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LOCAL ESTIMATES FOR GRADIENTS OF SOLUTIONS TO ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS

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Abstract. In this article we present local L^∞ estimates for the gradient of solutions to elliptic equations with variable exponents. Under proper conditions on the coefficients, we prove that

$$|\nabla u| \in L^\infty_{loc}$$

for all weak solutions of

$$\operatorname{div}(g(|\nabla u|^2, x)\nabla u) = 0 \quad \text{in } \Omega.$$

1. Introduction

Uhlenbeck [26] obtained the interior Hölder regularity estimates for weak solutions of

$$\operatorname{div}(\rho(|\nabla u|^2)\nabla u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ and $\rho \in C^1([0,\infty))$ is a non-negative function satisfying the ellipticity conditions

$$K^{-1}(\xi + c)^{\frac{p}{2}-1} \le \rho(\xi) + 2\rho'(\xi)\xi \le K(\xi + c)^{\frac{p}{2}-1}, \tag{1.2}$$

$$|\rho'(\xi_1)\xi_1 - \rho'(\xi_2)\xi_2| \le K(\xi_1 + \xi_2 + c)^{p/2-1-\alpha}(\xi_1 - \xi_2)^\alpha \tag{1.3}$$

for $c \ge 0$, $\alpha > 0$ and $p \ge 2$. Especially when $\rho(t) = t^{(p-2)/2}$, (1.1) is reduced to the well-known p-Laplace equation. In this paper we discuss the nonlinear elliptic equation of the form

$$\operatorname{div}(g(|\nabla u|^2, x)\nabla u) = 0 \quad \text{in } \Omega, \tag{1.4}$$

where $g(\xi, x) \in C^1([0,\infty) \times \Omega)$ satisfies the ellipticity conditions

$$C_1(\xi + c)^{\frac{p(x)}{2}-1} \le g(\xi, x) + 2\xi g_\xi(\xi, x) \le C_2(\xi + c)^{\frac{p(x)}{2}-1}, \tag{1.5}$$

$$|\nabla_x g(\xi, x)| \le C_3(\xi + c)^{\frac{p(x)-1}{2}}|\nabla p||\ln(\xi + c)| \tag{1.6}$$

for $c \ge 0$ and $C_1, C_2, C_3 > 0$. Here $p \in W^{1,s}(\Omega)$ for some $s > n$ satisfies

$$1 < p_1 = \inf_{\overline{\Omega}} p(x) \le p(x) \le \sup_{\overline{\Omega}} p(x) = p_2 < \infty. \tag{1.7}$$

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Especially when $g(t) = |t|^{\frac{p(x)-2}{2}}$, (1.4) is reduced to the $p(x)$-Laplace elliptic equation

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0 \quad \text{in } \Omega, \tag{1.8}$$

whose special case is the well-known elliptic p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega,$$

which can be derived from the variational problem

$$\Phi(u) = \min_{v|_{\partial\Omega}=\varphi} \Phi(v) =: \min_{v|_{\partial\Omega}=\varphi} \int_\Omega |\nabla v|^p dx.$$

We denote by $L^{p(x)}(\Omega)$ the variable exponent Lebesgue-Sobolev space

$$L^{p(x)}(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f|^{p(x)} dx < \infty\} \tag{1.9}$$

equipped with the Luxemburg type norm

$$\|f\|_{L^{p(x)}(\Omega)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{f}{\lambda}|^{p(x)} dx \le 1\big\}. \tag{1.10}$$

Furthermore, we define

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\} \tag{1.11}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}. \tag{1.12}$$

By $W_0^{1,p(x)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Actually, the $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ spaces are Banach spaces. There have been many investigations (see for example [9, 10, 11, 12, 16, 17, 19]) on properties of such variable exponent Sobolev spaces.

As usual, the solutions of (1.4) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.1. A function $u \in W_{\rm loc}^{1,p(x)}(\Omega)$ is a local weak solution of (1.4) in Ω if for any $\varphi \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega g(|\nabla u|^2, x)\nabla u \cdot \nabla\varphi dx = 0.$$

When $p(x)$ is a constant, many authors [3, 4, 8, 13, 14, 21, 20, 23] have studied the regularity estimates for weak solutions of quasilinear elliptic equations of p-Laplacian type and the general case. When $p(x)$ is not a constant, such elliptic problems (1.4) appear in mathematical models of various physical phenomena, such as the electro-rheological fluids (see, e.g., [1, 24, 25]). There have been many investigations [7, 15, 22] on Hölder estimates for the $p(x)$-Laplacian elliptic equation (1.4) and the more general case. Moreover, Acerbi and Mingione [2] proved that

$$|\mathbf{f}|^{p(x)} \in L^q_{\rm loc}(\Omega) \quad \Longrightarrow \quad |\nabla u|^{p(x)} \in L^q_{\rm loc}(\Omega) \quad \text{for } q > 1$$

of weak solutions of (1.4) under some assumptions. The purpose of this paper is to extend the results in [6], where Challal and Lyaghfouri obtained the local L^∞ estimates of $|\nabla u|$ for the weak solutions of (1.8).

We assume that $p(x) \in W^{1,s}(\Omega)$ for some $s > n$. Therefore, it follows from Sobolev embedding theorem that $p(x)$ is Hölder continuous with the exponent $\alpha = 1 - \frac{n}{s}$. Now let us state the main result of this work.

Theorem 1.2. *Let $u \in W^{1,p(x)}_{\rm loc}(\Omega)$ be a local weak solution of (1.4) in Ω under the assumptions (1.2)-(1.7). Then*

$$|\nabla u| \in L^\infty_{\rm loc}(\Omega).$$

Moreover, for each $\sigma > 0$ and $\delta \in (0, \frac{sq(1+\sigma)}{s-q} - 1)$ with a constant $q \in (n, s)$ there exists a positive constant R_0, depending only on n, p_1, p_2, s, σ and $\||\nabla u(\cdot)|^{p(\cdot)}\|_{L^1(\Omega)}$, such that, wherever $R \le R_0$ and the ball $B_{8R} \subset \Omega$,

$$\sup_{B_{R/2}} |\nabla u|^{p(x)} \le C\Big[\fint_{B_{2R}} |\nabla u|^{p(x)} dx + R^\alpha M^{(1+\sigma)\delta} \Big(\fint_{B_{8R}} |\nabla u|^{p(x)} + 1 dx\Big)^{1+\sigma}\Big],$$

where $M = \int_{B_{8R}} |\nabla u|^{p(x)} dx + 1$ and C depends on $n, p_1, p_2, s, \delta, \|p\|_{W^{1,s}(\Omega)}$.

2. Proof of main result

In this section we prove Theorem 1.2 by the approximation method. Our approach is much influenced by [2, 5, 6, 27]. We first consider the following approximation problem

$$\operatorname{div}(g(\epsilon + |\nabla u^\epsilon|^2, x)\nabla u^\epsilon) = 0, \quad x \in B_{R'}, \epsilon \in (0, 1], \tag{2.1}$$

where $B_{8R} \subset B_{R'} \subset \Omega$. It is standard that (2.1), with the boundary condition $u^\epsilon = u$ on $\partial B_{R'}$, has a unique solution u^ϵ for fixed $\epsilon > 0$. Similarly to [6], we know $u^\epsilon \in W^{2,2}_{\rm loc}(\Omega)$. From [15] we can get $u \in C^{1,\mu}_{\rm loc}(\Omega)$ for some $\mu \in (0, 1)$ and then have $u^\epsilon \in C^{1,\nu}(\overline{B}_{R'})$ for some $\nu \in (0, 1)$ and $\|u^\epsilon\|_{C^{1,\nu}(\overline{B}_{R'})} \le C$, where C is a constant independent of ϵ. It follows from Ascoli-Arzelà theorem that there exists a sequence of $\{\epsilon_k\}$ converging to 0 and satisfying $u^{\epsilon_k} \to u$ uniformly in $C^1(\overline{B}_{R'})$. Thus, we can get the result of Theorem 1.2 by passing to the limit as $\epsilon_k \to 0$ in (2.9) with u^{ϵ_k} replacing u. So it is sufficient to prove (2.9). For simplicity, we shall drop the index ϵ on u^ϵ in the exposition. Actually, from (2.1) we have

$$\begin{aligned} &\big[g(\epsilon + |\nabla u|^2, x)\delta_{ij} + 2g_\xi(\epsilon + |\nabla u|^2, x)u_i u_j\big]u_{ij} + g_{x_i}(\epsilon + |\nabla u|^2, x)u_i \\ &=: a_{ij}u_{ij} + b_i u_i = 0. \end{aligned} \tag{2.2}$$

Lemma 2.1. *If $g(\xi, x) \in C^1([0, \infty) \times \Omega)$ satisfies the conditions (1.2) and (1.6), then*

$$C_4(\xi + c)^{\frac{p(x)}{2} - 1} \le g(\xi, x) \le C_5(\xi + c)^{\frac{p(x)}{2} - 1}, \tag{2.3}$$

$$|g_\xi(\xi, x)\xi| \le C_6(\xi + c)^{\frac{p(x)}{2} - 1} \tag{2.4}$$

for the constants $0 < C_4 < C_1$, $C_5 > C_2 > 0, C_6 > 0$, and

$$C_4\Big(c + \epsilon + |\nabla u|^2\Big)^{\frac{p(x)}{2} - 1}|\xi|^2 \le a_{ij}\xi_i\xi_j \le C_5\Big(c + \epsilon + |\nabla u|^2\Big)^{\frac{p(x)}{2} - 1}|\xi|^2. \tag{2.5}$$

Proof. We prove only (1.3). First, we find that

$$\xi^{1/2} g(\xi, x) = \int_0^\xi (t^{1/2} g(t, x))_t dt = \int_0^\xi \frac{1}{2} t^{-1/2}\big[g(t, x) + 2t g_t(t, x)\big]dt.$$

Moreover, from (1.2) we deduce that

$$I_1 =: \frac{C_1}{2}\int_0^\xi t^{-1/2}(t + c)^{\frac{p(x)}{2} - 1} dt \le \xi^{1/2} g(\xi, x) \le \frac{C_2}{2}\int_0^\xi t^{-1/2}(t + c)^{\frac{p(x)}{2} - 1} dt =: I_2.$$

To estimate of I_1 and I_2, we consider two cases.

Case 1: $c \le \xi$**.** We have

$$\begin{aligned} I_1 &\ge \frac{C_1}{2}\int_0^{\xi}(t+c)^{\frac{p(x)-3}{2}}dt \\ &\ge \frac{C_1}{2}\big(\frac{1}{\frac{p(x)-1}{2}}\big)[(\xi+c)^{\frac{p(x)-1}{2}} - c^{\frac{p(x)-1}{2}}] \\ &\ge \frac{C_1}{p(x)-1}[(\xi+c)^{\frac{p(x)-1}{2}} - c^{\frac{p(x)-1}{2}}]. \end{aligned}$$

Since $1 < p_1 \le p(x) \le p_2$ and $c \le \frac{c+\xi}{2}$, we obtain

$$\begin{aligned} I_1 &\ge \frac{C_1}{p_2-1}\big[1-(\frac{1}{2})^{\frac{p(x)-1}{2}}\big](\xi+c)^{\frac{p(x)-1}{2}} \\ &\ge C_1'(\xi+c)^{\frac{p(x)-1}{2}} \\ &\ge C_1'(\xi+c)^{\frac{p(x)}{2}-1}\xi^{1/2}. \end{aligned}$$

Moreover, we deduce that

$$I_2 \le \frac{C_2}{2}(\xi+c)^{\frac{p(x)}{2}-1}\int_0^{\xi} t^{-1/2}dt = C_2(\xi+c)^{\frac{p(x)}{2}-1}\xi^{1/2} \quad \text{for } p(x) \ge 2$$

and

$$\begin{aligned} I_2 &\le \frac{C_2}{2}\int_0^{\xi} t^{\frac{p(x)-1}{2}-1}dt \\ &= \frac{C_2}{p(x)-1}\xi^{\frac{p(x)-1}{2}} \\ &\le \frac{C_2}{p_1-1}(\xi+c)^{\frac{p(x)-1}{2}} \\ &= \frac{C_2}{p_1-1}(\xi+c)^{\frac{p(x)}{2}-1}(\xi+c)^{1/2} \quad \text{for } 1 < p(x) < 2, \end{aligned}$$

which implies

$$I_2 \le \frac{C_2}{p_1-1}(\xi+c)^{\frac{p(x)}{2}-1}(2\xi)^{1/2} = \frac{\sqrt{2}C_2}{p_1-1}(\xi+c)^{\frac{p(x)}{2}-1}\xi^{1/2}$$

in view of the fact that $\xi + c \le 2\xi$.

Case 2: $c \ge \xi$**.** Then we have

$$I_1 \ge \frac{C_1}{2}\xi^{-1/2}(\xi+c)^{-1/2}\int_0^{\xi}(t+c)^{\frac{p(x)-1}{2}}dt.$$

Furthermore,

$$I_1 \ge \frac{C_1}{2}\xi^{-1/2}(\xi+c)^{\frac{p(x)}{2}-1}(\frac{1}{2})^{\frac{p(x)-1}{2}}\xi \ge C_1''(\xi+c)^{\frac{p(x)}{2}-1}\xi^{1/2}$$

since

$$t+c \ge c \ge \frac{1}{2}(2c) \ge \frac{1}{2}(\xi+c).$$

Since the result (1.3) is trivial when $c = 0$. Without loss of generality we may as well assume that $c > 0$. Moreover, we first have

$$I_2 \le \frac{C_2}{2}c^{-1/2}\int_0^{\xi} t^{-1/2}(t+c)^{\frac{p(x)-1}{2}}dt$$

$$\le \frac{C_2}{2} c^{-1/2}(\xi+c)^{\frac{p(x)-1}{2}} \int_0^{\xi} t^{-1/2} dt,$$

which implies

$$\begin{aligned} I_2 &\le C_2 c^{-1/2}(\xi+c)^{\frac{p(x)-1}{2}} \xi^{1/2} \\ &= C_2 (\frac{\xi+c}{c})^{1/2} (\xi+c)^{\frac{p(x)}{2}-1} \xi^{1/2} \\ &\le \sqrt{2} C_2 (\xi+c)^{\frac{p(x)}{2}-1} \xi^{1/2}. \end{aligned}$$

Thus, from Cases 1 and 2 we have

$$g(\xi,x) \ge \min\{C_1', C_1''\}(\xi+c)^{\frac{p(x)}{2}-1} =: C_4(\xi+c)^{\frac{p(x)}{2}-1}.$$

and

$$g(\xi,x) \le \max\{\frac{\sqrt{2}C_2}{p_1-1}, \sqrt{2}C_2\}(\xi+c)^{\frac{p(x)}{2}-1} =: C_5(\xi+c)^{\frac{p(x)}{2}-1},$$

which completes the proof. □

Now we denote

$$\widetilde{a_{ij}} = \frac{a_{ij}}{(c+\epsilon+|\nabla u|^2)^{p(x)/2-1}}. \tag{2.6}$$

Then, from the lemma above we have

$$C_4|\xi|^2 \le \widetilde{a_{ij}}\xi_i\xi_j \le C_5|\xi|^2 \quad \text{for each } \xi \in \mathbb{R}^n. \tag{2.7}$$

Lemma 2.2. *Let* $v = (c+\epsilon+|\nabla u|^2)^{p(x)/2}$. *Then*

$$\begin{aligned} \operatorname{div}(\frac{1}{p(x)}\widetilde{a_{ij}} \cdot \nabla v) &\ge \operatorname{div}\Big(a_{ij} \cdot (c+\epsilon+|\nabla u|^2) \cdot \ln(c+\epsilon+|\nabla u|^2)^{1/2} \cdot \frac{\nabla p(x)}{p(x)}\Big) \\ &=: \operatorname{div} F, \end{aligned}$$

where

$$|F| \le C\Big[1+(c+\epsilon+|\nabla u|^2)^{\frac{p(x)(1+\sigma)}{2}}\Big] \cdot |\nabla p| \quad \textit{for any } \sigma > 0.$$

Proof. We first find that

$$\begin{aligned} &v_{x_j} \\ &= \Big((c+\epsilon+|\nabla u|^2)\ln(c+\epsilon+|\nabla u|^2)^{1/2} p_{x_j}(x) + p(x)u_{kj}u_k\Big)(c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}-1} \end{aligned}$$

and

$$\begin{aligned} &\operatorname{div}(\frac{1}{p(x)}\widetilde{a_{ij}} \cdot \nabla v) \\ &= \operatorname{div}(a_{ij} \cdot (c+\epsilon+|\nabla u|^2) \cdot \ln(c+\epsilon+|\nabla u|^2)^{1/2} \cdot \frac{\nabla p(x)}{p(x)}) + (a_{ij}u_{kj}u_k)_{x_i}. \end{aligned}$$

Moreover, differentiating (2.1) with respect to x_k, we have

$$(a_{ij}u_{kj})_{x_i} + (b_k u_i)_{x_i} = 0, \tag{2.8}$$

where b_k is defined in (2.2). Therefore, from (2.8) and Lemma 2.1 we have

$$\begin{aligned} &\operatorname{div}\Big(\frac{1}{p(x)}\widetilde{a_{ij}} \cdot \nabla v\Big) \\ &= \operatorname{div}\Big(a_{ij} \cdot (c+\epsilon+|\nabla u|^2) \cdot \ln(c+\epsilon+|\nabla u|^2)^{1/2} \cdot \frac{\nabla p(x)}{p(x)}\Big) + a_{ij}u_{kj}u_{ki} - (b_k u_i)_{x_i} \end{aligned}$$

$$
\begin{aligned}
&\geq \operatorname{div}\Big(a_{ij}\cdot(c+\epsilon+|\nabla u|^2)\cdot\ln(c+\epsilon+|\nabla u|^2)^{1/2}\cdot\frac{\nabla p(x)}{p(x)}\Big)-(b_k u_i)_{x_i}\\
&=: \operatorname{div} F,
\end{aligned}
$$

where

$$
F = a_{ij}\cdot\Big(c+\epsilon+|\nabla u|^2\Big)\cdot\ln\Big(c+\epsilon+|\nabla u|^2\Big)^{1/2}\cdot\frac{\nabla p(x)}{p(x)} - b_k u_i
$$

Actually, for any $\sigma>0$, we deduce that

$$
\begin{aligned}
|F| &\leq C\big\{|a_{ij}|(c+\epsilon+|\nabla u|^2)|\ln(c+\epsilon+|\nabla u|^2)^{1/2}||\nabla p| + |b_k||\nabla u|\big\}\\
&\leq C\big(c+\epsilon+|\nabla u|^2\big)^{\frac{p(x)}{2}}|\ln(c+\epsilon+|\nabla u|^2)^{1/2}||\nabla p|\\
&\leq C\big(c+\epsilon+|\nabla u|^2\big)^{\frac{p(x)}{2}}|\ln(c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}}||\nabla p|\\
&\leq C\big[1+(c+\epsilon+|\nabla u|^2)^{\frac{p(x)(1+\sigma)}{2}}\big]|\nabla p|.
\end{aligned}
$$

By (1.6), Lemma 2.1 and

$$
|x||\ln x| \leq C(1+|x|^{1+\sigma}) \quad \text{for any } \sigma>0,
$$

we complete the proof. □

Next, we shall finish the proof of the main result.

Proof. Using [18, Theorem 8.17], we obtain

$$
\sup_{B_{R/2}}\Big(c+\epsilon+|\nabla u|^2\Big)^{p(x)/2} \leq C\Big(\frac{1}{R^n}\int_{B_{2R}}(c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}}dx + K(R)\Big),
$$

where

$$
K(R) = R^{1-\frac{n}{q}}\Big(\int_{B_{2R}}|F|^q dx\Big)^{1/q}
$$

and $q\in(n,s)$ is a positive constant. Moreover, we find that

$$
\int_{B_{2R}}|F|^q dx \leq C\big\{\int_{B_{2R}}|\nabla p|^q dx + \int_{B_{2R}}(c+\epsilon+|\nabla u|^2)^{\frac{p(x)(\sigma+1)q}{2}}|\nabla p|^q dx\big\}
$$

Furthermore, using Hölder's inequality and $p\in W^{1,s}$, we obtain

$$
\begin{aligned}
&\int_{B_{2R}}|F|^q dx\\
&\leq C\Big(\int_{B_{2R}}|\nabla p|^s dx\Big)^{q/s}\Big\{\Big(\int_{B_{2R}}(c+\epsilon+|\nabla u|^2)^{\frac{p(x)sq(1+\sigma)}{2(s-q)}}dx\Big)^{\frac{s-q}{s}} + R^{\frac{n(s-q)}{s}}\Big\}\\
&\leq C\Big\{\Big(\int_{B_{2R}}(c+\epsilon+|\nabla u|^2)^{\frac{p(x)sq(1+\sigma)}{2(s-q)}}dx\Big)^{\frac{s-q}{s}} + R^{\frac{n(s-q)}{s}}\Big\}\\
&\leq CR^{\frac{n(s-q)}{s}}\Big[\fint_{B_{2R}}(c+\epsilon+|\nabla u|^2)^{\frac{p(x)sq(1+\sigma)}{2(s-q)}}+1dx\Big]^{\frac{s-q}{s}}.
\end{aligned}
$$

From [2, Theorem 2], for any $\delta\in(0,\frac{sq(1+\sigma)}{s-q}-1)$ there exists a positive constant R_0, depending only on $n,p_1,p_2,s,\delta,\sigma$ and $\|\,|\nabla u(\cdot)|^{p(\cdot)}\|_{L^1(\Omega)}$, such that, wherever $R\leq R_0$,

$$
\int_{B_{2R}}|F|^q dx \leq CR^{\frac{n(s-q)}{s}}M^{q(1+\sigma)\delta}\Big(\fint_{B_{8R}}(c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}}+1dx\Big)^{q(1+\sigma)},
$$

where
$$M = \int_{B_{8R}} |\nabla u|^{p(x)} dx + 1.$$
Thus,
$$K(R) \le CR^\alpha M^{(1+\sigma)\delta} \Big(\fint_{B_{8R}} (c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}} + 1 dx \Big)^{1+\sigma},$$
where the exponent $\alpha = 1 - \frac{n}{s}$. Finally, we conclude that
$$\begin{aligned} &\sup_{B_{R/2}} (c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}} \\ &\le C \Big[\fint_{B_{2R}} (c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}} dx \\ &\quad + M^{(1+\sigma)\delta} R^\alpha \Big(\fint_{B_{8R}} (c+\epsilon+|\nabla u|^2)^{\frac{p(x)}{2}} + 1 dx \Big)^{1+\sigma} \Big], \end{aligned} \tag{2.9}$$
which completes our proof. □

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References

[1] E. Acerbi, G. Mingione; *Regularity results for a stationary electro-rheologicaluids*, Arch. Ration. Mech. Anal., 164(3)(2002), 213-259.
[2] E. Acerbi, G. Mingione; *Gradient estimates for the $p(x)$-Laplacean system*, J. reine angew. Math., 584(2005), 117-148.
[3] S. Byun, L. Wang; *Quasilinear elliptic equations with BMO coefficients in Lipschitz domains*, Trans. Amer. Math. Soc., 359(12)(2007), 5899-5913.
[4] S. Byun, F. Yao, S. Zhou; *Gradient estimates in Orlicz space for nonlinear elliptic equations*, J. Funct. Anal., 255(8)(2008), 1851-1873.
[5] S. Challal, A. Lyaghfouri; *Second order regularity for the $p(x)$-Laplace operator*, Math. Nachr., 284(10)(2011), 1270-1279.
[6] S. Challal, A. Lyaghfouri; *Gradient estimates for $p(x)$-harmonic functions*, Manuscripta Math., 131(3-4)(2010), 403-414.
[7] A. Coscia, G. Mingione; *Hölder continuity of the gradient of $p(x)$-harmonic mappings*, C. R. Acad. Sci. Paris Math., 328(4)(1999), 363-368.
[8] E. DiBenedetto, J. Manfredi; *On the higer integrability of the gradient of weak solutions of certain degenerate elliptic systems*, Amer. J. Math., 115(1993), 1107-1134.
[9] L. Diening; *Riesz potential and Sobolev embeddings of generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nach., 268(1)(2004), 31-43.
[10] L. Diening, M. Růžička; *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math., 563 (2003), 197-220.
[11] L. Diening, M. Růžička; *Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, part I*, J. Math. Anal. . Appl., 298(2)(2004), 559-571.
[12] L. Diening, M. Růžička; *Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, part II*, J. Math. Anal. . Appl., 298(2)(2004), 572-588.
[13] F. Duzaar, G. Mingione; *Gradient continuity estimates*, Calc. Var. Partial Differential Equations, 39(3-4) (2010), 379-418.
[14] F. Duzaar, G. Mingione; *Gradient estimates via non-linear potentials*, Amer. J. Math., 133(4) (2011), 1093-1149.
[15] X. Fan; *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations, 235 (2007), 397-417.
[16] X. Fan, J. Shen, D. Zhao; *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., 262(2001), 749-760.

[17] X. Fan, D. Zhao; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., 263(2001), 424-446.
[18] D. Gilbarg, N. Trudinger; *Elliptic Partial Diferential Equations of Second Order (3rd edition)*, Springer Verlag, Berlin, 1998.
[19] P. Harjulehto; *Variable exponent Sobolev spaces with zero boundary values*, Math. Bohem., 132(2007), 125-136.
[20] T. Iwaniec; *Projections onto gradient fields and L^p-estimates for degenerated elliptic operators*, Studia Math., 75(1983), 293-312.
[21] J. Kinnunen, S. Zhou; *A local estimate for nonlinear equations with discontinuous coefficients*, Comm. Partial Differential Equations, 24(1999), 2043-2068.
[22] A. Lyaghfouri; *Hölder continuity of $p(x)$-superharmonic functions*, Nonlinear Anal., 73(8)(2010), 2433-2444.
[23] G. Mingione; *Gradient estimates below the duality exponent*, Math. Ann., 346(3)(2010), 571-627.
[24] K. R. Rajagopal, M. Růžička; *Mathematical modeling of electro-rheological materials*, Contin. Mech. Thermodyn., 13(1)(2001), 59-78.
[25] M. Růžička; *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math., vol. 1748, Springer, Berlin, 2000.
[26] K. Uhlenbeck; *Regularity for a class of non-linear elliptic systems*, Acta Math., 138(1977), 219-240.
[27] L. Wang; *Compactness methods for certain degenerate elliptic equations*, J. Differential Equations, 107(2)(1994), 341-350.

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