the air and gas valves, ρ_a and ρ_g are the air and gas densities, and A_a and A_g are the effective flow areas of the air and gas valves. If $p_g \simeq p_a$ then

$$\dot{m}_R = (1+r)\dot{m}_g = \begin{cases} (1+r)\Gamma_g \sqrt{p_a - p}, & p < p_a \\ 0, & p \geq p_a, \end{cases} \tag{1.4}$$

where $r = \Gamma_a / \Gamma_g$ is the constant air-fuel ratio with $\Gamma_g = \sqrt{2\rho_g} C_{D_g} A_g$ and $\Gamma_a = \sqrt{2\rho_a} C_{D_a} A_a$. The relation (1.4) presents two difficulties: (i) it assumes an instantaneous opening (closing) of the valves at any time when the combustion pressure $p(t)$ falls below (rises above) p_a and (ii) it yields an $\dot{m}_R(p)$ which is not differentiable at any t where $p(t) = p_a$; to deal with these problems the authors in [2] replaced (1.4) by $\dot{m}_R^\epsilon = \Gamma_g(1+r) H_\epsilon(p - p_a)\sqrt{p - p_a}$ where H_ϵ, for $\epsilon > 0$, represents a smoothing of the usual Heaviside function. Using this approach, one can then either study the resulting model for finite $\epsilon > 0$, which leads to valve hysteresis, or, by imposing stability criteria associated with stable burner operation, extract explicit approximate expressions for the frequency ω and period T_p, of combustor pressure oscillations, as $\epsilon \to 0^+$; these stability criteria are equivalent to the statements

(i) there should be no net reactant accumulation or depletion over one cycle; i.e.,

$$\lim_{\epsilon \to 0^+} \int_0^{T_p} \dot{m}_R^\epsilon dt = \int_0^{T_p} \dot{m}_B dt\,, \tag{1.5a}$$

(ii) there should be no net pressure buildup or decay over successive cycles; i.e.,

$$\lim_{\epsilon \to 0^+} \int_0^{T_p} (p_\epsilon(t) - p_a) dt = 0\,. \tag{1.5b}$$

In [2] the actual flame structure in the chamber was idealized to consist of an equivalent plane flame sheet filling the combustion chamber cross-sectional area A_B; the plane flame propagates with a 'burning velocity' U_f, which is pressure-independent, relative to the unburned reactant mixture in the chamber. Under these assumptions $\dot{m}_B = \rho_R A_B U_f$ and, if the reactants (air and gas) are taken to be at the same constant temperature θ_a, the perfect gas law yields $\dot{m}_B = (A_B U_f / R\theta_a)p$.

In [8] it was shown that the model described above allows for the computation of analytical expressions for $p(t), v(t), U_f, T_p$ and the velocity v_0 at the inception of the first full stable chamber cycle, which display an explicit dependence on all relevant combustor physical and geometrical parameters; for a range of air-fuel ratios r these expressions yield a tailpipe velocity which exhibits flow reversal. In this model U_f is not the actual flame velocity, which depends on the specific diffusion, heat transfer, and chemical kinetics mechanisms at work during the burning process but is, rather, a system parameter whose value is compatible with the achievement of stable system oscillation.

Upon eliminating between (1.1) and (1.2) one obtains for $\tilde{p}(t)$ the nonlinear second order equation

$$\frac{d^2\tilde{p}}{dt^2} - \{\lambda^u + \lambda^d(\tilde{p})\}\frac{d\tilde{p}}{dt} + \omega_0^2\tilde{p} = 0\,, \tag{1.6}$$

where $\omega_0^2 = Rh_\theta A/c_v V_B L$ and

$$\lambda^u = \frac{\Delta H}{1+r}\Big(\frac{A_B U_f}{c_v \theta_a V_B}\Big), \quad \lambda^d(\tilde{p}) = \Big(\frac{Rh_R}{c_v V_B}\Big)\frac{d}{d\tilde{p}}\dot{m}_R(\tilde{p})\,. \tag{1.7}$$

In [8] an approximation to the solution of the initial-value problem for (1.6) was constructed which is periodic with period

$$T_p = \frac{\pi}{\omega_*} + \frac{\pi}{\sqrt{2\omega_0}}\Big(1 + \exp[\frac{-\lambda^u\pi}{2\omega_*}]\Big)^{1/2} \tag{1.8}$$

where $\omega_* = \omega_0\sqrt{1 - \frac{1}{4}\epsilon^2}$ with $\epsilon = \lambda^u/\omega_0$. For $\epsilon << 1$ $T_p \approx 2\pi/\omega_*$.

Further results have been obtained recently by one of the authors and his colleagues, in [7, 9, 10, 11, 12, 13, 14], by applying perturbation theory and dynamical systems analyses to study the behavior exhibited by spatially independent pulse combustor models of the type presented in this section; these results relate, e.g., to the effect of tailpipe friction on pressure and velocity oscillations, the influence of convective and radiative heat transfer, and the optimization of reactant flow rates and mass burning rates in lumped parameter pulse combustor models. For the balance of this paper, as well as in the follow-up paper [5], the focus will be on the pulse combustor models incorporating spatial dependence.

In the present paper we will formulate a one-dimensional model of pulse combustion; the model will be contrasted with earlier efforts in this direction, and the resulting set of governing equations will be shown to reduce, under an appropriate set of hypotheses, to the zero-space dimensional case introduced in this section. The initial-boundary value problem for the one-dimensional pulse combustor will also be compared to other problems in the broad realm of gas dynamics which have been treated extensively in the literature. In [5] we establish local and global existence of smooth solutions for the nonlinear initial-boundary value problem introduced in this paper. As the proof of local existence in [5] is dependent on a fixed-point argument, we establish, in this paper, the existence and uniqueness result for the relevant linearized hyperbolic-parabolic system; this is accomplished by first regularizing this system by introducing an artificial viscosity parameter δ, establishing existence and uniqueness for the resulting problem by using a Galerkin argument, and then employing energy estimates, which are independent of δ, that allow us to let $\delta \to 0$ in the regularized problem. The basic difficulty which must be overcome in both this paper, as well as in [5], is the influence of the time-dependent boundary conditions associated with the influx of reactants into the pulse combustion chamber.

2. Previous efforts at including spatial effects in pulse combustion modeling

There have been a few attempts to develop a mathematical model of a pulse combustor which incorporates spatial dependence of the physical quantities but there have been no known attempts to mathematically analyze the aforementioned models; in particular, the existence and uniqueness of solutions of the relevant initial-boundary value problems associated with these models has not been addressed. Moreover, the initial-boundary value problems associated with pulse combustion modeling differ from the majority of the gas-dynamics related initial boundary-value problems in the literature; they are often defined on a bounded domain and lead to situations involving time-dependent boundary conditions. In a pulse combustor reactants are added, and products are removed, periodically. These properties are not unique to pulse combustor modeling; similar initial-boundary value problems arise in many other physical applications. Therefore, the mathematical analysis presented in this paper may be of some significance for other physical problems as well. We will now present a summary of the three mathematical models of pulse combustion referenced above.

Many processes in a pulse combustor are three-dimensional and are dominated by turbulent transport phenomena. However, since a typical pulse combustor system has a large length to diameter ratio, the net influence of these processes results in an unsteady, one-dimensional wave system. The flow field in a pulse combustor can, over a large part of the combustor, be approximated by an oscillatory plug flow, thus, indicating that the flow can be simplified to be one-dimensional.

In [26] a one-dimensional model was formulated and analyzed numerically; the authors derive a coupled system of partial differential equations following the standard procedures of continuum mechanics; i.e., they begin with balance equations with a three-dimensional spatial dependence, namely,

conservation of mass:

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.1a}$$

conservation of momentum:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \mathbf{T} + \rho \mathbf{b} \tag{2.1b}$$

conservation of energy:

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \nabla \cdot (\rho \varepsilon \mathbf{v}) = \mathbf{T} \cdot \mathbf{D} - \nabla \cdot \mathbf{h} + \rho \sigma \tag{2.1c}$$

and the entropy inequality

$$\frac{\partial}{\partial t}(\rho\eta) + \nabla\cdot(\rho\eta\mathbf{v}) \geq -\nabla\cdot(\mathbf{h}/\theta) + \rho\sigma/\theta \tag{2.1d}$$

where ρ is the density, $\mathbf{v}$ is the velocity vector, ε is the specific internal energy, $\mathbf{h}$ is the heat conduction vector, $\mathbf{T}$ is the stress tensor, $\mathbf{b}$ is the specific body force vector, $\mathbf{D}$ is the deformation rate tensor, θ is the temperature, σ is the specific radiation, η is the specific entropy, and $\otimes$ is the standard tensor product of vectors.

Assuming a one-dimensional dependence for all of the variables involved, and introducing a heat conduction sink term $q_c^{\perp}$ to account for heat loss in the direction orthogonal to the axis of symmetry, the authors arrive at the system

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A v) = 0 \tag{2.2a}$$

$$\frac{\partial}{\partial t}(\rho A v) + \frac{\partial}{\partial x}(\rho A v^2) = \frac{\partial}{\partial x}(TA) \tag{2.2b}$$

$$\frac{\partial}{\partial t}(\rho A\varepsilon) + \frac{\partial}{\partial x}(\rho A\varepsilon v) = AT\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}(Ah) - q_c^{\perp} + \rho A\sigma + A\dot{Q} \tag{2.2c}$$

where x is the variable along the axis of the pulse combustion chamber and the tailpipe (see Figure 1), A is the cross sectional area of the pulse combustor, and $\dot{Q}$ is the heat generated by combustion.

As initial values the authors [26] assume atmospheric pressure, zero velocity, and room temperature; i.e.,

$$p(x,0) = p_0, \quad v(x,0) = v_0, \quad \theta(x,0) = \theta_0 \tag{2.3a}$$

The outlet of the tailpipe is considered to be the open end of an organ pipe; i.e., at $x = L$ there is a pressure node and a maximum amplitude of the velocity:

$$p(L,t) = p_0, \quad \frac{\partial v}{\partial x}(L,t) = 0, \quad \frac{\partial \rho}{\partial x}(L,t) = 0, \tag{2.3b}$$

Fureby and Lundgren [26] also distinguish between closed and open valves at the entrance to the pulse combustor; this results in two sets of boundary conditions at $x = 0$. In the case of a closed valve the pressure amplitude has a maximum and the mass transport is zero; i.e.,

$$\frac{\partial p}{\partial x}(0,t) = 0, \quad \{\rho A v\}(0,t) = 0, \quad \frac{\partial \theta}{\partial x}(0,t) = 0 \tag{2.3c}$$

In case of an open valve, the temperature and pressure are assumed to be the same as that in the gas supply line, and the mass transport is modeled separately, specifically,

$$p(0,t) = p_g, \quad \{\rho A v\}(0,t) = \dot{m}(t) \neq 0, \quad \theta(0,t) = \theta_g \tag{2.3d}$$

To close the system of equations the following set of constitutive relations is employed:

$$\mathbf{T} = (-p - \lambda tr\mathbf{D})\mathbf{I} + 2\mu\mathbf{D} + \nu\mathbf{g}\otimes\mathbf{g}, \tag{2.4a}$$

$$\mathbf{h} = (k + \beta tr\mathbf{D} + \delta(tr\mathbf{D})^2)\mathbf{g} + \gamma\mathbf{D}\mathbf{g}, \tag{2.4b}$$

$$\epsilon = \epsilon(\rho,\theta), \tag{2.4c}$$

$$\eta = \eta(\rho,\theta) \tag{2.4d}$$

where $\mathbf{g} = \nabla\theta$ and $\lambda, \mu, \nu, \beta, \delta, \gamma$, and k are material constants, e.g., μ is the viscosity, and k is the thermal conductivity. It is, however, unclear as to what specific form of these constitutive relations was used in the numerical experiments reported in [26]. In [26] submodels were also introduced to deal with the combustion processes, e.g., for the energy release term $\dot{Q}$ it was assumed that for some spatially varying amplitude function $K(x)$

$$\dot{Q} = \dot{Q}(x,t) = K(x)\sin\Big(2\pi\frac{t}{\tau} - \delta\Big) \tag{2.5}$$

where τ denotes the period and δ is the phase difference between the mass flow $\dot{m}$ and the energy release $\dot{Q}$. A heat transfer submodel of the form

$$q_c^{\perp} = O(x)h_r(x,t) \tag{2.6}$$

was incorporated into the model, where $O(x)$ is the circumference of the combustion chamber at x, and $h_r(x,t)$ is the radial component of the heat conduction vector $\mathbf{h}$. Finally, the following valve model was also introduced:

$$\dot{m}(t) = \{\rho A v\}(0,t) = \frac{\partial}{\partial x}\int_t^{t+\tau}(TA)(x,\xi)d\xi\Big|_{x=0} \tag{2.7}$$

where $\dot{m}(t)$ is the mass flow rate through the valve.

The other well-known contribution to the literature on one-dimensional pulse combustion modeling may be found in [3] where unsteady, one-dimensional equations of continuity, momentum, and energy were numerically solved; the model here also allows for a variable area geometry, and assumes the perfect gas equation of state. The full model has the form

$$\frac{\partial(\rho A)}{\partial x} = -\frac{\partial}{\partial x}(\rho u A) \tag{2.8a}$$

$$\frac{\partial(\rho u A)}{\partial x} = -\frac{\partial}{\partial x}(\rho u^2 A + pA) + p\frac{dA}{dx} - \rho A\frac{4f}{D}\frac{u^2}{2}\frac{u}{|u|} \tag{2.8b}$$

$$\frac{\partial(\rho A E_s)}{\partial x} = -\frac{\partial}{\partial x}[u(\rho A E_s + pA)] + \dot{q} - 4Dh(T - T_{air}) \tag{2.8c}$$

$$E_s = c_V T + \frac{u^2}{2}, \tag{2.8d}$$

$$p = \rho R T \tag{2.8e}$$

Here p, ρ, and T are, respectively, the pressure, density, and temperature of the gas, c_V is the specific heat at constant volume, u is the fluid velocity, D and A are the local side and cross-sectional area of the square combustor, T_{air} is the external temperature used to determine heat losses, f is the friction factor, and $\dot{q}$ is the heat generated due to the combustion process. The authors [3] use the following initial and boundary conditions in their numerical computations:

$$p(x,0) = p_{\rm atm}, \quad T(x,0) = T_{air}, \quad u(x,0) = 0 \tag{2.9a}$$

During injection, at the entrance, it was assumed that

$$\frac{\partial p}{\partial x}(0,t) = 0, \quad T(0,t) = T_{air}, \quad u(0,t) = \frac{\dot{m}(t)}{\rho A} \tag{2.9b}$$

and for the case where valve is closed the boundary conditions were

$$\frac{\partial p}{\partial x}(0,t) = 0, \quad \frac{\partial T}{\partial x}(0,t) = 0, \quad u(0,t) = 0 \tag{2.9c}$$

Finally, at the tailpipe exit, it was assumed that

$$p(L,t) = p_{\mathrm{atm}}, \quad \frac{\partial \rho}{\partial x}(L,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0 \tag{2.9d}$$

We note that, mathematically, only five boundary conditions are required in each of the models governed by systems (2.2) and (2.8). Specifying six boundary conditions makes these systems mathematically overdetermined; however, all six boundary conditions are necessary for a numerical treatment of the problems specified above and, indeed, these papers only employ numerical treatments of the models. It is not surprising that some of the boundary conditions are slightly different in the models developed in [3, 26]. Indeed, in both [3] and [26] a version of the McCormack predictor-corrector method is used to solve the system of partial differential equations subject to boundary and initial conditions; this method is mathematically equivalent to introducing an artificial viscosity parameter and then using standard numerical methods for parabolic equations. In each case the special pulse combustor data introduced for each model was used for calculations and many of the parameters were chosen specific to the particular pulse combustor. While [26] simply checks the consistency of the model with experiments, the authors of [3] also attempted to find an optimal frequency of operation for the pulse combustor.

3. A one-dimensional, lumped parameter, pulse combustion model

The general form of the equations describing the motion of a reactive gas is based on the following conservation laws:

(i) conservation of mass:

$$\frac{\partial}{\partial t}(\rho A) = -\frac{\partial}{\partial x}(\rho u A) \tag{3.1}$$

(ii) conservation of momentum:

$$\frac{\partial}{\partial t}(\rho u A) = -\frac{\partial}{\partial x}\left(\rho u^2 A - \sigma A\right) \tag{3.2}$$

and

(iii) conservation of energy:

$$\frac{\partial}{\partial t}(\rho A E) = -\frac{\partial}{\partial x}(u\rho A E + HA - u\sigma A) + \dot{q} \tag{3.3}$$

where, as in the previous section, $\rho(x,t)$ is the density of the gas, $u(x,t)$ is the velocity, $T(x,t)$ is the gas temperature, $A(x)$ is the cross sectional area of the pulse combustor, $\sigma(x,t)$ is the stress tensor, $E(x,t)$ is energy per unit mass, $H(x,t)$ is heat conduction in the axial direction, and $\dot{q}(x,t)$ is the heat released due to chemical reactions per unit time. The system of equations (3.1)-(3.3) is consistent with the system in [26] but there seems to be some inconsistency with the system in [3]. In particular the conservation of momentum equation (2.8)b has a form which seems to be inconsistent with the principles of continuum mechanics.

The system of conservation equations (3.1)-(3.3) are closed by the constitutive relations:

$$E = c_V T + \frac{u^2}{2}, \tag{3.4}$$

$$p = \rho R T, \tag{3.5}$$

$$\sigma = -p + \mu \frac{\partial u}{\partial x}, \tag{3.6}$$

$$H = -k \frac{\partial T}{\partial x} \tag{3.7}$$

where c_V is specific heat of the gas, R is the gas constant, μ is the gas viscosity, assumed to be constant, and k is the heat conduction coefficient, also assumed to be constant. This specific form of the constitutive relations is consistent with the forms proposed in [3, 26].

We choose as variables the density, velocity and temperature; all other functions will be assumed to be functions of ρ, u, and T, with the specific dependence expressed through the constitutive relations. Using the constitutive relations, we obtain from (3.1)-(3.3) the following evolution equations

$$A\frac{\partial \rho}{\partial t} = -Au\frac{\partial \rho}{\partial x} - A\rho\frac{\partial u}{\partial x} - \frac{\partial A}{\partial x}(\rho u) \tag{3.8}$$

$$\begin{aligned} &Au\frac{\partial \rho}{\partial t} + A\rho\frac{\partial u}{\partial t} \\ &= -\frac{\partial \rho}{\partial x}u^2 A - \frac{\partial u}{\partial x}2\rho u A - \frac{\partial(\rho RT)}{\partial x}A + \mu\frac{\partial^2 u}{\partial x^2}A - \frac{\partial A}{\partial x}\Big(\rho u^2 - \rho RT + \mu\frac{\partial u}{\partial x}\Big) \end{aligned} \tag{3.9}$$

$$\begin{aligned} &A\rho c_V\frac{\partial T}{\partial t} + A\rho u\frac{\partial u}{\partial t} + A\Big(c_V T + \frac{u^2}{2}\Big)\frac{\partial \rho}{\partial t} \\ &= -Auc_V\rho\frac{\partial T}{\partial x} - Au^2\rho\frac{\partial u}{\partial x} - Au\Big(c_V T + \frac{u^2}{2}\Big)\frac{\partial \rho}{\partial x} \\ &\quad - A\rho\Big(c_V T + \frac{u^2}{2}\Big)\frac{\partial u}{\partial x} - A\frac{\partial}{\partial x}\Big(-\rho RT\frac{\partial u}{\partial x} + \mu\big(\frac{\partial u}{\partial x}\big)^2\Big) + \dot{q} \end{aligned} \tag{3.10}$$

Using (3.8) to substitute for $A\frac{\partial \rho}{\partial t}$ in (3.9) we obtain

$$\begin{aligned} &u\Big(-Au\frac{\partial \rho}{\partial x} - A\rho\frac{\partial u}{\partial x} - \frac{\partial A}{\partial x}(\rho u)\Big) + A\rho\frac{\partial u}{\partial t} \\ &= -\frac{\partial \rho}{\partial x}u^2 A - \frac{\partial u}{\partial x}2\rho u A - \frac{\partial p}{\partial x}A + \mu\frac{\partial^2 u}{\partial x^2}A - \frac{\partial A}{\partial x}\Big(\rho u^2 - p + \mu\frac{\partial u}{\partial x}\Big) \end{aligned} \tag{3.11}$$

which can then be reduced to

$$A\rho\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x}\rho u A - \Big(R\rho\frac{\partial T}{\partial x} + RT\frac{\partial \rho}{\partial x}\Big)A + \mu\frac{\partial^2 u}{\partial x^2}A - \frac{\partial A}{\partial x}\Big(-\rho RT + \mu\frac{\partial u}{\partial x}\Big). \tag{3.12}$$

We now use (3.8) and (3.12) to substitute for $A\frac{\partial \rho}{\partial t}$ and $A\rho\frac{\partial u}{\partial t}$, respectively, on the right hand side of (3.10). Assuming $\rho(x,t) > 0$, we then obtain the following system of three partial differential equations for ρ, u, and T:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = -\rho u\frac{\partial}{\partial x}(\ln A), \tag{3.13a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + R\frac{\partial T}{\partial x} + \frac{RT}{\rho}\frac{\partial \rho}{\partial x} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial x^2} + \frac{\mu}{\rho}\frac{\partial u}{\partial x}\frac{\partial}{\partial x}(\ln A) - RT\frac{\partial}{\partial x}(\ln A), \tag{3.13b}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \frac{RT}{c_V}\frac{\partial u}{\partial x} = \frac{\mu}{c_V}\frac{1}{\rho}\Big(\frac{\partial u}{\partial x}\Big)^2 + \frac{k}{c_V}\frac{1}{\rho}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{Ac_V}\frac{1}{\rho} + \frac{k}{c_V\rho}\frac{\partial T}{\partial x}\frac{\partial}{\partial x}(\ln A). \tag{3.13c}$$

Using the assumptions introduced in [5] it will be shown that the vacuum state does not occur for initial data chosen sufficiently small; this will justify the assumption that $\rho > 0$ in (3.13). The valves at the entrance to the pulse combustor are assumed to be designed in such a way as to allow control over the reactant gas flow into the chamber. We assume, therefore, that the velocity of the gas entering the pulse combustor chamber is a known function of time; i.e., at $x = 0$ we have

$$u(0,t) = \alpha(t) \tag{3.14}$$

for some function $\alpha(t)$. The model does not differentiate between open and closed valves. We assume, instead that, due to valve inertia the valve never closes completely, which seems to be a physically realistic assumption; this allows for a continuous (although oscillating) flow of the reactants into the chamber. We impose the following conditions on the function α:

$$\alpha(t) \geq \alpha_0 > 0, \quad t \geq 0, \tag{3.15}$$

$$\alpha(t) \in C^3(0,\infty)\,. \tag{3.16}$$

The first condition reflects the fact that the flow is always directed towards the chamber and the valves never close completely. The second condition guarantees that the function $\alpha(t)$ is sufficiently smooth for the analysis to follow.

We also assume that we can control the temperature and pressure and, hence, the density of the reactants flowing into the chamber; this assumption is equivalent to the following entrance boundary conditions on ρ and T.

$$T(0,t) = T_{\rm in} > 0, \tag{3.17}$$

$$\rho(0,t) = \rho_{\rm in} = \frac{p_{\rm atm}}{RT_{\rm in}} > 0\,. \tag{3.18}$$

Remarks: The current model does not allow for flow reversal in the tailpipe. In order to incorporate flow reversal, the mathematical domain of the problem would need to be extended beyond the exit of the tailpipe, and some mixing mechanism would need to be introduced, likely requiring a model with at least a two dimensional spatial dependence.

As the system (3.13)a,b,c is first order ρ, no exit boundary condition can be imposed with respect to ρ at $x = L$; introducing such a boundary condition will result in an over determined system. The conditions for u and T at $x = L$ are the following: first of all, the flow of the gas exiting the tailpipe is incompressible; i.e.,

$$\frac{\partial u}{\partial x}(L,t) = 0 \tag{3.19}$$

Next, once a stable operating cycle of the pulse combustor has been established, the temperature of the combustion products coming out of the tailpipe remains constant, as it depends solely on the air-fuel ratio used. Therefore, we seek solutions for which

$$T(L,t) = T_{\rm out} = \text{const.} \tag{3.20}$$

This latter condition may also be substantiated as follows: the set of exit boundary conditions in the previous formulations of one-dimensional pulse combustion in [3, 26]; i.e.,

$$\rho_x(L,t) = 0, \quad u_x(L,t) = 0, \quad p(L,t) = p_{\rm atm}$$

yield, as has been noted, an overdetermined problem and does not contain a boundary condition for the temperature. However, for the exit boundary conditions in the current model (3.19), (3.20), the boundary condition for the temperature follows as a consequence of the boundary conditions for the density and the pressure which were used in [3, 26]. In fact, using the conservation of mass equation we obtain

$$\rho_t(L,t) + u(L,t)\rho_x(L,t) + \rho(L,t)u_x(L,t) = -\rho(L,t)u(L,t)\frac{A_x(L)}{A(L)} \tag{3.21}$$

If we then apply (3.19), and the fact that the cross-section has constant area along the length of the tailpipe, we obtain $\rho_t(L,t) = 0$ which, when combined with (3.19), implies that

$$\rho(L,t) = \text{const} \tag{3.22}$$

As a consequence of the ideal gas law, however,

$$T(L,t) = \frac{p_{\text{atm}}}{R\rho(L,t)} = \text{const}.$$

which is (3.20).

For the initial conditions at $t = 0$ we assume the specification of sufficiently smooth functions of x; i.e.,

$$u(x,0) = u_0(x) \tag{3.23a}$$

$$\rho(x,0) = \rho_0(x) \tag{3.23b}$$

$$T(x,0) = T_0(x) \tag{3.23c}$$

where $u_0(x)$, $\rho_0(x)$, $T_0(x) \in C^2[0,L]$.

The complete model considered in this paper, as well as in [5], consists of the system of equations (3.13)a,b,c for ρ, u, T, the boundary conditions (3.14), (3.17), (3.18), (3.19), and (3.20), and the initial data (3.23)a,b,c, a sketch of the pulse combustor configuration associated with this model is presented in Figure 3.

4. Some related work on problems in gas dynamics

Initial-boundary value problems associated with pulse combustion modeling differ from the majority of the gas-dynamics related initial boundary-value problems in the literature; such problems are often defined on a bounded domain and lead to situations involving time-dependent boundary conditions. In a pulse combustor reactants are added, and products are removed, periodically. These properties are not unique to pulse combustor modeling; similar initial-boundary value problems arise in other physical applications, e.g., blood flow [15] and the references contained therein.

Existence and uniqueness for initial and initial-boundary value problems associated with the motion of viscous, compressible fluids has been covered extensively in the literature [29, 30, 31, 33, 35, 43, 45, 46, 48, 49, 50, 53, 55, 57, 58, 59]. This includes work related to the gas dynamics equations with a three-dimensional spatial dependence [29, 45, 49, 53, 57]. In [45], the equations of motion of compressible viscous and heat-conductive fluids were investigated for initial boundary value problems in a half space and in the exterior domain of any bounded region. A globally unique solution (in time) was proved to exist and approach the stationary state as $t \to \infty$, provided the prescribed initial data and the external force were sufficiently small. The solutions, in fact, possess the following smoothness:

$$\begin{gathered}\rho \in C^0(0,\infty; H^3(\Omega)) \cap C^1(0,\infty; H^2(\Omega)),\\ u, \theta \in C^0(0,\infty; H^3(\Omega)) \cap C^1(0,\infty; H^1(\Omega))\end{gathered} \tag{4.1}$$

Tani [57] establishes existence and uniqueness results for the first initial-boundary value problem of compressible viscous fluid motion, and Itaya [29] provides a similar result for the Cauchy problem.

The system of gas dynamics equations with a two-dimensional spatial dependence was considered, for example, by Kazhikhov and Vaigant [58]. In particular the existence of a unique solution

$$u(x,y,t) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T), \quad \rho(x,y,t) \in C^{1+\alpha,1+\alpha/2}(\bar{Q}_T) \tag{4.2}$$

($Q_T = \Omega \times [0,T]$, Ω being the spatial domain) was established, provided the initial data were sufficiently small and satisfied

$$u^0(x,y) \in C^{2+\alpha}(\bar{\Omega}), \quad \rho^0(x,y) \in C^{1+\alpha}(\bar{\Omega}) \tag{4.3}$$

In [33], the existence of global classical solutions to initial boundary value problems in the dynamics of a one-dimensional, viscous, heat-conducting gas was established; in this work nonlinear dissipative effects turn out to be sufficiently strong to prevent the development of singularities. In [35], a system of equations for a viscous heat-conducting perfect gas was studied for the case of a one-dimensional motion with plane waves; the unique solvability of the problem of gas flow in a bounded region with impermeable thermally insulated boundaries was proven for the class of the both generalized (strong) and classical solutions. An existence theorem was established by using a priori estimates to extended the local (in time) solution to a global solution; a major role is played here by upper and lower bounds for the density and temperature. Kanel [30] provided an existence result for the Cauchy problem with one-dimensional spatial dependence. Finally, the review paper by Solonnikov and Kazhikhin [55] provides a good survey of existence results for the one-dimensional equations governing the motion of a compressible fluid.

All of the existence results cited above are, however, valid only for homogeneous systems of partial differential equations with homogeneous boundary conditions. For the case of the one-dimensional equations governing the motion of a compressible fluid, a change of variables from density to specific volume can be introduced (Lagrangian variables). This change of variables significantly simplifies the conservation equations; in particular, the conservation of mass equation reduces to $v_{,t} - u_{,x} = 0$, where $v = 1/\rho$ is the specific volume of the gas. However, this change of variables assumes that the density remains strictly positive; the possibility of a vacuum state must be addressed separately [55]. The Lagrangian change of variables can not be applied to the initial-boundary value problem formulated in section 3 as the problem introduced there does not involve a homogeneous system of equations, nor does it possess homogeneous boundary conditions. The analysis presented in this paper, as well as in [5], is closest in spirit to that of the energy argument employed in [54], albeit without the type of time-dependent boundary conditions which appear in the present work.

5. Relation of the current model to the spatially independent model

If one assumes spatial independence, as well as the other assumptions of the model described in section 3, we can expect the one-dimensional model to reduce to the spatially independent model described in [8]. In this section we will demonstrate that this, in fact, is the case.

The one-dimensional model is governed by the system of three conservation laws which result from combining (3.1)-(3.3) with (3.6), assuming $\mu = 0$; this procedure yields the system

$$\frac{\partial}{\partial t}(\rho A) = -\frac{\partial}{\partial x}(\rho u A), \tag{5.1}$$

$$\frac{\partial}{\partial t}(\rho u A) = -\frac{\partial}{\partial x}\left(\rho u^2 A + pA\right), \tag{5.2}$$

$$\frac{\partial}{\partial t}(\rho A E) = -\frac{\partial}{\partial x}\left[u\left(\rho A E + pA\right)\right] + \dot{q} \tag{5.3}$$

We also append the constitutive relations (3.4), (3.5) for the total energy and the pressure.

The spatially independent model involves two equations. The first one is a consequence of energy balance in the combustion chamber and is the precursor to (1.1), namely,

$$\frac{d}{dt}\left[\rho_R e_R V_R + \rho_P e_P V_P\right] = h_R \dot{m}_R + \frac{\Delta H}{1+r}\dot{m}_B - h_0 \dot{m}_0 \tag{5.4}$$

where ρ_R is the density of the reactants, ρ_P is the density of the combustion products, e_R and e_P are the internal energy of the reactants and products, respectively, (per unit mass of the mixture), $V_R(V_P)$ is the chamber volume occupied by reactants (products), $\dot{m}_R$ is the mass flow rate of the reactants, $\dot{m}_B$ is the mass burning rate of the reactant mixture, $\dot{m}_0$ is the mass flow rate of the combustion products in the tailpipe, r is the air fuel (mass) ratio, ΔH is the heat of combustion per unit mass of the fuel, h_R is the enthalpy, per unit mass of the reactant mixture entering the combustion chamber, and h_0 is the enthalpy, per unit mass of the mixture of combustion products leaving the chamber. We note that $V_B = V_R + V_P$, where V_B is the combustion chamber volume. The second equation in the model is the momentum equation in the tailpipe (1.2), which we repeat here as

$$\bar{\rho} L \frac{du}{dt} = p - p_a \tag{5.5}$$

with $\bar{\rho}$ the average density of the combustion products in the tailpipe, p the pressure in the tailpipe, p_a the atmospheric pressure, L the length of the tailpipe, and u the velocity of the gas in the tailpipe.

We first consider the energy equation (5.3) and recall the following three relationships from thermodynamics

$$c_P - c_V = R \tag{5.6}$$

where c_P is the specific heat at constant pressure, per unit mass of the gas,

$$h = c_P T \tag{5.7}$$

where h is the enthalpy of the gas, per unit mass, and

$$e = c_V T \tag{5.8}$$

where e is internal energy of the gas, per unit mass. In the spatially independent case the velocity inside the chamber is zero; therefore, in the chamber

$$E = c_V T + \frac{u^2}{2} = e \tag{5.9}$$

which can also be rewritten using (5.7) and the ideal gas law (3.5) as

$$E = c_V T = (c_P - R)T = h - RT = h - \frac{p}{\rho} \tag{5.10}$$

Substituting for E in (5.3) we obtain

$$\frac{\partial}{\partial t}(\rho A e) = -\frac{\partial}{\partial x}\Big[u\Big(\rho A(h - \frac{p}{\rho}) + pA\Big)\Big] + \dot{q} \tag{5.11}$$

which yields

$$\frac{\partial}{\partial t}(\rho A e) = -\frac{\partial}{\partial x}\left(\rho u A h\right) + \dot{q} \tag{5.12}$$

We now integrate equation (5.12) along the length of the chamber with the entrance of the chamber at $x = 0$ and the exit of the chamber at $x = l$:

$$\int_0^l \frac{\partial}{\partial t}(\rho A e)dx = -\int_0^l \frac{\partial}{\partial x}(\rho u A h)dx + \int_0^l \dot{q}dx \tag{5.13}$$

Remarks: In the model introduced in section 3, the entrance to the combustion chamber is at $x = 0$ and the exit from the tailpipe is at $x = L$; here we assume that the combustion chamber occupies the domain $0 \leq x \leq l$ and the tailpipe the domain $l < x \leq l + L$.

We now note that

$$e = c_V T = c_{VR} T \nu_R + c_{VP} T \nu_P = c_{VR} T \frac{\rho_R V_R}{\rho V} + c_{VP} T \frac{\rho_P V_P}{\rho V} \tag{5.14}$$

where ν_R and ν_P are mass fractions of the reactants and products, respectively. Since $V = Al$ we can write

$$\rho A e = c_{VR} T \frac{\rho_R V_R}{l} + c_{VP} T \frac{\rho_P V_P}{l} = \frac{1}{l}(e_R \rho_R V_R + e_P \rho_P V_P) \tag{5.15}$$

We have, from (5.13),

$$\frac{\partial}{\partial t}\int_0^l \frac{1}{l}(e_R \rho_R V_R + e_P \rho_P V_P)dx = (\rho u A h)\mid_0^l + \int_0^l \dot{q}dx \tag{5.16}$$

which yields

$$\frac{d}{dt}(e_R \rho_R V_R + e_P \rho_P V_P) = (\rho u A)_{\rm in} h_{\rm in} - (\rho u A)_{\rm out} h_{\rm out} + \dot{q}_{total} \tag{5.17}$$

where

$$\dot{q}_{total} = \int_0^l \dot{q}dx \tag{5.18}$$

is the total heat released in the chamber due to chemical reactions, per unit mass of the gas mixture, per unit time. In the spatially independent model, combustion of the fuel is completely achieved and hence

$$\dot{q}_{total} = \frac{\Delta H}{1 + r}\dot{m}_B \tag{5.19}$$

Also, because of the assumptions in the spatially independent model we have $(\rho u A)_{\rm in} = \dot{m}_R$, $(\rho u A)_{\rm out} = \dot{m}_0$, $h_{\rm in} = h_R$ and $h_{\rm out} = h_0$. Substituting these assumptions into (5.17) we obtain the zero-dimensional energy equation (5.4).

To arrive at the spatially-independent momentum equation (5.5) we consider the momentum equation (5.2) in the tailpipe. We integrate (5.2) along the length of the tailpipe so as to obtain

$$\frac{\partial}{\partial t}\int_l^{l+L}(\rho u A)dx = -\int_l^{l+L}\frac{\partial}{\partial x}\left(\rho u^2 A + pA\right)dx \tag{5.20}$$

where we used the fact that $\frac{\partial A}{\partial x}=0$ in the spatially independent case. In fact, as A is constant in the tailpipe, while the velocity u of the gas is spatially constant, we obtain

$$A\frac{\partial}{\partial t}\left(u\int_l^{l+L}\rho dx\right) = -\left((\rho u A)u + pA\right)\big|_l^{l+L} \tag{5.21}$$

However, the mass flow rate $\rho u A = \dot{m}_0$ is constant along the length of the tailpipe, therefore, with $\bar{\rho}$ the average density in the tailpipe, it follows from (5.21) that

$$A\bar{\rho}L\frac{du}{dt} = A(p(l) - p(L+l)) \tag{5.22}$$

which is, of course, equivalent to (5.5).

6. Existence and uniqueness for the linear system with artificial viscosity

The initial-boundary value problem for the one-dimensional pulse combustion model introduced in §3 consists of the mixed hyperbolic-parabolic system (3.13) a,b,c, the boundary conditions (3.14), (3.15), (3.16), (3.19), and (3.20), and the initial data (3.23) a,b,c.

We begin the analysis in this section by effecting a change of variables so as to obtain a problem with homogeneous boundary conditions; more specifically, we set

$$\hat{u} = u - \alpha(t) \tag{6.1a}$$

$$\hat{\rho} = \rho - \rho_{\rm in} \tag{6.1b}$$

$$\hat{T} = T - \frac{x}{L}T_{\rm out} - \frac{L-x}{L}T_{\rm in} \tag{6.1c}$$

and substitute (6.1)a,b,c in (3.13)a,b,c so as to obtain, after rearranging terms, the system

$$\hat{\rho}_t + a(x,t)\hat{u}_x + b(x,t)\hat{\rho}_x = c(x,t) \tag{6.2a}$$

$$\hat{u}_t + b_2(x,t)\hat{u}_x + R\hat{T}_x + d(x,t)\hat{\rho}_x = f(x,t)\hat{u}_{xx} + g(x,t) \tag{6.2b}$$

$$\hat{T}_t + b_3(x,t)\hat{T}_x + h(x,t)\hat{u}_x = i(x,t)\hat{T}_{xx} + j(x,t) \tag{6.2c}$$

The coefficients in (6.2)a,b,c are given, explicitly, by

$$a(x,t) = \hat{\rho} + \rho_{\rm in} \tag{6.3a}$$

$$b(x,t) = \hat{u} + \alpha(t) \tag{6.3b}$$

$$b_2(x,t) = \hat{u} + \alpha(t) - \frac{\mu}{\hat{\rho}+\rho_{\rm in}}(\ln A)_x \tag{6.3c}$$

$$b_3(x,t) = \hat{u} + \alpha(t) - \frac{k}{c_V(\hat{\rho}+\rho_{\rm in})}(\ln A)_x \tag{6.3d}$$

$$c(x,t) = -(\hat{\rho}+\rho_{\rm in})(\hat{u}+\alpha(t))(\ln A)_x \tag{6.3e}$$

$$d(x,t) = \frac{R}{\hat{\rho}+\rho_{\rm in}}\Big(\hat{T} + \frac{x}{L}T_{\rm out} + \frac{L-x}{L}T_{\rm in}\Big) \tag{6.3f}$$

$$f(x,t) = \frac{\mu}{\hat{\rho}+\rho_{\rm in}} \tag{6.3g}$$

$$g(x,t) = -\alpha'(t) - \frac{R}{L}(T_{\rm out}-T_{\rm in}) - R\Big(\hat{T} + \frac{x}{L}T_{\rm out} + \frac{L-x}{L}T_{\rm in}\Big)(\ln A)_x \tag{6.3h}$$

$$h(x,t)=\frac{\mu}{c_V}\frac{\hat u_x}{\hat\rho+\rho_{\rm in}}+\frac{R}{c_V}\Big(\hat T+\frac{x}{L}T_{\rm out}+\frac{L-x}{L}T_{\rm in}\Big) \tag{6.3i}$$

$$i(x,t)=\frac{k}{c_V(\hat\rho+\rho_{\rm in})} \tag{6.3j}$$

$$j(x,t)=-(\hat u+\alpha(t))\Big(\frac{T_{\rm out}-T_{\rm in}}{L}\Big)+\frac{\dot q}{c_V(\hat\rho+\rho_{\rm in})} \tag{6.3k}$$

We note that $h(x,t)$ is the only coefficient containing a derivative of one of the unknown functions. Also, in view of (6.1)a, b,c, the boundary data and initial conditions assume the following form: For the boundary conditions at $x=0$ we have, for all $t>0$,

$$\hat u(0,t)=u(0,t)-\alpha(t)=0 \tag{6.4a}$$

$$\hat\rho(0,t)=\rho(0,t)-\rho_{\rm in}=0 \tag{6.4b}$$

$$\hat T(0,t)=T(0,t)-T_{\rm in}=0 \tag{6.4c}$$

while those at $x=L$ assume the form

$$\hat u_x(L,t)=0 \tag{6.4d}$$

$$\hat T(L,t)=T(L,t)-T_{\rm out}=0 \tag{6.4e}$$

In terms of the new variables, the initial conditions are

$$\hat u(x,0)=u_0(x)-\alpha(0)=\hat u_0(x) \tag{6.5a}$$

$$\hat\rho(x,0)=\rho_0(x)-\rho_{\rm in}=\hat\rho_0(x) \tag{6.5b}$$

$$\hat T(x,0)=T_0(x)-\frac{x}{L}T_{\rm out}-\frac{L-x}{L}T_{\rm in}=\hat T_0(x) \tag{6.5c}$$

If we ignore the dependence of the coefficients in (6.3)a-k on $\hat\rho$, $\hat u$, $\hat T$ and instead assume that the coefficients a, b, b_2, b_3, c, d, f, g, h, i, j are known functions of x and t only, then by dropping the hats on $\hat\rho$, $\hat u$, and $\hat T$, including those on $\hat\rho_0$, $\hat u_0$, and $\hat T_0$, we obtain the linear initial-boundary value problem

$$\rho_t+au_x+b\rho_x=c \tag{6.6a}$$

$$u_t+b_2u_x+RT_x+d\rho_x=fu_{xx}+g \tag{6.6b}$$

$$T_t+b_3T_x+hu_x=iT_{xx}+j \tag{6.6c}$$

with initial data on $[0,L]$

$$u(x,0)=u_0(x) \tag{6.7a}$$

$$\rho(x,0)=\rho_0(x) \tag{6.7b}$$

$$T(x,0)=T_0(x) \tag{6.7c}$$

and, for all $t>0$, the boundary data

$$\rho(0,t)=0 \tag{6.8a}$$

$$u(0,t)=0 \tag{6.8b}$$

$$T(0,t)=0 \tag{6.8c}$$

$$u_x(L,t)=0 \tag{6.8d}$$

$$T(L,t)=0 \tag{6.8e}$$

The purpose of this paper is to prove an existence and uniqueness theorem for the system (6.6), (6.7), (6.8) (as well as for a regularized version of this system); the latter result will serve as the starting point for the local and global existence results for the original nonlinear problem (3.13)a,b,c, (3.14), (3.17)-(3.20), and (3.23)a,b,c in [5]. In fact, the existence and uniqueness result for (6.6), (6.7), (6.8), which is established in §7, depends on proving, in this section, a related result for the regularized version of this system which is introduced below; this regularized problem is treated by using the method of Galerkin approximations coupled with an energy argument. For the problem (6.6), (6.7), (6.8), as well as for the regularized version of this problem possessing an artificial viscosity, we will assume that

$$f(x,t) \geq f_c > 0, \quad \forall (x,t) \in [0,t] \times [0,L] \tag{6.9a}$$

$$i(x,t) \geq i_c > 0, \quad \forall (x,t) \in [0,t] \times [0,L] \tag{6.9b}$$

$$a, b, b_2, b_3, c, d, f, g, h, i, j \in C^\infty([0,t]; C^\infty[0,L]) \tag{6.9c}$$

To regularize the mixed, linear, hyperbolic-parabolic system (6.6) we add the viscous term $\delta\rho_{xx}$ to (6.6)a where $\delta > 0$ is an artificial viscosity; this produces the uniformly parabolic system

$$\rho_t + au_x + b\rho_x = \delta\rho_{xx} + c \tag{6.10a}$$

$$u_t + b_2u_x + RT_x + d\rho_x = fu_{xx} + g \tag{6.10b}$$

$$T_t + b_3T_x + hu_x = iT_{xx} + j\,. \tag{6.10c}$$

For the system (6.10)a,b,c we retain the initial data (6.7)a,b,c but, as we have increased the order of the equation governing the evolution of ρ, we append to the boundary data (6.8) the additional boundary condition

$$\rho_x(L,t) = 0, \quad t > 0 \tag{6.8f}$$

The full regularized problem now consists of (6.10)a,b,c, (6.7)a,b,c, and (6.8)a-f; to deal with this problem we begin by introducing the spaces which are used in the Galerkin approximations; i.e., we have the following definition.

Definition 6.1. *For $m \in \mathbb{N}$ define the finite dimensional spaces*

$$\mathcal{V}_m = \big\{v(x,t) : v = \sum_{l=0}^{m} \alpha_l(t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L}\big\},, \tag{6.11a}$$

$$\mathcal{W}_m = \big\{w(x,t) : w = \sum_{l=0}^{m} \beta_l(t)\sqrt{\frac{2}{L}}\sin\frac{l\pi x}{L}\big\}\,, \tag{6.11b}$$

where $\alpha_l(\tau)$, $\beta_l(\tau) \in C^1[0,t]$.

Recalling that the functions ρ, u, T which appear in (6.10)a,b,c, (6.7)a,b,c, and (6.8)a-f are, in fact, the $\hat\rho, \hat u, \hat T$ given by (6.1)a,b,c we state the following result.

Lemma 6.2. *Suppose $\rho(x,t), u(x,t) \in \mathcal{V}_m$ and $T(x,t) \in \mathcal{W}_m$ for some $m \in \mathbb{N}$, and*

$$\int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)v\,dx = 0 \tag{6.12a}$$

$$\int_0^L (u_t + b_2u_x + RT_x + d\rho_x - fu_{xx} - g)v\,dx = 0 \tag{6.12b}$$

for any $v \in \mathcal{V}_m$, while

$$\int_0^L (T_t + b_3 T_x + h u_x - i T_{xx} - j) w dx = 0 \tag{6.12c}$$

for any $w \in \mathcal{W}_m$; then, the following identity holds:

$$\begin{aligned}
&\frac{1}{2}\{\|\rho(\cdot,t)\|^2_{W^{1,2}} + \|u(\cdot,t)\|^2_{W^{1,2}} + \|T(\cdot,t)\|^2_{W^{1,2}}\} \\
&+ \int_0^t \int_0^L \Big\{ i(T_{xx}^2 + T_x^2) + T_t^2 + f(u_{xx}^2 + u_x^2) + u_t^2 + \delta(\rho_{xx}^2 + \rho_x^2) + \rho_t^2 \Big\}\, dx\, d\tau \\
&+ \frac{1}{2}\int_0^L f u_x^2 dx + \frac{1}{2}\int_0^L i T_x^2 dx \\
&= \frac{1}{2}\int_0^L f(x,0) u_x^2(x,0) dx + \frac{1}{2}\int_0^L i(x,0) T_x^2(x,0) dx \\
&\quad + \frac{1}{2}\{\|\rho(\cdot,0)\|^2_{W^{1,2}} + \|u(\cdot,0)\|^2_{W^{1,2}} + \|T(\cdot,0)\|^2_{W^{1,2}}\} \\
&\quad + \int_0^t \int_0^L \Big\{ - a\rho u_x - b\rho\rho_x - b_2 u u_x - R u T_x - d u \rho_x - b_3 T T_x - h T u_x \\
&\quad + \frac{1}{2} f_t u_x^2 + \frac{1}{2} i_t T_x^2 \Big\}\, dx\, d\tau + \int_0^t \int_0^L \Big\{ a u_x \rho_{xx} + b\rho_x \rho_{xx} + b_2 u_x u_{xx} \\
&\quad + R T_x u_{xx} + d\rho_x u_{xx} + b_3 T_x T_{xx} - a\rho_t u_x - b\rho_t \rho_x - b_2 u_t u_x - R u_t T_x \\
&\quad - d u_t \rho_x - b_3 T_t T_x - f_x u_t u_x - i_x T_t T_x - h T_t u_x + h u_x T_{xx} \Big\}\, dx\, d\tau \\
&\quad + \int_0^t \int_0^L \Big\{ c\rho + g u + j T - c\rho_{xx} - g u_{xx} \\
&\quad - j T_{xx} + c\rho_t + g u_t + j T_t \Big\}\, dx\, d\tau + \int_0^t \int_0^L \delta \rho_t \rho_{xx}\, dx\, d\tau
\end{aligned} \tag{6.13}$$

Remark 6.3. The hypotheses of Lemma 6.2 will hold for the Galerkin approximations to the solution of the regularized linear initial-boundary value problem which are constructed below.

Proof. We observe that as $\rho, u \in \mathcal{V}_m$ and $T \in \mathcal{W}_m$, all even order spatial derivatives of ρ, u and T will be zero at $x = 0$, while odd order spatial derivatives of ρ, u and even order spatial derivatives of T will vanish at $x = L$. As $\rho \in \mathcal{V}_m$, it follows from from (6.12)c that

$$\int_0^L \int_0^t (\rho_t + a u_x + b\rho_x - \delta\rho_{xx} - c)\rho\, dx\, d\tau = 0 \tag{6.14}$$

which, after integration by parts of the term $\delta\rho\rho_{xx}$, becomes

$$\begin{aligned}
&\int_0^L \frac{1}{2}\rho^2(x,t) dx - \int_0^L \frac{1}{2}\rho_0^2(x) dx + \int_0^t \int_0^L \delta\rho_x^2\, dx\, d\tau \\
&= \int_0^t \int_0^L \{-a\rho u_x - b\rho\rho_x + c\rho\}\, dx\, d\tau
\end{aligned} \tag{6.15}$$

Also, as $u \in \mathcal{V}_m$ we have, as a consequence of (6.12)b

$$\int_0^L (u_t + b_2 u_x + RT_x + d\rho_x - f u_{xx} - g) u dx = 0 \tag{6.16}$$

If we then integrate this last result over $[0, t]$ we obtain

$$\begin{aligned} &\int_0^L \frac{1}{2} u^2(x,t) dx - \int_0^L \frac{1}{2} u_0^2(x) dx + \int_0^t \int_0^L f u_x^2 \, dx \, d\tau \\ &= \int_0^t \int_0^L \{-b_2 u u_x - R u T_x - d u \rho_x + g u\} \, dx \, d\tau \end{aligned} \tag{6.17}$$

Next, as $T \in \mathcal{W}_m$, it follows from (6.12)c that

$$\int_0^L (T_t + b_3 T_x + h u_x - i T_{xx} - j) \, T dx = 0 \tag{6.18}$$

and integrating this result over $[0, t]$ we obtain

$$\begin{aligned} &\int_0^L \frac{1}{2} T^2(x,t) dx - \int_0^L \frac{1}{2} T_0^2(x) dx + \int_0^t \int_0^L i T_x^2 \, dx \, d\tau \\ &= \int_0^t \int_0^L \{-b_3 T T_x - h T u_x + j T\} \, dx \, d\tau \, . \end{aligned} \tag{6.19}$$

Since $\rho_{xx} \in \mathcal{V}_m$, (6.12)a yields

$$\int_0^L (\rho_t + a u_x + b \rho_x - c - \delta \rho_{xx}) \rho_{xx} dx = 0 \, . \tag{6.20}$$

Integration by parts in (6.20), coupled with the conditions $\rho_x(L,t) = 0$, $\rho_t(0,t) = 0$, and followed by integration over $[0, t]$, then yields

$$\begin{aligned} &\int_0^L \frac{1}{2} \rho_x^2(x,t) dx - \int_0^L \frac{1}{2} \rho_x^2(x,0) dx + \int_0^t \int_0^L \delta \rho_{xx}^2 \, dx \, d\tau \\ &= \int_0^t \int_0^L (a u_x \rho_{xx} + b \rho_x \rho_{xx} - c \rho_{xx}) \, dx \, d\tau \end{aligned} \tag{6.21}$$

As $u_{xx} \in \mathcal{V}_m$, (6.12)b implies that

$$\int_0^L (u_t + b_2 u_x + RT_x + d\rho_x - f u_{xx} - g) u_{xx} dx = 0 \tag{6.22}$$

If we integrate by parts in this last identity, use the conditions $u_x(L,t) = 0$ and $u_t(0,t) = 0$, and then integrate over $[0, t]$, we find that

$$\begin{aligned} &\int_0^L \frac{1}{2} u_x^2(x,t) dx - \int_0^L \frac{1}{2} u_x^2(x,0) dx + \int_0^t \int_0^L f u_{xx}^2 \, dx \, d\tau \\ &= \int_0^t \int_0^L (b_2 u_x u_{xx} + R T_x u_{xx} + d \rho_x u_{xx} - g u_{xx}) \, dx \, d\tau \, . \end{aligned} \tag{6.23}$$

Next, as $T_{xx} \in \mathcal{W}_m$, (6.12)c produces

$$\int_0^L (T_t + b_3 T_x + h u_x - i T_{xx} - j) T_{xx} dx = 0 \, . \tag{6.24}$$

Integrating by parts in (6.24), using the fact that $T_t(0,t) = T_t(L,t) = 0, t > 0$, and then integrating over $[0,t]$, we obtain

$$\begin{aligned} &\int_0^L \frac{1}{2}T_x^2(x,t)dx - \int_0^L \frac{1}{2}T_x^2(x,0)dx + \int_0^t \int_0^L iT_{xx}^2\,dx\,d\tau \\ &= \int_0^t \int_0^L (b_3T_xT_{xx} + hu_xT_{xx} - jT_{xx})\,dx\,d\tau\,. \end{aligned} \tag{6.25}$$

Since $\rho_t \in \mathcal{V}_m$, (6.12)a yields

$$\int_0^L (\rho_t + au_x + b\rho_x - c - \delta\rho_{xx})\rho_t dx = 0\,, \tag{6.26}$$

so that

$$\int_0^t \int_0^L \rho_t^2\,dx\,d\tau = \int_0^t \int_0^L (-a\rho_t u_x - b\rho_t\rho_x + c\rho_t - \delta\rho_t\rho_{xx})\,dx\,d\tau \tag{6.27}$$

Next, we note that as $u_t \in \mathcal{V}_m$, (6.12)b yields, after integration over $[0,t]$,

$$\int_0^t \int_0^L (u_t + b_2u_x + RT_x + d\rho_x - fu_{xx} - g)u_t dxdt = 0\,. \tag{6.28}$$

Integrating the next to the last term in (6.28) by parts, we find that

$$\int_0^t \int_0^L fu_{xx}u_t\,dx\,d\tau = -\int_0^L \frac{1}{2}fu_x^2dx\Big|_0^t + \int_0^t \int_0^L \frac{1}{2}f_tu_x^2\,dx\,d\tau - \int_0^t \int_0^L f_xu_xu_t\,dx\,d\tau \tag{6.29}$$

because $u_x(L,t) = u_t(0,t) = 0$. If we now substitute this last result back into (6.29), we obtain

$$\begin{aligned} &\int_0^t \int_0^L u_t^2\,dx\,d\tau + \int_0^L \frac{1}{2}fu_x^2dx \\ &= \int_0^L \frac{1}{2}f(x,0)u_x^2(x,0)dx \\ &\quad + \int_0^t \int_0^L (-b_2u_tu_x - Ru_tT_x - du_t\rho_x - f_xu_tu_x + \frac{1}{2}f_tu_x^2 - gu_t)\,dx\,d\tau\,. \end{aligned} \tag{6.30}$$

Finally, as $T_t \in \mathcal{W}_m$, (6.12)c produces, after integration over $[0,t]$,

$$\int_0^t \int_0^L (T_t + b_3T_x + hu_x - iT_{xx} - j)T_t dxdt = 0\,. \tag{6.31}$$

Integrating the next to the last term in (6.31) by parts yields

$$\int_0^t \int_0^L iT_{xx}T_t\,dx\,d\tau = -\int_0^L \frac{1}{2}iT_x^2dx\Big|_0^t + \int_0^t \int_0^L \frac{1}{2}i_tT_x^2\,dx\,d\tau - \int_0^t \int_0^L i_xT_xu_t\,dx\,d\tau \tag{6.32}$$

as $T_t(L,t) = T_t(0,t) = 0$. Substituting this last result back into (6.31) we find that

$$\begin{aligned} &\int_0^t \int_0^L T_t^2\,dx\,d\tau + \int_0^L \frac{1}{2}iT_x^2dx \\ &= \int_0^L \frac{1}{2}i(x,0)T_x^2(x,0)dx \\ &\quad + \int_0^t \int_0^L (-b_3T_tT_x - hT_tu_x - i_xT_tT_x + \frac{1}{2}i_tT_x^2 - jT_t)\,dx\,d\tau \end{aligned} \tag{6.33}$$

Adding together the results in equations (6.15), (6.17), (6.19), (6.21), (6.23), (6.25), (6.27), (6.30), and (6.33), and grouping like terms together, we obtain the result expressed by (6.13). □

We now introduce what will turn out to be an appropriate energy functional for the regularized system (6.10)a,b,c with artificial viscosity $\delta > 0$, namely, we have the following definition.

Definition 6.4. For $\rho(x,t)$, $u(x,t)$, and $T(x,t)$ we define the energy functional

$$\begin{aligned}\mathcal{E}_\delta(t) &= \frac{1}{2}\big\{\|\rho(\cdot,t)\|^2_{W^{1,2}} + \|u(\cdot,t)\|^2_{W^{1,2}} + \|T(\cdot,t)\|^2_{W^{1,2}}\big\}\\ &\quad + \frac{1}{2}\int_0^t\int_0^L \Big\{i_c(T_{xx}^2+T_x^2) + T_t^2 + f_c(u_{xx}^2+u_x^2) + u_t^2\\ &\quad + \delta(\rho_{xx}^2+\rho_x^2) + \rho_t^2\Big\}\,dx\,d\tau + \frac{1}{2}\int_0^L f_c u_x^2 dx + \frac{1}{2}\int_0^L i_c T_x^2 dx\end{aligned} \tag{6.34}$$

Lemma 6.5. *Under the hypotheses of Lemma 6.2 we have for $\rho,\ u \in \mathcal{V}_m$, $T \in \mathcal{W}_m$, and $0 < \delta < 1$,*

$$\mathcal{E}_\delta(t) \le \mathcal{E}(0) + \mathcal{G}t + \mathcal{K}\int_0^t \mathcal{E}_\delta(\tau)d\tau\,, \tag{6.35}$$

where $\mathcal{G}$ and $\mathcal{K}$ are positive constants.

Proof. By Lemma 6.2 the identity (6.13) holds. The terms on the right hand side of (6.13) have been separated by { } into four distinct groups; we now proceed to estimate these terms. In these estimates we will use generic positive constants C_i, K_i, G_i.

The terms from the first group involve functions and/or first derivatives with bounded coefficients; these can be estimated as in the following sample case:

$$\Big|\int_0^t\int_0^L a\rho u_x\,dx\,d\tau\Big| \le K_1\int_0^t\int_0^L(\rho^2+u_x^2)\,dx\,d\tau, \quad K_1 = \frac{\sup|a|}{2}\,. \tag{6.36}$$

The second group of terms involve second derivatives or time derivatives of the functions ρ, u, and T. These can be estimated as follows: for any $\eta > 0$,

$$\begin{gathered}\Big|\int_0^t\int_0^L a\rho_{xx}u_x\,dx\,d\tau\Big| \le \eta\sup|a|\int_0^t\int_0^L \rho_{xx}^2\,dx\,d\tau + \frac{\sup|a|}{4\eta}\int_0^t\int_0^L u_x^2\,dx\,d\tau\,,\\ \Big|\int_0^t\int_0^L a\rho_{xx}u_x\,dx\,d\tau\Big| \le \eta C_1\int_0^t\int_0^L \rho_{xx}^2\,dx\,d\tau + K_2(\eta)\int_0^t\int_0^L u_x^2\,dx\,d\tau\end{gathered} \tag{6.37}$$

One further example of this type, in the first { }, would be

$$\begin{aligned}&\Big|\int_0^t\int_0^L a\rho_t u_x\,dx\,d\tau\Big| \le \eta\sup|a|\int_0^t\int_0^L \rho_t^2\,dx\,d\tau + \frac{\sup|a|}{4\eta}\int_0^t\int_0^L u_x^2\,dx\,d\tau\\ &= \eta C_2\int_0^t\int_0^L \rho_t^2\,dx\,d\tau + K_4(\eta)\int_0^t\int_0^L u_x^2\,dx\,d\tau\,.\end{aligned} \tag{6.38}$$

For the third group of { } terms on the right-hand side of (6.13) we have estimates which conform to the pattern in the following two examples:

$$
\begin{aligned}
\Big|\int_0^t\int_0^L c\rho\,dx\,d\tau\Big| &\le \frac{1}{2}\int_0^t\int_0^L \rho^2\,dx\,d\tau + \frac{1}{2}\int_0^t\int_0^L c^2\,dx\,d\tau \\
&\le K_3\int_0^t\int_0^L \rho^2\,dx\,d\tau + G_1\cdot t\,,
\end{aligned} \tag{6.39}
$$

$$
\begin{aligned}
\Big|\int_0^t\int_0^L c\rho_{xx}\,dx\,d\tau\Big| &\le \eta\int_0^t\int_0^L \rho_{xx}^2\,dx\,d\tau + \frac{1}{4\eta}\int_0^t\int_0^L c^2\,dx\,d\tau \\
&\le \eta\int_0^t\int_0^L \rho_{xx}^2\,dx\,d\tau + G_2(\eta)\cdot t, \quad G_2(\eta)=\frac{\text{const.}}{\eta}
\end{aligned} \tag{6.40}
$$

Terms involving second order spatial derivatives or time derivatives on the right hand side of, say, (6.40) can be moved to the left hand side of (6.13) to be absorbed by those terms with a similar structure, if η is chosen sufficiently small.

Finally the last term on the right hand side of (6.13) may be estimated as follows:

$$
\Big|\int_0^t\int_0^L \delta\rho_t\rho_{xx}\,dx\,d\tau\Big| \le \int_0^t\int_0^L \frac{\delta}{2}\rho_t^2\,dx\,d\tau + \int_0^t\int_0^L \frac{\delta}{2}\rho_{xx}^2\,dx\,d\tau\,. \tag{6.41}
$$

Both terms on the right-hand side of (6.41) can be brought over to the left-hand side of (6.13) and absorbed by those terms with a similar structure; this is true for the first term on the right-hand side of (6.41) provided $\delta < 2$.

We observe that on the left-hand side of (6.13),

$$
\int_0^t\int_0^L iT_{xx}^2\,dx\,d\tau \ge \int_0^t\int_0^L i_cT_{xx}^2\,dx\,d\tau \tag{6.42a}
$$

and similarly for term involving fu_{xx}^2. Then, for η chosen small enough

$$
\int_0^t\int_0^L (i_c - C\eta)T_{xx}^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}i_cT_{xx}^2\,dx\,d\tau \tag{6.42b}
$$

$$
\int_0^t\int_0^L (f_c - C\eta)u_{xx}^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}f_cu_{xx}^2\,dx\,d\tau \tag{6.42c}
$$

$$
\int_0^t\int_0^L \Big(\frac{\delta}{2} - C\eta\Big)\rho_{xx}^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{4}\delta\rho_{xx}^2\,dx\,d\tau \tag{6.42d}
$$

$$
\int_0^t\int_0^L (1 - C\eta)T_t^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}T_t^2\,dx\,d\tau \tag{6.42e}
$$

$$
\int_0^t\int_0^L (1 - C\eta)u_t^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}u_t^2\,dx\,d\tau \tag{6.42f}
$$

$$
\int_0^t\int_0^L \Big(\frac{1}{2} - C\eta\Big)\rho_t^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{4}\rho_t^2\,dx\,d\tau\,, \tag{6.42g}
$$

where $C = \sum C_i$. We note that once η is chosen, the K_i and G_i, $i = 1, 2, \dots$ are constants. Adding all our estimates, and making use of (6.42)a-g, we obtain (6.35) with $\mathcal{K} = \sum K_i$ and $\mathcal{G} = \sum G_i$. □

As a consequence of Lemma 6.5, we have the following a priori estimate for the energy functional $\mathcal{E}_\delta(t)$.

Lemma 6.6. *For some $C_{t_0} > 0$, and all t, $0 \le t \le t_0$, we have, under the hypotheses of Lemma 6.5,*

$$\mathcal{E}_\delta(t) \le C_{t_0}\Big(\frac{\mathcal{G}}{\mathcal{K}} + \mathcal{E}(0)\Big) \tag{6.43}$$

Proof. The proof is a consequence of Gronwall's inequality [25]. By virtue of (6.35),

$$\mathcal{E}_\delta(t) + \frac{\mathcal{G}}{\mathcal{K}} \le \mathcal{E}(0) + \frac{\mathcal{G}}{\mathcal{K}} + \mathcal{G}t + \mathcal{K}\int_0^t \mathcal{E}_\delta(\tau)d\tau, \quad 0 \le t \le t_0\,. \tag{6.44}$$

If we set

$$\bar{\mathcal{E}}_\delta(t) = \mathcal{E}_\delta(t) + \frac{\mathcal{G}}{\mathcal{K}} \tag{6.45}$$

it follows that

$$\bar{\mathcal{E}}_\delta(t) \le \bar{\mathcal{E}}(0) + \mathcal{K}\int_0^t \bar{\mathcal{E}}_\delta(\tau)d\tau \tag{6.46}$$

Applying Gronwall's inequality to $\bar{\mathcal{E}}_\delta$ we obtain

$$\mathcal{E}_\delta(t) \le \mathcal{E}_\delta(t) + \frac{\mathcal{G}}{\mathcal{K}} \le \Big(\frac{\mathcal{G}}{\mathcal{K}} + \mathcal{E}(0)\Big)e^{\mathcal{K}t}, \quad 0 \le t \le t_0 \tag{6.47}$$

from which (6.43) follows, for $0 \le t \le t_0$, with $C_{t_0} = \exp(\mathcal{K}t_0)$. $\square$

As a prelude to the introduction of the Galerkin approximations, we first extend the initial data symmetrically to $[0, 2L]$; i.e., for $L \le x \le 2L$ we define

$$u_0(x) = u_0(2L - x) \tag{6.48a}$$
$$\rho_0(x) = \rho_0(2L - x) \tag{6.48b}$$
$$T_0(x) = T_0(2L - x) \tag{6.48c}$$

and then extend u_0, ρ_0, T_0 periodically to the entire line with period $2L$. A complete orthonormal set of functions on $[0, 2L]$, with respect to the inner product $< f, g >= \int_0^{2L} fg dx$, is given by

$$\{\frac{1}{\sqrt{2L}},\ \frac{1}{\sqrt{L}}\sin\frac{l\pi x}{2L},\ \frac{1}{\sqrt{L}}\cos\frac{l\pi x}{2L};\ l = 1, 2, \dots\} \tag{6.49}$$

We also observe that the set

$$\{\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L},\ l = 0, 1, 2, \dots\} \tag{6.50}$$

is a complete orthonormal set with respect to

$$\{u(\cdot)|u(\cdot) \in L^2[0, L],\ u(0) = 0,\ u_x(L) = 0\}.$$

in the trace sense [1, 25]. Similarly,

$$\{\sqrt{\frac{2}{L}}\sin\frac{l\pi x}{L},\ l = 0, 1, 2, \dots\} \tag{6.51}$$

is a complete orthonormal set with respect to

$$\{u(\cdot)|u(\cdot) \in L^2[0, L],\ u(0) = 0,\ u(L) = 0\}.$$

in the trace sense.

To proceed, we define candidate "approximate solutions"; i.e., Galerkin approximations for the system consisting of (6.10)a,b,c, (6.7)a,b,c and (6.8)a-f of the form

$$\rho_m = \sum_{l=0}^{m} A_{lm}(t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L} \tag{6.52a}$$
$$u_m = \sum_{l=0}^{m} B_{lm}(t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L} \tag{6.52b}$$
$$T_m = \sum_{l=0}^{m} C_{lm}(t)\sqrt{\frac{2}{L}}\sin\frac{l\pi x}{L}\,, \tag{6.52c}$$

where for $0 \le l \le m, m = 0, 1, 2, \dots,$

$$A_{lm}(0) = \xi_l \tag{6.53a}$$
$$B_{lm}(0) = \eta_l \tag{6.53b}$$
$$C_{lm}(0) = \nu_l \tag{6.53c}$$

and ξ_l, η_l, ν_l are determined as the coefficients in the following expansions of the initial data:

$$\sum_{l=0}^{m} \xi_l\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L} \to \rho_0(x), \quad \text{in } L^2(0,L) \tag{6.54a}$$
$$\sum_{l=0}^{m} \eta_l\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L} \to u_0(x), \quad \text{in } L^2(0,L) \tag{6.54b}$$
$$\sum_{l=0}^{m} \nu_l\sqrt{\frac{2}{L}}\sin\frac{l\pi x}{L} \to T_0(x), \quad \text{in } L^2(0,L), \tag{6.54c}$$

as $m \to \infty$. The ξ_l, η_l and ν_l are uniquely determined, once the functions $\rho_0(x)$, $u_0(x)$, and $T_0(x)$ have been extended as described above. In (6.54)a,b,c, $\rho_0(x)$, $u_0(x)$, and $T_0(x)$ are actually the functions $\hat\rho_0(x)$, $\hat u_0(x)$, $\hat T_0(x)$, the hats having been dropped. We now require that the coefficients $A_{km}(t)$, $B_{km}(t)$, and $C_{km}(t)$ satisfy the linear system of ordinary differential equations:

$$\begin{aligned}
A'_{km}(t) = &-\sum_{l=0}^{m} B_{lm}(t)\frac{(2l+1)\pi}{2L^2}\langle a(x,t)\cos\frac{(2l+1)\pi x}{2L}, \sin\frac{(2k+1)\pi x}{2L}\rangle \\
&-\sum_{l=0}^{m} A_{lm}(t)\frac{(2l+1)\pi}{2L^2}\langle b(x,t)\cos\frac{(2l+1)\pi x}{2L}, \sin\frac{(2k+1)\pi x}{2L}\rangle \\
&- A_{km}(t)\delta\frac{(2l+1)^2\pi^2}{2L^3} + \langle c(x,t), \frac{1}{\sqrt{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle\,,
\end{aligned} \tag{6.55a}$$

$$
\begin{aligned}
&B_{km}'(t)\\
&= -\sum_{l=0}^{m} B_{lm}(t)\frac{(2l+1)\pi}{2L^2}\langle b_2(x,t)\cos\frac{(2l+1)\pi x}{2L}, \sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad -\sum_{l=0}^{m} A_{lm}(t)\frac{(2l+1)\pi}{2L^2}\langle d(x,t)\cos\frac{(2l+1)\pi x}{2L}, \sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad -\sum_{l=0}^{m} C_{lm}(t)R\frac{l\pi}{L^2}\langle \cos\frac{l\pi x}{L}, \sin\frac{(2k+1)\pi x}{2L}\rangle + \langle g(x,t), \frac{1}{\sqrt{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad -\sum_{l=0}^{m} B_{lm}(t)\frac{(2l+1)^2\pi^2}{4L^3}\langle f(x,t)\sin\frac{(2l+1)\pi x}{2L}, \sin\frac{(2k+1)\pi x}{2L}\rangle\,,
\end{aligned} \tag{6.55b}
$$

$$
\begin{aligned}
C_{km}'(t) = &-\sum_{l=0}^{m} B_{lm}(t)\frac{(2l+1)\pi}{2L^2}\langle h(x,t)\cos\frac{(2l+1)\pi x}{2L}, \sin\frac{k\pi x}{L}\rangle\\
&-\sum_{l=0}^{m} C_{lm}(t)\frac{l\pi}{L^2}\langle b_3(x,t)\cos\frac{l\pi x}{L}, \sin\frac{k\pi x}{L}\rangle\\
&-\sum_{l=0}^{m} C_{lm}(t)\frac{l^2\pi^2}{L^3}\langle i(x,t)\sin\frac{l\pi x}{2L}, \sin\frac{k\pi x}{L}\rangle\\
&+\langle j(x,t), \frac{1}{\sqrt{L}}\sin\frac{k\pi x}{L}\rangle
\end{aligned} \tag{6.55c}
$$

and the initial conditions, for $0 \le l \le m$,

$$A_{lm}(0) = \xi_l \tag{6.56a}$$

$$B_{lm}(0) = \eta_l \tag{6.56b}$$

$$C_{lm}(0) = \nu_l \tag{6.56c}$$

By standard ODE theory the system (6.55)a,b,c, (6.56)a,b,c possesses a unique solution. Moreover, we have the following result.

Lemma 6.7. *Let $\{A_{km}(t), B_{km}(t), C_{km}(t)\}$, $0 \le k \le m$, be the unique solution of the initial value problem (6.55)a,b,c, (6.56)a,b,c. Then the "approximate solutions" ρ_m, u_m and T_m, as defined by (6.52)a,b,c, satisfy the hypotheses of Lemma 6.2 (and, hence, those of Lemma 6.5 as well).*

Proof. We have

$$
\begin{aligned}
&\rho_{mt} + au_{mx} + b\rho_{mx} - \delta\rho_{mxx} - c\\
&= \sum_{l=0}^{m} A_{lm}'(t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L} + \sum_{l=0}^{m} B_{lm}(t)a(x,t)\frac{(2l+1)\pi}{2L}\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L}\\
&\quad + \sum_{l=0}^{m} A_{lm}(t)b(x,t)\frac{(2l+1)\pi}{2L}\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L}\\
&\quad - c(x,t) + \sum_{l=0}^{m} A_{lm}(t)\delta\frac{(2l+1)^2\pi^2}{4L^2}\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L}
\end{aligned} \tag{6.57}
$$

If we now multiply (6.57) by $\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}$, $0\le k\le m$, and integrate from 0 to L, we obtain, for $0\le k\le m$,

$$\begin{aligned}
&\int_0^L(\rho_{mt}+au_{mx}+b\rho_{mx}-\delta\rho_{mxx}-c)\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}dx\\
&=A'_{km}(t)+\sum_{l=0}^m B_{lm}(t)\frac{(2l+1)\pi}{2L}\langle a(x,t)\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L},\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad+\sum_{l=0}^m A_{lm}(t)\frac{(2l+1)\pi}{2L}\langle b(x,t)\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L},\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad+A_{km}(t)\delta\frac{(2l+1)^2\pi^2}{2L^3}-\langle c(x,t),\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle=0
\end{aligned}\tag{6.58}$$

(for each $0\le k\le m$) as a consequence (6.55)a. Therefore, for any $v\in\mathcal{V}_m$,

$$\int_0^L(\rho_{mt}+au_{mx}+b\rho_{mx}-\delta\rho_{mxx}-c)v\,dx=0\,.\tag{6.59}$$

Next, we compute that

$$\begin{aligned}
&u_{mt}+b_2u_{mx}+RT_{mx}+d\rho_{mx}-fu_{mxx}-g\\
&=\sum_{l=0}^m B'_{lm}(t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L}+\sum_{l=0}^m B_{lm}(t)b_2(x,t)\frac{(2l+1)\pi}{2L}\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L}\\
&\quad+\sum_{l=0}^m C_{lm}(t)R\frac{l\pi}{L}\sqrt{\frac{2}{L}}\cos\frac{l\pi x}{L}+\sum_{l=0}^m A_{lm}(t)d(x,t)\frac{(2l+1)\pi}{2L}\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L}\\
&\quad+\sum_{l=0}^m B_{lm}(t)\left(\frac{(2l+1)\pi}{2L}\right)^2 f(x,t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L}-g(x,t)
\end{aligned}\tag{6.60}$$

Multiplying (6.60) by $\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}$, $0\le k\le m$, and integrating over $[0,L]$, we obtain, for $0\le k\le m$,

$$\begin{aligned}
&\int_0^L(u_{mt}+b_2u_{mx}+RT_{mx}+d\rho_{mx}-fu_{mxx}-g)\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}dx\\
&=\sum_{l=0}^m B'_{km}(t)\\
&\quad+\sum_{l=0}^m B_{lm}(t)\frac{(2l+1)\pi}{2L}\langle b_2(x,t)\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L},\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad+\sum_{l=0}^m A_{lm}(t)\frac{(2l+1)\pi}{2L}\langle d(x,t)\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L},\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad+\sum_{l=0}^m C_{lm}(t)R\frac{l\pi}{L}\langle\sqrt{\frac{2}{L}}\cos\frac{l\pi x}{L},\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle\\
&\quad-\langle g(x,t),\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle
\end{aligned}$$

$$\begin{aligned} &+\sum_{l=0}^{m} B_{lm}(t)\left(\frac{(2l+1)\pi}{2L}\right)^2 \langle f(x,t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L}, \sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L}\rangle \\ &= 0 \end{aligned} \tag{6.61}$$

as a consequence of (6.55)b. Thus, for any $v \in \mathcal{V}_m$,

$$\int_0^L (u_{mt} + b_2 u_{mx} + RT_{mx} + d\rho_{mx} - f u_{mxx} - g)\sqrt{\frac{2}{L}}\sin\frac{(2k+1)\pi x}{2L} dx = 0\,. \tag{6.62}$$

Finally, we have

$$\begin{aligned} &T_t + b_3 T_x + h u_x - i T_{xx} - j \\ &= \sum_{l=0}^{m} C'_{lm}(t)\sqrt{\frac{2}{L}}\sin\frac{l\pi x}{L} + \sum_{l=0}^{m} B_{lm}(t) h(x,t)\frac{(2l+1)\pi}{2L}\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L} \\ &\quad+\sum_{l=0}^{m} C_{lm}(t)\frac{l\pi}{L} b_3(x,t)\sqrt{\frac{2}{L}}\cos\frac{l\pi x}{L} \\ &\quad+\sum_{l=0}^{m} C_{lm}(t)\left(\frac{l\pi}{L}\right)^2 i(x,t)\sqrt{\frac{2}{L}}\sin\frac{l\pi x}{2L} - j(x,t) \end{aligned} \tag{6.63}$$

Multiplying (6.63) by $\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}$, $0 \le k \le m$, and integrating over $[0, L]$, we obtain

$$\begin{aligned} &\int_0^L (T_t + b_3 T_x + h u_x - i T_{xx} - j)\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L} dx \\ &= \sum_{l=0}^{m} C'_{km}(t) \\ &\quad+\sum_{l=0}^{m} B_{lm}(t)\frac{(2l+1)\pi}{2L}\langle h(x,t)\sqrt{\frac{2}{L}}\cos\frac{(2l+1)\pi x}{2L}, \sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}\rangle \\ &\quad+\sum_{l=0}^{m} C_{lm}(t)\frac{l\pi}{L}\langle b_3(x,t)\sqrt{\frac{2}{L}}\cos\frac{l\pi x}{L}, \sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}\rangle \\ &\quad+\sum_{l=0}^{m} C_{lm}(t)\left(\frac{l\pi}{L}\right)^2 \langle i(x,t)\sqrt{\frac{2}{L}}\sin\frac{l\pi x}{2L}, \sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}\rangle \\ &\quad- \langle j(x,t), \sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}\rangle = 0\,, \end{aligned} \tag{6.64}$$

as a consequence of (6.55)c. Thus, we conclude that for any $w \in \mathcal{W}_m$,

$$\int_0^L (T_t + b_3 T_x + h u_x - i T_{xx} - j) w dx = 0 \tag{6.65}$$

which completes the proof of Lemma 6.7 □

We are now in a position to state the main result of this section.

Theorem 6.8. *The initial-boundary value problem* (6.10)*a,b,c,* (6.7)*a,b,c,* (6.8)*a-f, has a unique solution* (ρ, u, T)*, for any* $t > 0$*, such that*

$$\rho, u, T \in L^2([0,t];\ W^{2,2}[0,L]), \tag{6.66a}$$

$$\rho_t, u_t, T_t \in L^2([0,t];\ L^2[0,L])\,. \tag{6.66b}$$

Proof. The approximations ρ_m, u_m, and T_m defined by (6.52)a,b,c satisfy the hypotheses of Lemma 6.6. Therefore, the *a priori* estimate (6.43) applies to ρ_m, u_m, and T_m; i.e., on any interval $[0,t_0]$ we have

$$\mathcal{E}_{\delta m}(t) \le C_{t_0}\Big(\frac{\mathcal{G}}{\mathcal{K}} + \mathcal{E}_m(0)\Big) \tag{6.67}$$

Therefore,

$$\rho_m,\ u_m,\ T_m\ \in L^2([0,t];W^{2,2}[0,L]); \tag{6.68a}$$

$$\rho_m,\ u_m,\ T_m\ \in W^{1,2}([0,t];L^2[0,L]) \tag{6.68b}$$

for $0 \le t \le t_0$, which implies that ρ_m, u_m, and T_m are continuous in both time and space. By the choice at $t = 0$ of the coefficients in the Galerkin approximations; i.e., (6.53)a,b,c, ρ_m, u_m, and T_m converge as $t \to 0$ to $\rho_0(x)$, $u_0(x)$, and $T_0(x)$, so by the continuity of ρ_m, u_m, and T_m with respect to time we conclude that $\mathcal{E}_m(0) \to \mathcal{E}(0)$. Thus, $\mathcal{E}_m(0)$ is a bounded sequence and (6.67) then implies that $\mathcal{E}_{\delta m}(t)$ is also a bounded sequence, for $0 \le t \le t_0$. Therefore, as each of the sequences ρ_m, u_m, and T_m is bounded in $L^2([0,t];W^{2,2}[0,L])$ they have convergent subsequences ρ_{m_k}, u_{m_k}, and T_{m_k} (which we will also denote as ρ_m, u_m, and T_m) that converge weakly in $L^2([0,t];W^{2,2}[0,L])$ to unique limits ρ, u, and T

We want to show that the limiting set $\{\rho,u,T\}$ is a solution of the initial-boundary value problem (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f, with (6.10)a,b,c being satisfied in the sense of distributions. We observe that $\{\rho_m,u_m,T_m\}$ satisfy

$$\int_0^t\int_0^L (\rho_{mt} + au_{mx} + b\rho_{mx} - \delta\rho_{mxx} - c)v\,dx\,d\tau = 0, \tag{6.69a}$$

$$\int_0^t\int_0^L (u_{mt} + b_2u_{mx} + RT_{mx} + d\rho_{mx} - fu_{mxx} - g)v\,dx\,d\tau = 0 \tag{6.69b}$$

for any $v \in \mathcal{V}_m$ with coefficients $\alpha_l(t) \in C^1(0,t_0)$; while

$$\int_0^t\int_0^L (T_{mt} + b_3T_{mx} + hu_{mx} - iT_{mxx} - j)w\,dx\,d\tau = 0 \tag{6.69c}$$

for any $w \in \mathcal{W}_m$, with coefficients $\beta(t) \in C^1(0,t_0)$, for $m = 1,2,\dots$. Consider an arbitrary function $\psi \in L^2([0,t_0];C_0^\infty[0,L])$ with compact support on $[0,t_0]\times[0,L]$; this function can be expanded in a series

$$\psi(x,t) = \sum_{l=0}^{\infty} p_l(t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L} \tag{6.70}$$

which converges uniformly to ψ. The series obtained by differentiating (6.70) term by term k times also converges uniformly to the respective k^{th} derivative of ψ. For the approximation to ψ given by

$$\psi_n(x,t) = \sum_{l=0}^{n} p_l(t)\sqrt{\frac{2}{L}}\sin\frac{(2l+1)\pi x}{2L}\,, \tag{6.71}$$

we have $\psi_n \to \psi$ in $L^2([0,t_0];C_0^\infty[0,L])$, as $n \to \infty$. We observe that $\psi_n \in \mathcal{V}_m$, for $m \ge n$, with $p_l(t) = 0$, for $n < l \le m$, so that ψ_n satisfies, for each $m = 1,2,\dots$

$$\int_0^t\int_0^L (\rho_{mt} + au_{mx} + b\rho_{mx} - \delta\rho_{mxx} - c)\psi_n\,dx\,d\tau = 0\,. \tag{6.72}$$

Letting $m \to \infty$ in (6.72) we obtain, in the limit,

$$\int_0^t \int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)\psi_n \, dx \, d\tau = 0 \,. \tag{6.73}$$

Taking the limit in (6.73) as $n \to \infty$ yields

$$\int_0^t \int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)\psi \, dx \, d\tau = 0 \,. \tag{6.74a}$$

In a similar manner it follows that

$$\int_0^t \int_0^L (u_t + b_2 u_x + RT_x + d\rho_x - fu_{xx} - g)\psi \, dx \, d\tau = 0, \tag{6.74b}$$

$$\int_0^t \int_0^L (T_t + b_3 T_x + hu_x - iT_{xx} - j)\psi \, dx \, d\tau = 0 \tag{6.74c}$$

for any arbitrary function $\psi \in L^2([0, t_0]; C_0^\infty[0, L])$ with compact support in $[0, t_0] \times [0, L]$. By a standard density argument we conclude that the limiting set $\{\rho, u, T\}$ is a distribution solution of (6.10)a,b,c in the interior of the rectangle $[0, t_0] \times [0, L]$. The boundary conditions (6.8)a-f are satisfied by each member of the sequence $\{\rho_m, u_m, T_m\}$. However, each member $\{\rho_m, u_m, T_m\}$ of this sequence is continuous with respect to x at each $t, 0 \le t \le t_0$, and therefore so is $\{\rho, u, T\}$. Thus, the boundary conditions (6.8)a-f are also satisfied by $\{\rho, u, T\}$. Finally, the initial conditions (6.7)a,b,c are satisfied by $\{\rho, u, T\}$, since each member $\{\rho_m, u_m, T_m\}$ is continuous with respect to t, at each $0 \le x \le L$, and $\{\rho_m(x, 0), u_m(x, 0), T_m(x, 0)\}$ converges to the prescribed initial data as $m \to \infty$. $\square$

To establish higher regularity for the solution (ρ, u, T) of (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f, than that which is given by (6.6)a,b, we must differentiate the equations (6.10)a,b,c; differentiation here is understood in the sense of distributions. As an example of such differentiation consider equation (6.10)a. For any test function $\psi(x, t)$ the derivative ψ_x is also a test function and we have

$$\int_0^t \int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)\psi_x \, dx \, d\tau = 0 \,. \tag{6.75}$$

Integrating this last expression by parts (in space) we obtain

$$\int_0^t (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)\psi\Big|_0^L d\tau - \int_0^t \int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)_x \psi \, dx \, d\tau = 0 \,. \tag{6.76}$$

In view of the compact support of ψ, in the rectangle $[0, L] \times [0, t]$, it follows from (6.76) that

$$\int_0^t \int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)_x \psi \, dx \, d\tau = 0 \tag{6.77}$$

for any test function ψ. Therefore, in the sense of distributions

$$(\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)_x = 0 \tag{6.78}$$

and (6.78) implies, e.g., the validity of results such as

$$\int_0^t \int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)_x \rho_x \, dx \, d\tau = 0, \tag{6.79}$$

because for any sequence of test functions ψ_n such that $\psi_n \to \rho_x$ we have

$$\int_0^t \int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)_x \psi_n \, dx \, d\tau = 0 \,. \tag{6.80}$$

Theorem 6.9. *Let $\rho, u, T \in L^2([0,t]; W^{2,2}[0,L])$ be the unique solution of* (6.10) *a,b,c,* (6.7) *a,b,c,* (6.8) *a-f, for $t > 0$, whose existence was established in Theorem 6.8. Then, in fact,*

$$\rho, u, T \in C^\infty([0,t]; C^\infty[0,L]) \tag{6.81}$$

Proof. We rewrite system (6.10)a,b,c in the form

$$\rho_t - \delta\rho_{xx} + au_x + b\rho_x = c \tag{6.82a}$$

$$u_t - fu_{xx} + b_2 u_x + RT_x + d\rho_x = g \tag{6.82b}$$

$$T_t - iT_{xx} + b_3 T_x + hu_x = j \tag{6.82c}$$

As $c, g, j \in L^2([0,t]; W^{1,2}[0,L])$, standard parabolic theory [42] impplies that ρ, u, T belong to $L^2([0,t]; W^{3,2}[0,L])$ and, because $\rho, u, T \in L^2([0,t]; W^{3,2}[0,L])$, we can differentiate the equations in (6.10)a,b,c with respect to x to obtain the system

$$(\rho_x)_t - \delta(\rho_x)_{xx} + a(u_x)_x + b(\rho_x)_x = -a_x u_x - b_x \rho_x + c_x \tag{6.83a}$$

$$(u_x)_t - f(u_x)_{xx} + b_2(u_x)_x + R(T_x)_x + d(\rho_x)_x = -f_x u_{xx} - b_{2x} u_x - d_x \rho_x + g_x \tag{6.83b}$$

$$(T_x)_t - i(T_x)_{xx} + b_3(T_x)_x + h(u_x)_x = -i_x T_{xx} - b_{3x} T_x - h_x u_x + j_x \tag{6.83c}$$

which is a system of equations for ρ_x, u_x, T_x with the same principal part as (6.82)a,b,c. Also, each forcing term on the right hand side of (6.83)a,b,c is again in $L^2([0,t]; W^{1,2}[0,L])$. Therefore, $\rho_x, u_x, T_x \in L^2([0,t]; W^{3,2}[0,L])$, in which case $\rho, u, T \in L^2([0,t]; W^{4,2}[0,L])$.

By differentiating (6.82)a,b,c with respect to time, we obtain a system of parabolic equations for ρ_t, u_t and T_t which also has the same principal part as (6.82)a,b,c and for this system each forcing term is, again, in $L^2([0,t]; W^{1,2}[0,L])$. This leads to the result that $\rho_t, u_t, T_t \in L^2([0,t]; W^{3,2}[0,L])$. By continuing this argument we may establish that the spatial, time, and mixed derivatives of ρ, u, and T, of all orders, are in $L^2([0,t]; W^{3,2}[0,L])$ which, in turn, implies the result (6.81). $\square$

7. Existence of solutions to the Linear hyperbolic-parabolic initial-boundary value problem

In this section we will establish existence of a unique solution for the mixed hyperbolic-parabolic initial boundary value problem (6.6)a,b,c, (6.7)a,b,c, (6.8)a-e. Our assumptions on the coefficients in (6.6)a,b,c are those stated as (6.9)a,b,c; in addition, we will require that

$$c(0,t) = c(L,t) = 0, t > 0, \tag{7.1a}$$

$$b(0,t) \geq \zeta, \quad \text{for some } \zeta > 0 \text{ and all } t > 0. \tag{7.1b}$$

For the coefficient $c(x,t)$ defined by the nonlinear problem; i.e., (6.3)e, (7.1)a is satisfied as $(\ln A)_x = 0$ at $x = 0, L, \forall t \geq 0$; however, (7.1)b is not satisfied for $b(x,t)$ as defined by (6.3)b. This is, however, of little concern here as the results presented in [5], for the original nonlinear problem, depend only on the existence and uniqueness theorem proven in §6 for the linear system with artificial viscosity without the hypotheses (7.1)a,b. In the present section, we will again prove an existence and uniqueness theorem for the problem (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f; however, the addition of the hypotheses (7.1)a,b will enable us to use an energy functional which does not depend explicitly on the artificial viscosity parameter δ and this, in turn, will allow us to extract the limit, as $\delta \to 0$, of the solutions of the regularized problem so as to obtain the desired solution of (6.6)a,b,c, (6.7)a,b,c, (6.8)a-e.

We begin with the following definition.

Definition 7.1. For $\rho(x,t), u(x,t)$, and $T(x,t)$ we define the energy functional

$$\begin{aligned}\hat{\mathcal{E}}_\delta(t) &= \frac{1}{2}\{\|\rho(\cdot,t)\|^2_{W^{1,2}} + \|u(\cdot,t)\|^2_{W^{1,2}} + \|T(\cdot,t)\|^2_{W^{1,2}}\}\\ &+ \frac{1}{2}\int_0^t\int_0^L \{i_c(T_{xx}^2 + T_x^2) + T_t^2 + f_c(u_{xx}^2 + u_x^2) + u_t^2 + \rho_t^2\}\,dx\,d\tau\\ &+ \frac{1}{2}\int_0^L f_c u_x^2 dx + \frac{1}{2}\int_0^L i_c T_x^2 dx\end{aligned} \tag{7.2}$$

We note that $\hat{\mathcal{E}}_\delta(t)$ is almost identical with $\mathcal{E}_\delta(t)$ in (6.34), except that in $\hat{\mathcal{E}}_\delta(t)$ the term

$$\frac{\delta}{2}\int_0^t\int_0^L (\rho_x^2 + \rho_{xx}^2)\,dx\,d\tau$$

has been deleted; thus $\hat{\mathcal{E}}_\delta(t)$ depends, implicitly, on δ because ρ, u, T eventually will (as solutions of the regularized linear problem) but $\hat{\mathcal{E}}_\delta$ does not depend explicitly on δ. With $\mathcal{V}_m$ and $\mathcal{W}_m$ defined as in §6; i.e., (6.11)a,b we now have the following counterpart to Lemma 6.2:

Lemma 7.2. *Suppose $\rho(x,t), u(x,t) \in \mathcal{V}_m$ and $T(x,t) \in \mathcal{W}_m$ for some $m \in \mathbb{N}$, and*

$$\int_0^L (\rho_t + au_x + b\rho_x - \delta\rho_{xx} - c)v dx = 0 \tag{7.3a}$$

$$\int_0^L (u_t + b_2 u_x + RT_x + d\rho_x - fu_{xx} - g)v dx = 0 \tag{7.3b}$$

for any $v \in \mathcal{V}_m$, while

$$\int_0^L (T_t + b_3 T_x + hu_x - iT_{xx} - j)w dx = 0 \tag{7.3c}$$

for any $w \in \mathcal{W}_m$; then the following identity holds:

$$\begin{aligned}&\frac{1}{2}\{\|\rho(\cdot,t)\|^2_{W^{1,2}} + \|u(\cdot,t)\|^2_{W^{1,2}} + \|T(\cdot,t)\|^2_{W^{1,2}}\}\\ &+ \int_0^t\int_0^L \{i(T_{xx}^2 + T_x^2) + T_t^2 + f(u_{xx}^2 + u_x^2) + u_t^2 + \delta(\rho_{xx}^2 + \rho_x^2) + \rho_t^2\}\,dx\,d\tau\\ &+ \frac{1}{2}\int_0^L fu_x^2 dx + \frac{1}{2}\int_0^L iT_x^2 dx + \frac{1}{2}\int_0^t b(0,\tau)\rho_x(0,\tau)^2 d\tau\\ &= -\int_0^t a(0,\tau)u_x(0,\tau)\rho_x(0,\tau)d\tau + \frac{1}{2}\int_0^L f(x,0)u_x^2(x,0)dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^L i(x,0)T_x^2(x,0)dx+\frac{1}{2}\{\|\rho(\cdot,0)\|_{W^{1,2}}^2+\|u(\cdot,0)\|_{W^{1,2}}^2+\|T(\cdot,0)\|_{W^{1,2}}^2\}\\
&+\int_0^t\int_0^L\Big\{-a\rho u_x-b\rho\rho_x-b_2uu_x-RuT_x-du\rho_x-b_3TT_x-hTu_x\\
&+\frac{1}{2}f_tu_x^2+\frac{1}{2}i_tT_x^2-\frac{1}{2}b_x\rho_x^2\Big\}dx\,d\tau+\int_0^t\int_0^L\Big\{-au_{xx}\rho_x+b_2u_xu_{xx}\\
&+RT_xu_{xx}+d\rho_xu_{xx}+b_3T_xT_{xx}-a\rho_tu_x-b\rho_t\rho_x-b_2u_tu_x-Ru_tT_x\\
&-du_t\rho_x-b_3T_tT_x-f_xu_tu_x-i_xT_tT_x-hT_tu_x+hu_xT_{xx}\Big\}\,dx\,d\tau\\
&+\int_0^t\int_0^L\{c\rho+gu+jT-c\rho_{xx}-gu_{xx}-jT_{xx}+c\rho_t+gu_t+jT_t\}\,dx\,d\tau\\
&+\int_0^t\int_0^L\delta\rho_t\rho_{xx}\,dx\,d\tau\,. \qquad (7.4)
\end{aligned}
$$

Proof. We observe that as $\rho, u\in\mathcal{V}_m$ and $T\in\mathcal{W}_m$, all even order spatial derivatives of ρ, u and T will be zero at $x=0$, while odd order spatial derivatives of ρ, u and even order spatial derivatives of T will vanish at $x=L$.

As $\rho\in\mathcal{V}_m$, it follows from (7.3)a that

$$
\int_0^L\int_0^t(\rho_t+au_x+b\rho_x-\delta\rho_{xx}-c)\rho\,dx\,d\tau=0
$$

or

$$
\int_0^L\frac{1}{2}\rho^2(t,x)dx-\int_0^L\frac{1}{2}\rho_0^2(x)dx=\int_0^t\int_0^L\{-a\rho u_x-b\rho\rho_x+c\rho+\delta\rho_{xx}\rho\}\,dx\,d\tau
$$

which, after integration by parts of the term $\delta\rho\rho_{xx}$, becomes

$$
\begin{aligned}
&\int_0^L\frac{1}{2}\rho^2(t,x)dx-\int_0^L\frac{1}{2}\rho_0^2(x)dx+\int_0^t\int_0^L\delta\rho_x^2\,dx\,d\tau\\
&=\int_0^t\int_0^L\{-a\rho u_x-b\rho\rho_x+c\rho\}\,dx\,d\tau
\end{aligned}
\qquad (7.5)
$$

Also, as $u\in\mathcal{V}_m$ we have, as a consequence of (7.3)b

$$
\int_0^L(u_t+b_2u_x+RT_x+d\rho_x-fu_{xx}-g)udx=0\,.
$$

If we then integrate this last result over $[0,t]$ we obtain

$$
\begin{aligned}
&\int_0^L\frac{1}{2}u^2(t,x)dx-\int_0^L\frac{1}{2}u_0^2(x)dx+\int_0^t\int_0^L fu_x^2\,dx\,d\tau\\
&=\int_0^t\int_0^L\{-b_2uu_x-RuT_x-du\rho_x+gu\}\,dx\,d\tau
\end{aligned}
\qquad (7.6)
$$

Next, as $T\in\mathcal{W}_m$, it follows from (7.3)c that

$$
\int_0^L(T_t+b_3T_x+hu_x-iT_{xx}-j)Tdx=0
$$

and integrating this result over $[0,t]$ we obtain

$$\begin{aligned}&\int_0^L \frac{1}{2}T^2(t,x)dx - \int_0^L \frac{1}{2}T_0^2(x)dx + \int_0^t\int_0^L iT_x^2\,dx\,d\tau\\&= \int_0^t\int_0^L \{-b_3TT_x - hTu_x + jT\}\,dx\,d\tau\,.\end{aligned}\tag{7.7}$$

Since $\rho_{xx}\in\mathcal{V}_m$, (7.3)a yields

$$\int_0^L (\rho_t + au_x + b\rho_x - c - \delta\rho_{xx})\rho_{xx}dx = 0,$$

and an integration by parts produces

$$\rho_t\rho_x\big|_0^L - \int_0^L \rho_x\rho_{xt}dx + \int_0^L (au_x\rho_{xx} + b\rho_x\rho_{xx} - c\rho_{xx} - \delta\rho_{xx}^2)dx = 0\,.\tag{7.8}$$

Integrating by parts the first three terms in last integral in (7.8) we obtain

$$\begin{aligned}&\int_0^L \Big(au_x\rho_{xx} + \frac{1}{2}b(\rho_x)_x^2 - c\rho_{xx}\Big)dx\\&= \Big(au_x\rho_x + \frac{1}{2}b\rho_x^2 - c\rho_x\Big)\Big|_0^L - \int_0^L \Big(au_{xx}\rho_x + \frac{1}{2}b_x\rho_x^2 - c_x\rho_x\Big)dx\,.\end{aligned}$$

But $\rho_x(L,t)=0$, and $c(0,t)=c(L,t)=0$, so (7.8) and this last identity yields

$$\begin{aligned}&\rho_t\rho_x\big|_0^L - \int_0^L \rho_x\rho_{xt}dx + \int_0^L \Big(-au_{xx}\rho_x - \frac{1}{2}b_x\rho_x^2 + c_x\rho_x - \delta\rho_{xx}^2\Big)dx\\&- a(0,t)u_x(0,t)\rho_x(0,t) - \frac{1}{2}b(0,t)\rho_x^2 = 0\end{aligned}\tag{7.9}$$

The first term on the left-hand side of (7.9) vanishes as $\rho_x(L,t)=\rho_t(0,t)=0$, for all $t\ge 0$, and this reduces (7.9) to

$$\begin{aligned}\int_0^L \rho_x\rho_{xt}dx &= -a(0,t)u_x(0,t)\rho_x(0,t) - \frac{1}{2}b(0,t)\rho_x^2\\&\quad + \int_0^L \Big(-au_{xx}\rho_x - \frac{1}{2}b_x\rho_x^2 + c_x\rho_x - \delta\rho_{xx}^2\Big)dx\end{aligned}$$

which after integration over $[0,t]$ becomes the identity

$$\begin{aligned}&\int_0^L \frac{1}{2}\rho_x^2(t,x)dx - \int_0^L \frac{1}{2}\rho_x^2(x,0)dx + \int_0^t\int_0^L \delta\rho_{xx}^2\,dx\,d\tau + \frac{1}{2}\int_0^t b(0,\tau)\rho_x(0,\tau)^2d\tau\\&= -\int_0^t a(0,\tau)u_x(0,\tau)\rho_x(0,\tau)d\tau + \int_0^t\int_0^L \Big(-au_{xx}\rho_x - \frac{1}{2}b_x\rho_x^2 + c_x\rho_x\Big)\,dx\,d\tau\,.\end{aligned}\tag{7.10}$$

We note that, in (7.10), we have $b(0,\tau)\ge\zeta>0$, $0\le\tau\le t$. As $u_{xx}\in\mathcal{V}_m$, (7.3)b yields

$$\int_0^L (u_t + b_2u_x + RT_x + d\rho_x - fu_{xx} - g)u_{xx}dx = 0\,.$$

Integrating by parts in this last result, and using the fact that $u_x(L,t)=u_t(0,t)=0$, $t\ge 0$, yields

$$\int_0^L u_xu_{xt}dx = \int_0^L (b_2u_x + RT_x + d\rho_x - fu_{xx} - g)u_{xx}dx$$

which upon integration over $[0,t]$ produces the identity

$$\begin{aligned} &\int_0^L \frac{1}{2}u_x^2(t,x)dx - \int_0^L \frac{1}{2}u_x^2(x,0)dx + \int_0^t\int_0^L f u_{xx}^2\,dx\,d\tau \\ &= \int_0^t\int_0^L (b^2 u_x u_{xx} + RT_x u_{xx} + d\rho_x u_{xx} - g u_{xx})\,dx\,d\tau\,. \end{aligned} \tag{7.11}$$

Next, as $T_{xx} \in \mathcal{W}_m$, (7.3)c produces

$$\int_0^L (T_t + b_3 T_x + h u_x - i T_{xx} - j) T_{xx} dx = 0\,.$$

In this last equation we integrate by parts, apply the conditions $T_t(0,t) = T_t(L,t) = 0, t \geq 0$, and then integrate over $[0,t]$ so as to obtain

$$\begin{aligned} &\int_0^L \frac{1}{2}T_x^2(t,x)dx - \int_0^L \frac{1}{2}T_x^2(x,0)dx + \int_0^t\int_0^L iT_{xx}^2\,dx\,d\tau \\ &= \int_0^t\int_0^L (b_3 T_x T_{xx} + h u_x T_{xx} - jT_{xx})\,dx\,d\tau \end{aligned} \tag{7.12}$$

Since $\rho_t \in \mathcal{V}_m$, (7.3)a yields

$$\int_0^L (\rho_t + a u_x + b\rho_x - c - \delta\rho_{xx})\rho_t dx = 0\,;$$

so that

$$\int_0^t\int_0^L \rho_t^2\,dx\,d\tau = \int_0^t\int_0^L (-a\rho_t u_x - b\rho_t\rho_x + c\rho_t + \delta\rho_t\rho_{xx})\,dx\,d\tau\,. \tag{7.13}$$

Next, we note that as $u_t \in \mathcal{V}_m$, (7.3)b yields

$$\int_0^t\int_0^L (u_t + b_2 u_x + RT_x + d\rho_x - f u_{xx} - g)u_t\,dx\,d\tau = 0\,. \tag{7.14}$$

Integrating the next to last term in this integral by parts we find that, as $u_x(L,t) = u_t(0,t) = 0$,

$$\int_0^t\int_0^L f u_{xx} u_t\,dx\,d\tau = -\int_0^L \frac{1}{2} f u_x^2 dx\big|_0^t + \int_0^t\int_0^L \frac{1}{2} f_t u_x^2\,dx\,d\tau - \int_0^t\int_0^L f_x u_x u_t\,dx\,d\tau$$

and if we now substitute this last result back into (7.14) we obtain

$$\begin{aligned} &\int_0^t\int_0^L u_t^2\,dx\,d\tau + \int_0^L \frac{1}{2} f u_x^2 dx \\ &= \int_0^L \frac{1}{2} f(x,0) u_x^2(x,0) dx \\ &\quad + \int_0^t\int_0^L \Big(-b_2 u_t u_x - R u_t T_x - d u_t \rho_x - f_x u_t u_x + \frac{1}{2} f_t u_x^2 - g u_t \Big)\,dx\,d\tau\,. \end{aligned} \tag{7.15}$$

Using the fact that $T_t \in \mathcal{W}_m$ we obtain from (7.3)c

$$\int_0^t\int_0^L (T_t + b_3 T_x + h u_x - i T_{xx} - j) T_t\,dx\,d\tau = 0\,. \tag{7.16}$$

Integration of the next to the last term in (7.16) by parts, and use of the conditions $T_t(L,t) = T_t(0,t) = 0$, produces the identity

$$\int_0^t \int_0^L iT_{xx}T_t \,dx\,d\tau = -\int_0^L \frac{1}{2} iT_x^2 dx\Big|_0^t + \int_0^t \int_0^L \frac{1}{2} i_t T_x^2 \,dx\,d\tau - \int_0^t \int_0^L i_x T_x u_t \,dx\,d\tau$$

which, when substituted in (7.16) yields

$$\begin{aligned} &\int_0^t \int_0^L T_t^2 \,dx\,d\tau + \int_0^L \frac{1}{2} iT_x^2 dx \\ &= \int_0^L \frac{1}{2} i(x,0) T_x^2(x,0) dx \\ &\quad + \int_0^t \int_0^L \Big(-b_3 T_t T_x - hT_t u_x - i_x T_t T_x + \frac{1}{2} i_t T_x^2 - jT_t \Big) \,dx\,d\tau\,. \end{aligned} \tag{7.17}$$

Adding together the results in (7.5), (7.6), (7.7), (7.10), (7.11), (7.12), (7.13), (7.15), and (7.17), and then grouping like terms together, we obtain the identity (7.4). □

From the energy identity (7.4) we are now able to obtain for $\hat{\mathcal{E}}_\delta(t)$ an energy inequality entirely analogous to (6.35) for $\mathcal{E}_\delta(t)$.

Lemma 7.3. *Under the conditions in Lemma 7.2, we have for $\rho, u \in \mathcal{V}_m, T \in \mathcal{W}_m$, and $0 < \delta < 1$,*

$$\hat{\mathcal{E}}_\delta(t) \le \hat{\mathcal{E}}(0) + \mathcal{G}t + \mathcal{K} \int_0^t \hat{\mathcal{E}}_\delta(\tau) d\tau \tag{7.18}$$

for some positive constants $\mathcal{G}$ and $\mathcal{K}$, where $\hat{\mathcal{E}}_\delta(0)$, being independent of δ, has been denoted as $\hat{\mathcal{E}}(0)$.

Remarks: (i) In view of the definition of $\hat{\mathcal{E}}_\delta(t), \hat{\mathcal{E}}(0)$ is independent of δ. (ii) The terms on the right-hand side of (7.4) have been grouped by { } into four distinct subsets of terms; estimates for typical terms in each of these four groupings are derived in the proof of Lemma 7.3 and generic positive constants C_i, K_i, G_i will be used in these estimates. (iii) For δ, satisfying $0 < \delta < 1$, a stronger result than (7.18) actually follows from the proof of the Lemma, namely,

$$\hat{\mathcal{E}}_\delta(t) + \mathcal{I} \le \hat{\mathcal{E}}(0) + \mathcal{G}t + \mathcal{K} \int_0^t \hat{\mathcal{E}}(\tau) d\tau \tag{7.19}$$

with

$$\mathcal{I} = \frac{1}{2} \int_0^t \int_0^L \delta \rho_{xx}^2 \,dx\,d\tau \tag{7.20}$$

Proof of Lemma 7.3. The terms in the first grouping on the right-hand side of (7.4) involve functions and/or first derivatives of functions with bounded coefficients; these may be estimated as in the following sample case:

$$\Big| \int_0^t \int_0^L a\rho u_x \,dx\,d\tau \Big| \le \frac{\sup|a|}{2} \int_0^t \int_0^L (\rho^2 + u_x^2) \,dx\,d\tau = K_1 \int_0^t \int_0^L (\rho^2 + u_x^2) \,dx\,d\tau\,. \tag{7.21}$$

The second group of terms involves second derivatives or time derivatives of the functions ρ, u, and T. These can be estimated as follows: for any $\eta > 0$,

$$\Big| \int_0^t \int_0^L Ru_{xx}T_x \,dx\,d\tau \Big| \le \eta R \int_0^t \int_0^L u_{xx}^2 \,dx\,d\tau + \frac{R}{4\eta} \int_0^t \int_0^L T_x^2 \,dx\,d\tau, \tag{7.22a}$$

$$\Big|\int_0^t\int_0^L Ru_{xx}T_x\,dx\,d\tau\Big| \le \eta C_1\int_0^t\int_0^L u_{xx}^2\,dx\,d\tau + K_2(\eta)\int_0^t\int_0^L T_x^2\,dx\,d\tau \quad (7.22b)$$

where $K_2(\eta)$ indicates that the constant K_2 depends on the choice of the parameter η. One further example of this type would be

$$\begin{aligned}\Big|\int_0^t\int_0^L a\rho_t u_x\,dx\,d\tau\Big| &\le \eta\sup|a|\int_0^t\int_0^L \rho_t^2\,dx\,d\tau + \frac{\sup|a|}{4\eta}\int_0^t\int_0^L u_x^2\,dx\,d\tau\\ &= \eta C_2\int_0^t\int_0^L \rho_t^2\,dx\,d\tau + K_4(\eta)\int_0^t\int_0^L u_x^2\,dx\,d\tau\,.\end{aligned} \quad (7.22c)$$

The estimates for the third group of terms are similar to those in the following two examples:

$$\begin{aligned}\Big|\int_0^t\int_0^L c\rho\,dx\,d\tau\Big| &\le \frac{1}{2}\int_0^t\int_0^L \rho^2\,dx\,d\tau + \frac{1}{2}\int_0^t\int_0^L c^2\,dx\,d\tau\\ &\le K_3\int_0^t\int_0^L \rho^2\,dx\,d\tau + G_1\cdot t\,,\end{aligned} \quad (7.23a)$$

$$\begin{aligned}\Big|\int_0^t\int_0^L gu_{xx}\,dx\,d\tau\Big| &\le \eta\int_0^t\int_0^L u_{xx}^2\,dx\,d\tau + \frac{1}{4\eta}\int_0^t\int_0^L g^2\,dx\,d\tau\\ &\le \eta\int_0^t\int_0^L u_{xx}^2\,dx\,d\tau + \frac{\text{const.}}{4\eta}\cdot t \quad (7.23b)\\ &= \eta C_4\int_0^t\int_0^L u_{xx}^2\,dx\,d\tau + G_2(\eta)\cdot t\,.\end{aligned}$$

Finally, the last term on the right-hand side of (7.4); i.e., $\int_0^t\int_0^L \delta\rho_t\rho_{xx}\,dx\,d\tau$ is estimated exactly as in (6.41), which we rewrite here as

$$\Big|\int_0^t\int_0^L \delta\rho_t\rho_{xx}\,dx\,d\tau\Big| \le \int_0^t\int_0^L \frac{\delta}{2}\rho_t^2\,dx\,d\tau + \mathcal{I}\,. \quad (7.24)$$

Now, terms which appear on the right-hand sides of estimates such as (7.22)a,b,c, (7.23)b, et.al., and which are multiplied by η, may be absorbed by the similar terms on the left-hand side of (7.4) because of (6.9)a,b; in particular, for η sufficiently small,

$$\int_0^t\int_0^L (i_c - C\eta)T_{xx}^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}i_cT_{xx}^2\,dx\,d\tau \quad (7.25a)$$

$$\int_0^t\int_0^L (f_c - C\eta)u_{xx}^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}f_cu_{xx}^2\,dx\,d\tau \quad (7.25b)$$

$$\int_0^t\int_0^L (1 - C\eta)T_t^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}T_t^2\,dx\,d\tau \quad (7.25c)$$

$$\int_0^t\int_0^L (1 - C\eta)u_t^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{2}u_t^2\,dx\,d\tau \quad (7.25d)$$

$$\int_0^t\int_0^L \Big(\frac{1}{2} - C\eta\Big)\rho_t^2\,dx\,d\tau \ge \int_0^t\int_0^L \frac{1}{4}\rho_t^2\,dx\,d\tau\,, \quad (7.25e)$$

where $C = \sum C_i$. Note that once η is chosen, the K_i and G_i, $i = 1,2,\dots$ are constants.

In (7.24), the term

$$\mathcal{J} = \frac{1}{2}\delta \int_0^t \int_0^L \rho_t^2 \, dx \, d\tau \tag{7.26}$$

may be absorbed by the similar term on the left-hand side of (7.4) and, as $0 < \delta < 1$, we will have $1 - \frac{\delta}{2} > \frac{1}{2}$. Also, the integral $\mathcal{I}$, on the right-hand side of (7.24) will be absorbed by the term

$$\delta \int_0^t \int_0^L \rho_{xx}^2 \, dx \, d\tau \equiv 2\mathcal{I}$$

on the left-hand side of (7.4) leaving a balance of $\mathcal{I}$ among the terms on the left-hand side of (7.4). After all the above-referenced terms are absorbed (on the left-hand side of (7.4)) we see that as a lower bound for the left-hand side of (7.4) we have the expression

$$\hat{\mathcal{E}_\delta}(t) + \mathcal{I} + \frac{1}{2}\int_0^t b(0,t)\rho_x(0,\tau)^2 d\tau \tag{7.27a}$$

while, adding all the estimates referenced above, yields an upper bound for the remaining terms on the right-hand side of (7.4) of the form

$$\hat{\mathcal{E}}(0) + \mathcal{G}t + \mathcal{K}\int_0^t \hat{\mathcal{E}_\delta}(\tau) d\tau \, , \tag{7.27b}$$

where $\mathcal{K} = \sum K_i$ and $\mathcal{G} = \sum G_i$. Combining the lower bound for the left-hand side of (7.4); i.e., (7.27)a, with the upper bound for the surviving terms on the right-hand side of (7.4), and taking note of the fact that $b(0,t) \geq \zeta > 0$ in (7.27)a, we are led to the estimate (7.19). $\square$

From (7.18) and Gronwall's inequality we may now conclude, as in Lemma 6.6, the following result.

Lemma 7.4. *Under the conditions stated in Lemma 7.2 we have for $\rho, u \in \mathcal{V}_m, T \in \mathcal{W}_m$, $0 < \delta < 1$, and all $t, 0 \leq t \leq t_0$, that there exists $\hat{\mathcal{C}_{t_0}} > 0$ such that*

$$\hat{\mathcal{E}_\delta}(t) \leq \hat{\mathcal{C}_{t_0}}\Big(\frac{\mathcal{G}}{\mathcal{K}} + \hat{\mathcal{E}}(0)\Big) \tag{7.28}$$

To obtain the required existence and uniqueness result for the regularized problem consisting of (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f, we proceed exactly as in §6, namely, (i) we introduce the Galerkin approximations (6.52)a,b,c with coefficients A_{lm}, B_{lm}, C_{lm} satisfying (6.53)a,b,c, where $\zeta_l, \eta_l, \gamma_l$ satisfy (6.54)a,b,c, (ii) we require that the coefficients in the Galerkin approximations satisfy the coupled system of ordinary differential equations (6.55)a,b,c, (iii) we invoke the result of Lemma 6.7 and show that the approximate solutions ρ_m, u_m, T_m defined by (6.52)a,b,c satisfy the hypotheses of Lemma 7.2 and 7.3, and (iv) using the estimate for (ρ_m, u_m, T_m); i.e., on any interval $[0, t_0]$

$$\hat{\mathcal{E}}_{\delta m}(t) \leq \hat{\mathcal{C}}_{t_0}\Big(\frac{\mathcal{G}}{\mathcal{K}} + \hat{\mathcal{E}}_m(0)\Big) \, , \tag{7.29}$$

we conclude that ρ_m, u_m, T_m satisfy (6.68)a,b for $0 \leq t \leq t_0$. The remaining parts of the proof of Theorem 6.8 remain unchanged in the present circumstances and, thus, we are able to conclude, once again, that the regularized initial-boundary value problem (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f has, for each $\delta > 0$, a unique solution $(\rho^\delta, u^\delta, T^\delta)$, for any $t > 0$, such that (6.66)a,b are satisfied; furthermore, the higher regularity result expressed by Theorem 6.9; i.e., (6.81), also holds in the present circumstances in which we have imposed the hypothesis (7.1)a,b.

As the Galerkin approximations ρ_m, u_m, T_m, converge in $W^{1,2}([0,t]; L^2[0,L]) \cap L^2([0,t]; W^{2,2}[0,L])$ to the unique solution of (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f, we have for the limit $\rho^\delta, u^\delta, T^\delta$ of these sequences the estimate

$$\hat{\mathcal{E}}_\delta(t) \leq \hat{\mathcal{C}}_{t_0}\Big(\frac{\mathcal{G}}{\mathcal{K}} + \hat{\mathcal{E}}(0)\Big) \equiv \mathcal{C}'_{t_0} \tag{7.30}$$

for all $\delta > 0$ and $t \in (0, t_0)$. Therefore, the solution set $(\rho^\delta, u^\delta, T^\delta)$ for the problem (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f satisfies the following estimates:

$$\|\rho^\delta\|^2_{L^2([0,t];W^{1,2}([0,L]))} + \|\rho^\delta_{,t}\|^2_{L^2([0,t];L^2([0,L]))} \leq \mathcal{C}_1\,, \tag{7.31a}$$

$$\|u^\delta\|^2_{L^2([0,t];W^{2,2}([0,L]))} + \|u^\delta_{,t}\|^2_{L^2([0,t];L^2([0,L]))} \leq \mathcal{C}_2\,, \tag{7.31b}$$

$$\|T^\delta\|^2_{L^2([0,t];W^{2,2}([0,L]))} + \|T^\delta_{,t}\|^2_{L^2([0,t];L^2([0,L]))} \leq \mathcal{C}_3 \tag{7.31c}$$

for positive generic constants $\mathcal{C}_i$, $i = 1, 2, 3$, which are independent of δ. From (7.31)a,b,c it follows that there exists a triplet (ρ, u, T) with

$$\rho \in L^2([0,t]; W^{1,2}([0,L]), \quad \rho_t \in L^2([0,t]; L^2([0,L]))\,, \tag{7.32a}$$

$$u \in L^2([0,t]; W^{2,2}([0,L])), \quad u_t \in L^2([0,t]; L^2([0,L]))\,, \tag{7.32b}$$

$$T \in L^2([0,t]; W^{2,2}([0,L])), \quad T_t \in L^2([0,t]; L^2([0,L])) \tag{7.32c}$$

and a sequence $(\rho^{\delta_k}, u^{\delta_k}, T^{\delta_k})$ of solutions to the problem (6.10)a,b,c, (6.7)a,b,c, (6.8)a-f such that

$$\rho^{\delta_k} \rightharpoonup \rho \quad \text{weakly in } L^2([0,t]; W^{1,2}([0,L])) \tag{7.33a}$$

$$\rho^{\delta_k}_{,t} \rightharpoonup \rho_t \quad \text{weakly in } L^2([0,t]; L^2([0,L])) \tag{7.33b}$$

$$u^{\delta_k} \rightharpoonup u \quad \text{weakly in } L^2([0,t]; W^{2,2}([0,L])\,, \tag{7.34a}$$

$$u^{\delta_k}_{,t} \rightharpoonup u_t \quad \text{weakly in } L^2([0,t]; L^2([0,L]) \tag{7.34b}$$

and

$$T^{\delta_k} \rightharpoonup T \quad \text{weakly in } L^2([0,t]; W^{2,2}([0,L])\,, \tag{7.35a}$$

$$T^{\delta_k}_{,t} \rightharpoonup T_{,t} \quad \text{weakly in } L^2([0,t]; L^2([0,L])\,. \tag{7.35b}$$

However, the triplet $(\rho^{\delta_k}, u^{\delta_k}, T^{\delta_k})$, satisfies (6.74)a, (6.74)b, (6.74)c, with $\delta = \delta^k$ for any $\psi \in L^2([0,t]; \mathcal{C}^\infty_0([0,L]))$; letting $\delta^k \to 0$ we conclude that

$$(\rho, u, T) = \lim_{\delta^k \to 0} (\rho^{\delta_k}, u^{\delta_k}, T^{\delta_k}) \tag{7.36}$$

satisfies

$$\int_0^t \int_0^L (\rho_t + au_x + b\rho_x - c)\psi \, dx \, d\tau = 0\,, \tag{7.37a}$$

$$\int_0^t \int_0^L (u_t + b_2 u_x + RT_x + d\rho_x - fu_{xx} - g)\psi \, dx \, d\tau = 0\,, \tag{7.37b}$$

$$\int_0^t \int_0^L (T_t + b_3 T_x + hu_x - iT_{xx} - j)\psi \, dx \, d\tau = 0 \tag{7.37c}$$

for all $\psi \in L^2([0,t];\mathcal{C}_0^\infty([0,L]))$. Thus, (ρ,u,T) is a weak solution of the problem (6.6)a,b,c. Furthermore, by virtue of the usual trace theorem and (7.31)a,b,c we can conclude that (ρ,u,T) also satisfies the initial conditions (6.7)a,b,c as well as the boundary conditions (6.8)a-e. The argument delineated above has established the following result.

Theorem 7.5. *Given the hypotheses* (6.8)*a,b,c and* (7.1)*a,b, the mixed hyperbolic-parabolic initial boundary-value problem* (6.6)*a,b,c,* (6.7)*a,b,c,* (6.8)*a-e has a unique solution* (ρ,u,T) *which satisfies* (7.32)*a,b,c.*

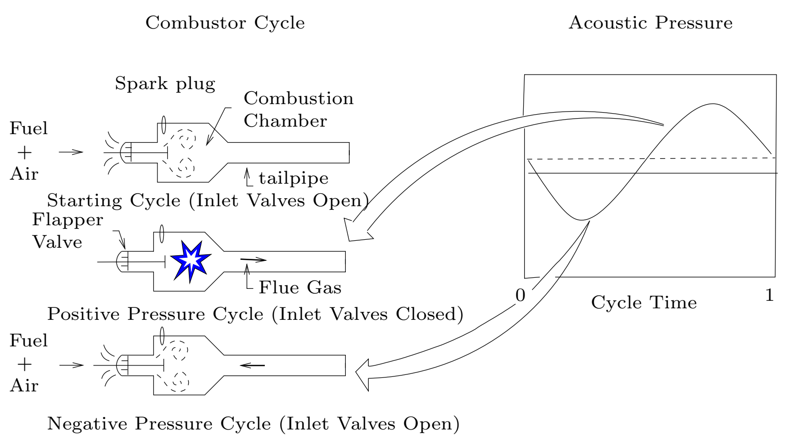

Figure 1. The limit of the integrand does not exist

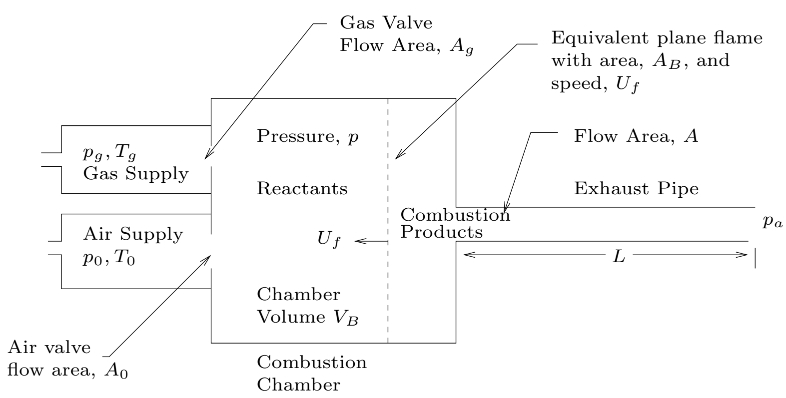

Figure 2. Parameters in the Pulse Combustor Model

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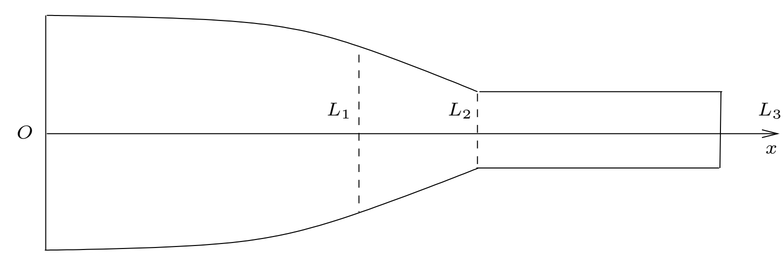

FIGURE 3. Pulse combustor configuration

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