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EXISTENCE AND STABILITY OF SOLUTIONS TO NEUTRAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this article, by using a fixed point theorem, we study the existence and regularity of mild solutions for a class of abstract neutral functional differential equations with infinite delay. The fraction power theory and α-norm is used to discuss the problem so that the obtained results can be applied to equations with terms involving spatial derivatives. A stability result for the autonomous case is also established. We conclude with an example that illustrates the applications of the results obtained.

1. Introduction

In this article, we study the existence, regularity and stability of mild solutions for the following abstract neutral functional evolution equation with infinite delay:

$$\begin{gathered}\frac{d}{dt}[x(t)+F(t,x_t)]+Ax(t)=G(t,x_t),\quad 0\le t\le a,\\ x_0=\phi\in\mathscr{B}_\alpha.\end{gathered}\tag{1.1}$$

where $x(\cdot)$ takes values in a subspace of Banach space X, the operator $-A : D(A) \to X$ generates an analytic semigroup $(S(t))_{t\geq 0}$, and $F, G : [0, a] \times \mathscr{B}_\alpha \to X$ are appropriate functions, $\mathscr{B}_\alpha \subset \mathscr{B}$, and $\mathscr{B}$ is the phase space to be specified later.

Since many practical functional differential models can be studied by rewritten to abstract equation (1.1), in these years there has been an increasing interest in the study of semilinear evolution equations of form (1.1), such as existence and asymptotic behavior of solutions (mild solutions, strong solutions and classical solutions), and existence of (almost) periodic solutions, etc. Here we mention the work of Travis and Webb [25], Rankin III [21], Bátkai1 and Piazzera [4] for the case of finite delay, and Henríquez [12], Adimy et al [1]-[3], Liu [19], Diagana and Hernández[6], Hernández et al [15, 16, 17] and Ye [28] for the case of infinite delay. In [13] and [14] Hernández and Henríquez have extended the problem studied in [12] to neutral equations and established the corresponding existence results of solutions and periodic solutions. In their work, the operator A generates an analytic semigroup so that the theory of fractional power has been used effectively there to obtain the existence of mild solutions, strong solutions and periodic solutions

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for (1.1). In the subsequent years, various similar results have been established by many mathematicians. In paper [1]-[3] Adimy et al have discussed this topic for the equations where the linear parts are non-densely defined and have also achieved some similar results. Particularly, in paper [2] the authors have discussed the following functional differential system with infinite delay:

$$\begin{gathered}\frac{d}{dt}[x(t)+F(t,x_t)] = A[x(t)+F(t,x_t)]+G(t,x_t), \quad 0\le t\le a,\\ x_0=\phi\in\mathscr{B},\end{gathered} \tag{1.2}$$

where A is non-densely defined Hille-Yosida operator and generates an integrated semigroup. The authors have proved there the existence, uniqueness and the regularity of integral solutions, and have investigated the stability near an equilibrium associated to the autonomous case of (1.2).

The purpose of this article is to extend the work in [13] and [2] so that the corresponding results can be applied to the system

$$\begin{gathered}\frac{\partial}{\partial t}[u(t,x)+f(t,u(\cdot,x),\frac{\partial}{\partial x}u(\cdot,x))]+\frac{\partial^2}{\partial x^2}u(t,x) = g(t,u(\cdot,x),\frac{\partial}{\partial x}u(\cdot,x)),\\ z(t)=z(t,\pi)=0,\\ z(\theta,x)=\phi(\theta,x),\theta\le 0, \quad 0\le x\le\pi.\end{gathered} \tag{1.3}$$

Evidently, this system can be treated as the abstract equation (1.1), however, the results established in [13] become invalid for this situation, since the functions f, g in (1.3) involve spatial derivatives. As one will see in Section 5, if take $X = L^2([0,\pi])$, then the third variables of f and g are defined on $X_{\frac{1}{2}}$ and so the solutions can not be discussed on X like in [13]. In this paper, inspired by the work in [26],[27] and [8], we shall discuss this problem by using fractional power operators theory and $\alpha-$norm, that is, we shall restrict this equation in a Banach space $X_\alpha(\subset X)$ and investigate the existence and regularity of mild solutions for (1.1), as well as the stability for the autonomous equation via $\|\cdot\|_\alpha$. We mention here that, for the regularity of mild solutions, other than paper [13], we obtain the existence of strict solutions (not strong solutions) for Eq. (1.1) under Hölder continuous conditions, see Section 3.2.

This article is organized as follows: we firstly introduce some preliminaries about analytic semigroup and phase space for infinite delay in Section 2, particularly, to make them to be still valid in our situation, we have restated the axioms of phase space on the space X_α. The existence and uniqueness results of mild solutions are discussed in Section 3 by applying fixed point theorem. In this section we also provide some sufficient conditions to guarantee the regularity of mild solutions, that is, we obtain the existence of strict solutions. In section 4, we are concerned with the stability of mild solutions. As in [2], we state in this part some properties of the solution operator associated to the autonomous case of (1.1). Also, we investigate here the stability near an equilibrium for this situation by using linearized technique. Finally, an example is presented in Section 5 to show the applications of the results obtained.

2. Preliminaries

Throughout this paper X is a Banach space with norm $\|\cdot\|$. And, $-A : D(-A) \to X$ is the infinitesimal generator of a compact analytic semigroup $(S(t))_{t\geq 0}$ of uniformly bounded linear operators. Let $0 \in \rho(A)$. Then it is possible to define the fractional power A^α , for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Hereafter we denote by X_α the Banach space $D(A^\alpha)$ normed with $\|x\|_\alpha$. Then for each $\alpha > 0$, X_α is a Banach space, and $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta < \alpha$ and the imbedding is compact whenever the resolvent operator of A is compact.

For the semigroup $(S(t))_{t\geq 0}$, the following properties will be used:

(a) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that
$$\|S(t)\| \leq Me^{\omega t}, \quad \text{for all } t \geq 0; \tag{2.1}$$
(b) For any $\alpha > 0$, there exists a constant $C_\alpha > 0$ such that
$$\|A^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha} e^{\omega t}, \quad t > 0. \tag{2.2}$$
(c) For every $\alpha > 0$, there exists a constant $C'_\alpha > 0$ such that
$$\|(S(t) - I)A^{-\alpha}\| \leq C'_\alpha t^\alpha, \quad 0 < t \leq a. \tag{2.3}$$

In the sequel, we will use directly the estimates $\|S(t)\| \leq M$ and $\|A^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha}$ on finite intervals. For more details about the theory of operator semigroups and fraction powers of operators, we refer to [7] and [22].

To study (1.1), we assume that the histories $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belong to some abstract phase space $\mathscr{B}$, which is defined axiomatically. In this article, we employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato [10] and follow the terminology used in [18]. Thus, $\mathscr{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$. We assume that $\mathscr{B}$ satisfies the following axioms:

(A1) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma + a)$ the following statements hold:
  (i) x_t is in $\mathscr{B}$;
  (ii) $\|x(t)\| \leq H\|x_t\|_{\mathscr{B}}$;
  (iii) $\|x_t\|_{\mathscr{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathscr{B}_\alpha}$. Here $H \geq 0$ is a constant, $K, M : [0, +\infty) \to [0, +\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and H, $K(\cdot)$, $M(\cdot)$ are independent of $x(t)$.

(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathscr{B}$-valued continuous function on $[\sigma, \sigma + a]$.

(B1) The space $\mathscr{B}$ is complete.

We denote by $\mathscr{B}_\alpha$ the set of all the elements in $\mathscr{B}$ that take values in space X_α; that is,

$$\mathscr{B}_\alpha := \{\phi \in \mathscr{B} : \phi(\theta) \in X_\alpha \text{ for all } \theta \leq 0\}.$$

Then$\mathscr{B}_\alpha$ becomes a subspace of $\mathscr{B}$ endowed with the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ which is induced by $\|\cdot\|_{\mathscr{B}}$ through $\|\cdot\|_\alpha$. More precisely, for any $\phi \in \mathscr{B}_\alpha$, the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ is defined by $\|A^\alpha\phi(\theta)\|$, instead of $\|\phi(\theta)\|$. For example, let the phase space

$\mathscr{B} = C_r \times L^p(g : X)$, $r \geq 0$, $1 \leq p < \infty$ (cf. [18]), which consists of all classes of functions $\phi : (\infty, 0] \to X$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\phi(\cdot)\|^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\mathscr{B}$ is defined by

$$\|\phi\|_{\mathscr{B}} = \sup\{\phi(\theta) : -r \leq \theta \leq 0\} + \Big(\int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|^p d\theta\Big)^{1/p}.$$

Then the seminorm in $\mathscr{B}_\alpha$ is defined by

$$\|\phi\|_{\mathscr{B}_\alpha} = \sup\{\|A^\alpha\phi(\theta)\| : -r \leq \theta \leq 0\} + \Big(\int_{-\infty}^{-r} g(\theta)\|A^\alpha\phi(\theta)\|^p d\theta\Big)^{1/p}.$$

See also the space $\mathscr{C}_{g,\frac{1}{2}}$ presented in Section 5. Hence, since X_α is still a Banach space, we will assume that the subspace $\mathscr{B}_\alpha$ also satisfies the following conditions:

- (A1') If $x : (-\infty, \sigma + a) \to X_\alpha$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ (in $\alpha-$norm) and $x_\sigma \in \mathscr{B}_\alpha$, then for every $t \in [\sigma, \sigma + a)$ the followings hold:
  - (i) x_t is in $\mathscr{B}_\alpha$;
  - (ii) $\|x(t)\|_\alpha \leq H\|x_t\|_{\mathscr{B}_\alpha}$;
  - (iii) $\|x_t\|_{\mathscr{B}_\alpha} \leq K(t - \sigma)\sup\{\|x(s)\|_\alpha : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathscr{B}_\alpha}$.

  Here H, $K(\cdot)$ and $M(\cdot)$ are as in $(A)(iii)$ above.
- (A2') For the function $x(\cdot)$ in (A), x_t is a $\mathscr{B}_\alpha$-valued continuous function on $[\sigma, \sigma + a]$.
- (B1') The space $\mathscr{B}_\alpha$ is complete.

Finally we conclude this section by stating the following two theorems, which play an essential role for our proofs in the next section.

Theorem 2.1 ([24]). *Let P be a condensing operator on a Banach space X; i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \leq \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$. If $P(H) \subset H$ for a convex, closed and bounded set H of X, then P has a fixed point in H (where $\alpha(\cdot)$ denotes Kuratowski's measure of non-compactness).*

Theorem 2.2 ([5]). *Let $(V(t))_{t\geq 0}$ be a nonlinear strongly continuous semigroup on subset Ω of a Banach space X. Assume that $x_0 \in \Omega$ is an equilibrium of $(V(t))_{t\geq 0}$ and $V(t)$ is Fréchet-differentiable at x_0 for $t \geq 0$, with $W(t)$ the Fréchet derivative at x_0 of $V(t)$. Then $(W(t))_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on X. Moreover, if the zero equilibrium of $(W(t))_{t\geq 0}$ is exponentially stable, then x_0 is a locally exponentially stable equilibrium of $(U(t))_{t\geq 0}$.*

3. Existence results

We devote this section to study the existence and regularity of mild solutions for (1.1).

3.1. **Existence of mild solutions.** A mild solution of (1.1) is defined as follows.

Definition 3.1. A function $x(\cdot) : (-\infty, b] \to D(A^\alpha)$, $b > 0$, is a mild solution of (1.1), if $x_0 = \phi$, the restriction of $x(\cdot)$ to the interval $[0, b]$ is continuous and for each $0 \leq t \leq b$, the function $AS(t - s)F(s, x_s)$, $s \in [0, t)$ is integrable and the following integral equality is satisfied:

$$\begin{aligned} x(t) = S(t)[\phi(0) + F(0,\phi)] - F(t, x_t) + \int_0^t AS(t-s)F(s, x_s)ds \\ + \int_0^t S(t-s)G(s, x_s)ds, \quad 0 \le t \le b. \end{aligned} \tag{3.1}$$

The last two terms are integrals in sense of Bocher (see [20]).

We now give the basic assumptions for (1.1) in our discussion. Let $\Omega \subset \mathscr{B}_\alpha$ be an open set.

(H1) $F : [0,a] \times \Omega \to D(A^{\alpha+\beta})$ is a continuous function for some $\beta \in (0,1)$ with $\alpha + \beta \le 1$, and there exists $l > 0$ such that the function $A^\beta F$ satisfies:
$$\|A^\beta F(s_1, \phi_1) - A^\beta F(s_2, \phi_2)\|_\alpha \le l(|s_1 - s_2| + \|\phi_1 - \phi_2\|_{\mathscr{B}_\alpha}) \tag{3.2}$$
for any $0 \le s_1, s_2 \le a$, $\phi_1, \phi_2 \in \Omega$, and the inequality
$$M_1 l K(0) < 1 \tag{3.3}$$
holds, where $M_1 := \|A^{-\beta}\|$.

(H2) The function $G : [0,a] \times \Omega \to X$ is continuous.

Theorem 3.2. *Let $\phi \in \Omega$. If assumptions* (H1), (H2) *are satisfied, then* (1.1) *admits at least one mild solution on $(-\infty, b_\phi]$ for some $b_\phi < a$.*

Proof. Let $y(\cdot) : (-\infty, a] \to X_\alpha$ be the function defined by
$$y(t) := \begin{cases} S(t)\phi(0), & t \ge 0, \\ \phi(t), & -\infty < t < 0, \end{cases}$$
then $y_0 = \phi$, $y_t \in \mathscr{B}_\alpha$ for any $t \in [0,a]$, and it is easy to prove that the map $t \to y(t)$ is continuous in $\alpha-$ norm on $[0,a]$, hence $t \to y_t$ is continuous in seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$. We denote $N_1 := \sup\{\|y_t\|_{\mathscr{B}_\alpha} : 0 \le t \le a\}$. Since $A^\beta F(\cdot,\cdot)$ satisfies Lipschitz condition, G is continuous and Ω is open, there exists $r > 0$ such that $B_r(\phi) \subset \Omega$ and $\|A^\beta F(t,\psi)\| \le N_2$ and $\|G(t,\psi)\| \le N_3$ for constants $N_2,\ N_3 \ge 0$ and all $(t,\psi) \in [0,a] \times B_r(\phi)$. In the sequel, we always denote
$$K_t := \sup_{s\in[0,t]} K(s), \quad M_t := \sup_{s\in[0,t]} M(s).$$
As $y_0 = \phi$, we may choose $0 < b_1 < a$ such that $\|y_t - \phi\|_{\mathscr{B}_\alpha} \le r/2$ for all $0 \le t \le b_1$.

Let $\rho = \frac{r}{2K_{b_1}}$, and define the set
$$S(\rho) := \{z \in C([0,b_\phi]; X_\alpha) : z(0) = 0,\ \|z(t)\|_\alpha \le \rho,\ 0 \le t \le b_\phi\},$$
where b_ϕ $(< b_1)$ will be determined below. Then $S(\rho)$ is clearly a non-empty bounded, closed and convex subset of $C([0,b_\phi]; X_\alpha)$. For each $z \in S(\rho)$, we denote by $\bar z$ the function defined by
$$\bar z(t) := \begin{cases} z(t), & 0 \le t \le b_\phi, \\ 0, & -\infty < t < 0. \end{cases}$$
Obviously, if $x(\cdot)$ satisfies (3.1), we can decompose it as $x(t) = z(t) + y(t)$, $0 \le t \le b_\phi$, which implies $x_t = \bar z_t + y_t$ for every $0 \le t \le b_\phi$ and the function $z(\cdot)$ satisfies
$$z(t) = S(t)F(0,\phi) - F(t, \bar z_t + y_t) + \int_0^t AS(t-s)F(s, \bar z_s + y_s)ds$$

$$+\int_0^t S(t-s)G(s,\bar{z}_s+y_s)ds, \quad 0\le t\le b_\phi.$$

Let P, P_1, P_2 be the operators on $S(\rho)$ defined, respectively, by

$$\begin{aligned}(Pz)(t) &:= S(t)F(0,\phi)-F(t,\bar{z}_t+y_t)+\int_0^t AS(t-s)F(s,\bar{z}_s+y_s)ds\\ &\quad +\int_0^t S(t-s)G(s,\bar{z}_s+y_s)ds,\\ (P_1z)(t) &:= S(t)F(0,\phi)-F(t,\bar{z}_t+y_t)+\int_0^t AS(t-s)F(s,\bar{z}_s+y_s)ds\end{aligned}$$

and

$$(P_2z)(t) := \int_0^t S(t-s)G(s,\bar{z}_s+y_s)ds.$$

Then, the assertion that (1.1) admits a mild solution is equivalent to $P=P_1+P_2$ has a fixed point. Next we prove that P has a fixed point by using Theorem 2.1. For this purpose, we will show that P maps $S(\rho)$ into itself and P_1 verifies a contraction condition while P_2 is a completely continuous operator.

Initially, we see that if $z(t)\in S(\rho)$, then $\bar{z}_t+y_t\in B_r(\phi)$ for all $0\le t\le b_\phi$. In fact, Axiom (A1') of the phase space $\mathscr{B}_\alpha$ yields that

$$\begin{aligned}\|\bar{z}_t+y_t-\phi\|_{\mathscr{B}_\alpha} &\le \|\bar{z}_t\|_{\mathscr{B}_\alpha}+\|y_t-\phi\|_{\mathscr{B}_\alpha}\\ &\le K(t)\sup_{0\le s\le t}\|z(s)\|_\alpha+\|y_t-\phi\|_{\mathscr{B}_\alpha}\\ &\le K_{b_\phi}\rho+\frac{r}{2}\le r.\end{aligned}$$

To show that P maps $S(\rho)$ into $S(\rho)$, let $z\in S(\rho)$. Then

$$\begin{aligned}(P_1z)(t) &= S(t)A^{-\beta}[A^\beta F(0,\phi)-A^\beta F(t,y_t)]\\ &\quad +(S(t)-I)F(t,y_t)\\ &\quad +A^{-\beta}[A^\beta F(t,y_t)-A^\beta F(t,\bar{z}_t+y_t)]\\ &\quad +\int_0^t AS(t-s)F(s,\bar{z}_s+y_s)ds,\end{aligned}$$

then from assumption (H1), (2.1) and (2.3) it follows that

$$\begin{aligned}\|(P_1z)(t)\|_\alpha &\le \|S(t)\|\|A^{-\beta}\|\|A^\alpha(A^\beta F(0,\phi)-A^\beta F(t,y_t))\|\\ &\quad +\|A^\alpha(S(t)-I)A^{-\beta}A^\beta F(t,y_t)\|\\ &\quad +\|A^{-\beta}\|\|A^\alpha(A^\beta F(t,y_t)-A^\beta F(t,\bar{z}_t+y_t))\|\\ &\quad +\int_0^t\|A^{1-\beta}S(t-s)\|\|A^\alpha A^\beta F(s,\bar{z}_s+y_s)\|ds\\ &\le MM_1l(t+\|y_t-\phi\|_{\mathscr{B}_\alpha})+M_1C'_\beta N_2t^\beta\\ &\quad +M_1lK_{b_\phi}\rho+C_{1-\beta}N_2\int_0^t\frac{1}{(t-s)^\beta}ds.\end{aligned}$$

for $0\le t\le b_\phi$. And we have also that

$$\|(P_2z)(t)\|_\alpha=\|\int_0^t A^\alpha S(t-s)G(s,\bar{z}_s+y_s)ds\|\le C_\alpha N_3\int_0^t\frac{1}{(t-s)^\alpha}ds.$$

Therefore, by (3.3) we may choose b_ϕ, $0 < b_\phi < b_1$ such that

$$\begin{aligned} &MM_1 l(t + \|y_t - \phi\|_{\mathscr{B}_\alpha}) + M_1 C'_\beta N_2 t^\beta + \frac{C_{1-\beta} N_2}{\beta} t^\beta + \frac{C_\alpha N_3}{1-\alpha} t^{1-\alpha} \\ &\le (1 - M_1 l K_{b_\phi})\rho \end{aligned} \tag{3.4}$$

for all $0 < t \le b_\phi$, and

$$l^* := l K_{b_\phi}(M_1 + C_{1-\beta} \frac{b_\phi^{1-\beta}}{1-\beta}) < 1. \tag{3.5}$$

Hence from (3.4) we obtain that

$$\begin{aligned} \|(Pz)(t)\|_\alpha &\le \|(P_1 z)(t)\|_\alpha + \|(P_2 z)(t)\|_\alpha \\ &\le (1 - M_1 l K_{b_\phi})\rho + M_1 l K_{b_\phi} \rho = \rho, \end{aligned}$$

which shows P maps $S(\rho)$ into itself.

Now we prove that P_1 is a contraction map. Take $z_1, z_2 \in S(\rho)$, then for each $t \in [0, b_\phi]$ and by Axiom (A1)(ii) and (3.2), we have

$$\begin{aligned} &\|(P_1 z_1)(t) - (P_1 z_2)(t)\|_\alpha \\ &\le \|F(t, \bar z_{1,t} + y_t) - F(t, \bar z_{2,t} + y_t)\|_\alpha \\ &+ \|\int_0^t AS(t-s)[F(s, \bar z_{1,s} + y_s) - F(s, \bar z_{2,s} + y_s)] ds\|_\alpha \\ &\le M_1 l \|\bar z_{1,t} - \bar z_{2,t}\|_{\mathscr{B}_\alpha} + \int_0^t \frac{C_{1-\beta}}{(t-s)^\beta} l \|\bar z_{1,s} - \bar z_{2,s}\|_{\mathscr{B}_\alpha} ds \\ &\le l K_{b_\phi}(M_1 + C_{1-\beta} \frac{b_\phi^{1-\beta}}{1-\beta}) \sup_{0 \le s \le b_\phi} \|z_1(s) - z_2(s)\|_\alpha \\ &= l^* \sup_{0 \le s \le b_\phi} \|z_1(s) - z_2(s)\|_\alpha, \end{aligned}$$

where $l^* < 1$ by (3.5). Thus

$$\|P_1 z_1 - P_1 z_2\|_\alpha < l^* \|z_1 - z_2\|_\alpha,$$

and so P_1 is a contraction.

To prove that P_2 is a completely continuous operator, first we note that P_2 is obviously continuous on $S(\rho)$. Then we prove that the family $\{P_2 z : z \in S(\rho)\}$ is a family of equi-continuous functions. To do this, let $0 < t \le b_\phi$, $h > 0$ be sufficient small, then

$$\begin{aligned} &\|(P_2 z)(t+h) - (P_2 z)(t)\|_\alpha \\ &= \|\int_0^{t+h} A^\alpha S(t+h-s) G(s, \bar z_s + y_s) ds - \int_0^t A^\alpha S(t-s) G(s, \bar z_s + y_s) ds\| \\ &\le \int_0^{t-\epsilon} \|A^\alpha (S(t+h-s) - S(t-s))\| \|G(s, \bar z_s + y_s)\| ds \\ &\quad + \int_{t-\epsilon}^t \|A^\alpha (S(t+h-s) - S(t-s))\| \|G(s, \bar z_s + y_s)\| ds \\ &\quad + \int_t^{t+h} \|A^\alpha S(t+h-s)\| \|G(s, \bar z_s + y_s)\| ds. \end{aligned}$$

$$
\begin{aligned}
&\le N_3\|S(h+\epsilon)-S(\epsilon)\|\int_0^{t-\epsilon}\|A^\alpha S(t-s-\epsilon)\|ds\\
&\quad + N_3\int_{t-\epsilon}^{t}\|A^\alpha[S(t+h-s)-S(t-s)]\|ds\\
&\quad + N_3\|\int_t^{t+h}A^\alpha T(t+h-s)ds\|\\
&\le \frac{C_\alpha}{1-\alpha}N_3(t-\epsilon)^{1-\alpha}\|S(h+\epsilon)-S(\epsilon)\|\\
&\quad + \frac{C_\alpha}{1-\alpha}N_3[h^{1-\alpha}-(h-\epsilon)^{1-\alpha}+\epsilon^{1-\alpha}]+\frac{C_\alpha}{1-\alpha}N_3h^{1-\alpha}.
\end{aligned}
$$

The right-hand side tends to zero as $h\to 0$ with ϵ sufficiently small, since $S(t)$ is strongly continuous, and the compactness of $S(t)$, $t>0$, implies the continuity in the uniform operator topology. Hence, P_2 maps $S(\rho)$ into a family of equicontinuous functions.

It remains to prove that $V(t)=\{(P_2z)(t) : z\in S(\rho)\}$ is relatively compact in X_α. Obviously it is true in the case $t=0$. Observe that for $0<\alpha<\alpha_1<1$, $t>0$,

$$
\begin{aligned}
\|A^{\alpha_1}(P_2z)(t)\| &= \Big\|\int_0^t A^{\alpha_1}S(t-s)G(s,\bar z_s+y_s)ds\Big\|\\
&\le C_{\alpha_1}N_3\int_0^t\frac{1}{(t-s)^{\alpha_1}}ds,
\end{aligned}
$$

which implies that $A^{\alpha_1}(P_2z)(t)$ is bounded in X, Hence, by the compactness of operator $A^{-\alpha_1}:X\to X_\alpha$ (note the imbedding $X_{-\alpha_1}\hookrightarrow X_\alpha$ is compact), we infer that the set $V(t)$ is relatively compact in X_α. Thus, by Arzela-Ascoli theorem P_2 is a completely continuous operator. These arguments enable us to conclude that $P=P_1+P_2$ is a condensing map on $S(\rho)$, and by Theorem 2.1 there exists a fixed point $z(\cdot)$ for P on $S(\rho)$, which implies equation (1.1) admits a mild solution on $(-\infty,b_\phi]$. Then the proof is complete. □

We can easily prove the following result on uniqueness of solutions.

Theorem 3.3. *Assume the condition* (H1) *of the preceding theorem holds. If there exists $l'>0$ such that*

$$
\|G(t,\phi_1)-G(t,\phi_2)\|\le l'\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}
$$

for all $0\le t\le a$, and $\phi_1,\phi_2\in\Omega$. Then, for any $\phi\in\Omega$, the problem (1.1) *has a unique mild solution on $(-\infty,b_\phi]$ for some $b_\phi\in(0,a)$.*

The extension of solutions to (1.1) can also be obtained by standard arguments. Here we only state the result as a theorem, the proof is very similar to that in Paper [12] and [26].

Theorem 3.4. *Assume that the conditions of Theorem 3.2 or Theorem 3.3 are satisfied. Then, for any $\phi\in\Omega$, the equation* (1.1) *has a solution $x(t)$ on a maximal interval of existence $(-\infty,b_{\max})$. And, if $b_{\max}<\infty$, then $\overline{\lim}_{t\to b_{\max}^-}\|x(t)\|_\alpha=\infty$.*

3.2. **Existence of strict solutions.** In this subsection, we discuss the regularity of mild solutions for (1.1); that is, we will provide conditions to allow the differentiability of mild solutions of (1.1). For this purpose we need some additional properties of the phase subspace $\mathscr{B}_\alpha$. Let $\mathcal{BC}_\alpha$ be the set of bounded and continuous functions mapping $(-\infty, 0]$ into X_α, and C_{00} its subset consisting of functions with compact support. If $\mathscr{B}_\alpha$ also satisfies the additional axiom:

(C1) If a uniformly bounded sequence $\{\phi^n(\theta)\}$ in C_{00} converges to a function $\phi(\theta)$ uniformly on every compact set on $(-\infty, 0]$, then $\phi \in \mathscr{B}_\alpha$ and
$$\lim_{n\to+\infty} \|\phi^n - \phi\|_{\mathscr{B}_\alpha} = 0.$$

Then $\mathcal{BC}_\alpha$ is continuously imbedded into $\mathscr{B}_\alpha$. Put
$$\|\phi\|_\infty = \sup\{\|\phi(\theta)\|_\alpha : \theta \le 0\},$$
for $\phi \in \mathcal{BC}_\alpha$, then one has the following result.

Lemma 3.5 ([10]). *If the phase space $\mathscr{B}_\alpha$ satisfies the axiom* (C1)*, then $\mathcal{BC}_\alpha \subset \mathscr{B}_\alpha$, and there exists a constant $J > 0$ such that $\|\phi\|_{\mathscr{B}_\alpha} \le J\|\phi\|_\infty$ for all $\phi \in \mathcal{BC}_\alpha$*

Definition 3.6. A function $x(\cdot) : (-\infty, b] \to X_\alpha$, $b > 0$, is said to be a strict solution of problem (1.1), if

(1) $x(t) + F(t, x_t) \in C([0, b]; X_\alpha) \cap C^1((0, b]; X)$;
(2) $x(\cdot) \in D(A)$ satisfies
$$\frac{d}{dt}[x(t) + F(t, x_t)] + Ax(t) = G(t, x_t),$$
on $[0, b]$ and
$$x_0 = \phi \in \mathscr{B}_\alpha.$$

Theorem 3.7. *Let the phase space $\mathscr{B}_\alpha$ satisfies the axiom* (C1) *additionally. Suppose that condition* (H1) *and* (H2) *are satisfied. Also the following conditions hold:*

(H1') *Let $x : (-\infty, a] \to X_\alpha$ be a function such that $x_t \in \Omega$ for $t \in [0, a]$ and $x(\cdot)$ is continuous on $[0, a]$, then the map $t \to A^\beta F(t, x_t)$ is Hölder continuous in $\alpha-$norm with exponent $0 < \theta_1 < 1$ satisfying $\theta_1 > 1 - \alpha - \beta$.*

(H2') *Function $G(\cdot, \cdot)$ is locally Hölder continuous; i.e., for each $(t^0, \phi^0) \in [0, a] \times \Omega$, there exists a neighborhood W of (t^0, ϕ^0), and constants $l_2 > 0$, $0 < \theta_2 < 1$, such that*
$$\|G(s_2, \phi_2) - G(s_1, \phi_1)\| \le l_2[|s_2 - s_1|^{\theta_2} + \|\phi_2 - \phi_1\|^{\theta_2}_{\mathscr{B}_\alpha}]$$
for $(s_i, \phi_i) \in W \subset ([0, a] \times \Omega)$, $i = 1, 2$;

(H3) *The initial function $\phi \in \Omega$ is Hölder continuous, and $\phi(0)+F(0, \phi) \in D(A)$.*

Then the equation (1.1) *has a strict solution on $(-\infty, b_\phi]$ for some $b_\phi > 0$.*

Proof. By Theorem 3.2, we see that (1.1) has a mild solution $x(\cdot)$ on $(-\infty, b_\phi]$. For this $x(\cdot)$, let
$$m(t) = S(t)[\phi(0) + F(0, \phi)],$$
$$p(t) = \int_0^t AS(t - s)F(s, x_s)ds,$$
$$q(t) = \int_0^t S(t - s)G(s, x_s)ds.$$
It follows from (2.1) and (2.3) that
$$\|m(t + h) - m(t)\|_\alpha = \|S(t)(S(h) - I)A^{-(\alpha'-\alpha)}A^{\alpha'}[\phi(0) + F(0, \phi)]\|$$

$$\le MC'_\alpha h^{\alpha'-\alpha}\|\phi(0)+F(0,\phi)\|_{\alpha'},$$

where $\alpha'>0$ is a constant chosen to satisfy $\alpha<\alpha'<\alpha+\beta$,

$$\begin{aligned}
&\|p(t+h)-p(t)\|_\alpha\\
&\le \|\int_0^t AS(t-s)[S(h)-I]F(s,x_s)ds\|_\alpha + \|\int_t^{t+h} AS(t+h-s)F(s,x_s)ds\|_\alpha\\
&\le \|\int_0^t A^{1-(\alpha'-\alpha)}S(t-s)[S(h)-I]A^{-(\alpha+\beta-\alpha')}A^\alpha A^\beta F(s,x_s)ds\|\\
&\quad + \|\int_t^{t+h} A^{1-\beta}S(t+h-s)A^\alpha A^\beta F(s,x_s)ds\|\\
&\le [\int_0^t C_{1-(\alpha'-\alpha)}C'_{\alpha+\beta-\alpha'}(t-s)^{(\alpha'-\alpha)-1}h^{\alpha+\beta-\alpha'}ds\\
&\quad + \int_t^{t+h} C_{1-\beta}(t+h-s)^{\beta-1}ds]\max_{0\le s\le b_\phi}\|A^\beta F(s,x_s)\|_\alpha\\
&\le [\frac{C_{1-(\alpha'-\alpha)}C'_{\alpha+\beta-\alpha'}}{\alpha'-\alpha}t^{\alpha'-\alpha}h^{\alpha+\beta-\alpha'} + \frac{C_{1-\beta}}{\beta}h^\beta]\max_{0\le s\le b_\phi}\|A^\beta F(s,x_s)\|_\alpha,
\end{aligned}$$

and

$$\begin{aligned}
&\|q(t+h)-q(t)\|_\alpha\\
&\le \|\int_0^t S(t-s)[S(h)-I]G(s,x_s)ds\|_\alpha + \|\int_t^{t+h} S(t+h-s)G(s,x_s)ds\|_\alpha\\
&\le \|\int_0^t A^{\alpha'}S(t-s)[S(h)-I]A^{\alpha-\alpha'}G(s,x_s)ds\|\\
&\quad + \|\int_t^{t+h} A^\alpha S(t+h-s)G(s,x_s)ds\|\\
&\le \Big[\int_0^t C_{\alpha'}C'_{\alpha'-\alpha}(t-s)^{-\alpha'}h^{\alpha'-\alpha}ds + \int_t^{t+h} C_\alpha(t+h-s)^{-\alpha}ds\Big]\max_{0\le s\le b_\phi}\|G(s,x_s)\|\\
&\le [\frac{C_{\alpha'}C/-\alpha'-\alpha}{1-\alpha'}t^{1-\alpha'}h^{\alpha'-\alpha} + \frac{C_\alpha}{1-\alpha}h^{1-\alpha}]\max_{0\le s\le b_\phi}\|G(s,x_s)\|,
\end{aligned}$$

from which we see that $m(t)$, $p(t)$ and $q(t)$ are all Hölder continuous on $[0,b_\phi]$. So combined condition (H_1') it is easy to deduce that $x(\cdot)$ is Hölder continuous on $[0,b_\phi]$. Since ϕ is Hölder continuous on $(-\infty,0]$ we infer that $x(\cdot)$ is Hölder continuous on $(-\infty,b_\phi]$. Thus, by Lemma 3.5 the map $t\to x_t(\cdot,\phi)$ is also Hölder continuous on $[0,b_\phi]$. Hence the map

$$s\to G(s,x_s)$$

is Hölder continuous on $[0,b_\phi]$. Therefore, from the proof of [22, Corollary 4.3.3] it is not difficult to see that $q(t)\in D(A)$, and

$$q'(t)=G(t,x_t)-A\int_0^t S(t-s)G(s,x_s)ds.$$

On the other hand, we can also show $p(t)$ has the similar property as $q(t)$. Indeed, let $t \in [0, b_\phi)$ and $h > 0$, then

$$\begin{aligned} &\frac{S(h)p(t) - p(t)}{h} \\ &= \frac{1}{h}[S(h)\int_0^t AS(t-s)F(s,x_s)ds - \int_0^t AS(t-s)F(s,x_s)ds] \\ &= \frac{1}{h}(p(t+h) - p(t)) - \frac{1}{h}\int_t^{t+h} AS(t+h-s)F(s,x_s)ds, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\|\frac{1}{h}\int_t^{t+h} AS(t+h-s)F(s,x_s)ds - AF(t,x_t)\| \\ &\leq \|\frac{1}{h}\int_t^{t+h} AS(t+h-s)[F(s,x_s) - F(t,x_t)]ds\| \\ &\quad + \|\frac{1}{h}\int_t^{t+h} AS(t+h-s)F(t,x_t)ds - AF(t,x_t)\| \to 0, \quad \text{as } h \to 0^+. \end{aligned} \tag{3.7}$$

Let

$$\begin{aligned} p(t) &= p_1(t) + p_2(t) \\ &:= \int_0^t AS(t-s)[F(s,x_s) - F(t,x_t)]ds + \int_0^t AS(t-s)F(t,x_t)ds. \end{aligned}$$

Then $S(t)X \subset \cap_{n=1}^{+\infty} D(A^n)$ and A^α is closed for any $\alpha > 0$ imply that $p_2(t) \in D(A)$. Since

$$Ap_2(t) = \int_\delta^t A^2S(t-s)F(t,x_t)ds + \int_0^\delta A^2S(t-s)F(t,x_t)ds, \tag{3.8}$$

the first term on the right side of (3.8) is clearly continuous and the second term is $O(\delta)$, this means $Ap_2(t)$ is continuous. Set

$$p_{1,\epsilon}(t) := \int_0^{t-\epsilon} AS(t-s)[F(s,x_s) - F(t,x_t)]ds,$$

then condition (H1') yields

$$\begin{aligned} Ap_{1,\epsilon}(t) &= \int_0^{t-\epsilon} A^2S(t-s)[F(s,x_s) - F(t,x_t)]ds \\ &= \int_0^{t-\epsilon} A^{2-\alpha-\beta}S(t-s)A^{\alpha+\beta}[F(s,x_s) - F(t,x_t)]ds \\ &\to \int_0^t A^2S(t-s)[F(s,x_s) - F(t,x_t)]ds, \quad \text{as } \epsilon \to 0. \end{aligned}$$

Hence from the closure of A it follows that $p_1(t) \in D(A)$ and

$$Ap_1(t) = \int_0^t A^2S(t-s)[F(s,x_s) - F(t,x_t)]ds.$$

The continuity of $Ap(t)$ can be shown as that of $Ap_2(t)$. Hence we deduce that $p(t) \in D(A)$ and $Ap(t)$ is continuous. Thus, (3.6) and (3.7) indicate that $p(t)$ is differentiable and

$$p'(t) = Ap(t) + AF(t, x_t).$$

Therefore, $x(t) + F(t, x_t)$ is differentiable in t on $[0, b_\phi]$ and satisfies that

$$\begin{aligned}\frac{d}{dt}[x(t) + F(t, x_t)] &= \frac{d}{dt}S(t)[\phi(0) + F(0, \phi)] + p'(t) + q'(t) \\ &= -A(t)S(t)[\phi(0) + F(0, \phi)] \\ &\quad + A(t)F(t, x_t) - A(t)p(t) + G(t, x_t) - A(t)q(t) \\ &= -A(t)x(t) + G(t, x_t).\end{aligned}$$

This shows that $x(\cdot)$ is a strict solution of the Cauchy problem (1.1). Thus the proof is complete. □

4. Stability of mild solutions

In this section, we study the stability of mild solutions for (1.1) with F and G autonomous; namely, we discuss the stability of the equilibrium of the autonomous equation

$$\begin{gathered}\frac{d}{dt}[x(t) + F(x_t)] + Ax(t) = G(x_t), \quad t\geq 0, \\ x_0 = \phi \in \mathscr{B}_\alpha.\end{gathered} \tag{4.1}$$

In this equation F and G satisfy the following conditions:

(H4) F, G satisfy the Lipschitz condition; i.e,

$$\begin{gathered}\|A^\beta F(\phi_1) - A^\beta F(\phi_2)\|_\alpha \le l_3 \|\phi_1 - \phi_2\|_{\mathscr{B}_\alpha}, \\ \|G(\phi_1) - G(\phi_2)\| \le l_4 \|\phi_1 - \phi_2\|_{\mathscr{B}_\alpha},\end{gathered}$$

for $\phi_1, \phi_2 \in \mathscr{B}_\alpha$.

(H5) There holds

$$K(0)(l_3\|A^{-\beta}\| + l_3 C_{1-\beta}\Gamma(\beta) + l_4 C_\alpha \Gamma(1-\alpha)) < 1, \tag{4.2}$$

where $\Gamma(\cdot)$ is the gamma function satisfying the formula $\int_0^{+\infty} e^{-\beta t} t^{-\alpha} dt = \Gamma(1-\alpha)\beta^{\alpha-1}$, for $0 < \alpha < 1$ and $\beta > 0$.

First we consider the solution semigroup for (4.1). For each $t \ge 0$, define the nonlinear operator semigroup $(U(t))_{t\ge 0}$ as

$$U(t)(\phi) = x_t(\cdot, \phi),$$

where $x_t(\cdot, \phi)$ denotes the unique mild solution of (4.1) through $(0, \phi)$. Then $(U(t))_{t\ge 0}$ is a nonlinear strongly continuous semigroup on $\mathscr{B}_\alpha$; that is,

(i) $U(0) = I$;
(ii) $U(t+s) = U(t)U(s)$ for all $t,\ s \ge 0$;
(iii) For all $\phi \in \mathscr{B}_\alpha$, the map $t \to U(t)(\phi)$ is continuous in $\mathscr{B}_\alpha$.

And it satisfies the translation property

$$(U(t)(\phi))(\theta) = \begin{cases}(U(t+\theta)(\phi))(0), & t+\theta \ge 0, \\ \phi(t+\theta), & -\infty < t+\theta < 0,\end{cases}$$

for $t \ge 0$ and $\theta \in (-\infty, 0]$. Moreover, we have the following result.

Theorem 4.1. *For the nonlinear semigroup $(U(t))_{t\geq 0}$, there exist a $\mu\in\mathbb{R}^+$ and a function $P(\cdot,\mu)\in L^\infty((0,+\infty);\mathbb{R}^+)$ such that, for $\phi_1,\phi_2\in\mathscr{B}_\alpha$,*
$$\|U(t)\phi_1-U(t)\phi_2\|_{\mathscr{B}_\alpha}\leq P(t,\mu)e^{\mu t}\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}.$$

Proof. Let $t_0>0$, $K_{t_0}=\max_{0\leq s\leq t_0}K(s)$, $M_{t_0}=\sup_{0\leq s\leq t_0}M(s)$, and $x^1(\cdot)=x(\cdot,\phi_1)$, $x^2(\cdot)=x(\cdot,\phi_2)$. For $t\in[0,t_0]$, there holds
$$\begin{aligned}
&\|U(t)\phi_1-U(t)\phi_2\|_{\mathscr{B}_\alpha}\\
&=\|x_t^1-x_t^2\|_{\mathscr{B}_\alpha}\\
&\leq K(t)\sup_{0\leq s\leq t}\|x^1(s)-x^2(s)\|_\alpha+M(t)\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}\\
&\leq K_{t_0}\sup_{0\leq s\leq t}\{\|S(s)[\phi_1(0)-\phi_2(0)+F(\phi_1)-F(\phi_2)]\|_\alpha+\|F(x_s^1)-F(x_s^2)\|_\alpha\\
&\quad+\|\int_0^s AS(s-\tau)[F(x_\tau^1)-F(x_\tau^2)]d\tau\|_\alpha\\
&\quad+\|\int_0^s S(s-\tau)[G(x_\tau^1)-G(x_\tau^2)]d\tau\|_\alpha\}+M_{t_0}\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}\\
&\leq K_{t_0}\sup_{0\leq s\leq t}(Me^{\omega s}H+l_3\|A^{-\beta}\|)\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}\\
&\quad+K_{t_0}\|A^{-\beta}\|l_3\sup_{0\leq s\leq t}\|x_s^1-x_s^2\|_{\mathscr{B}_\alpha}\\
&\quad+K_{t_0}l_3\sup_{0\leq s\leq t}[\int_0^s C_{1-\beta}(s-\tau)^{\beta-1}e^{\omega(s-\tau)}\|x_\tau^1-x_\tau^2\|_{\mathscr{B}_\alpha}d\tau\\
&\quad+K_{t_0}l_4\sup_{0\leq s\leq t}\int_0^s C_\alpha(s-\tau)^{-\alpha}e^{\omega(s-\tau)}\|x_\tau^1-x_\tau^2\|_{\mathscr{B}_\alpha}d\tau]+M_{t_0}\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}.
\end{aligned}$$

Choose $\mu\in\mathbb{R}^+$ such that $\omega-\mu<-1$, then the above estimate implies that
$$\begin{aligned}
e^{-\mu t}\|x_t^1-x_t^2\|_{\mathscr{B}_\alpha}&\leq e^{-\mu t}K_{t_0}\sup_{0\leq s\leq t}(Me^{\omega s}H+l_3\|A^{-\beta}\|)\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}\\
&\quad+K_{t_0}l_3\|A^{-\beta}\|e^{-\mu t}\sup_{0\leq s\leq t}\|x_s^1-x_s^2\|_{\mathscr{B}_\alpha}\\
&\quad+K_{t_0}l_3\sup_{0\leq s\leq t}[\int_0^s C_{1-\beta}(s-\tau)^{\beta-1}e^{\omega(s-\tau)}e^{-\mu t}\|x_\tau^1-x_\tau^2\|_{\mathscr{B}_\alpha}d\tau\\
&\quad+K_{t_0}l_4\sup_{0\leq s\leq t}\int_0^s C_\alpha(s-\tau)^{-\alpha}e^{\omega(s-\tau)}e^{-\mu t}\|x_\tau^1-x_\tau^2\|_{\mathscr{B}_\alpha}d\tau]\\
&\quad+M_{t_0}e^{-\mu t}\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}.
\end{aligned}$$
Put $W(s):=e^{-\mu s}\|x_s^1-x_s^2\|_{\mathscr{B}_\alpha}$, then
$$\begin{aligned}
&\sup_{0\leq s\leq t}W(s)\\
&\leq K_{t_0}\sup_{0\leq s\leq t}\big(Me^{\omega s}H+l_3\|A^{-\beta}\|\big)\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}+K_{t_0}l_3\|A^{-\beta}\|\sup_{0\leq s\leq t}W(s)\\
&\quad+K_{t_0}l_3\sup_{0\leq s\leq t}\Big[\int_0^s C_{1-\beta}(s-\tau)^{\beta-1}e^{(\omega-\mu)(s-\tau)}W(\tau)d\tau\\
&\quad+K_{t_0}l_4\sup_{0\leq s\leq t}\int_0^s C_\alpha(s-\tau)^{-\alpha}e^{(\omega-\mu)(s-\tau)}W(\tau)d\tau\Big]+M_{t_0}\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\le K_{t_0} \sup_{0\le s\le t} \big(Me^{\omega s}H + l_3\|A^{-\beta}\|\big)\, \|\phi_1-\phi_2\|_{\mathscr{B}_\alpha} \\
&\quad + K_{t_0} l_3 \|A^{-\beta}\| \sup_{0\le s\le t} W(s) + K_{t_0} l_3 C_{1-\beta}\Gamma(\beta)(\mu-\omega)^{-\beta} \sup_{0\le s\le t} W(s) \\
&\quad + K_{t_0} l_4 C_\alpha \Gamma(1-\alpha)(\mu-\omega)^{\alpha-1} \sup_{0\le s\le t} W(s) + M_{t_0}\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}.
\end{aligned}
$$

So, by (4.2) we may take $t_0>0$ sufficiently small such that $1-K_{t_0}(l_3\|A^{-\beta}\| + l_3C_{1-\beta}\Gamma(\beta)(\mu-\omega)^{-\beta} + l_4C_\alpha\Gamma(1-\alpha)(\mu-\omega)^{\alpha-1})>0$, then

$$
\begin{aligned}
&\sup_{0\le s\le t} W(s) \\
&\le \Big(K_{t_0} \sup_{0\le s\le t} (Me^{\omega s}H + l_3\|A^{-\beta}\|) + M_{t_0}\Big)\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha} \\
&\div \Big(1-K_{t_0}(l_3\|A^{-\beta}\| + l_3C_{1-\beta}\Gamma(\beta)(\mu-\omega)^{-\beta} + l_4C_\alpha\Gamma(1-\alpha)(\mu-\omega)^{\alpha-1})\Big) \\
&:= P(t,\mu)\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha},
\end{aligned}
$$

or

$$
\|x_t^1 - x_t^2\|_{\mathscr{B}_\alpha} \le P(t,\mu)e^{\mu t}\|\phi_1-\phi_2\|_{\mathscr{B}_\alpha}, \tag{4.3}
$$

for all $t\in[0,t_0]$. For any $t>t_0$, find an $n\in\mathbb{N}$ such that $t\in(nt_0,(n+1)t_0]$, then we may repeat the above computation for n times and obtain that (4.3) holds for $t>t_0$. Thus we complete the proof of the assertion. □

In what follows, we investigate the stability of an equilibrium of (4.1). For each $u\in X_\alpha$, the corresponding constant function $\tilde u\in\mathscr{B}_\alpha$ is defined by $\tilde u(\theta)\equiv u$, $\theta\in(-\infty,0]$. Here by an equilibrium of (4.1) we mean a constant function $\tilde u\in\mathscr{B}_\alpha$ satisfying

$$
u\in D(A) \quad\text{and}\quad -Au + G(\tilde u) = 0.
$$

If $\tilde u$ is an equilibrium of (4.1), then it is trivial to verify that 0 is the equilibrium solution of the equation

$$
\frac{d}{dt}[x(t)+F_1(x_t)] + Ax(t) = G_1(x_t),
$$

where $F_1(\phi)=F(\phi+\tilde u)-F(\tilde u)$, $G_1(\phi)=G(\phi+\tilde u)-G(\tilde u)$. Accordingly, without loss of generality we may assume that $\tilde u=0$ and $G(0)=F(0)=0$. Moreover, we suppose that

- (H6) $A^\beta F$ is Fréchet-differentiable at 0 in space X_α, G is Fréchet-differentiable at 0 in X.

Let $L_1=A^{-\beta}(A^\beta F)'(0)$, $L_2=G'(0)$. Then the linearized equation of Equation (4.1) around the equilibrium 0 is

$$
\begin{gathered}
\frac{d}{dt}[x(t)+L_1x_t] + Ax(t) = L_2x_t, \quad t\ge 0, \\
x_0=\phi\in\mathscr{B}_\alpha.
\end{gathered} \tag{4.4}
$$

Denote by $(T(t))_{t\ge0}$ the linear solution semigroup associated to (4.4). Then we have the following result.

Theorem 4.2. *Suppose that conditions* (H4)–(H6) *are satisfied. Then, for $t\ge0$, the Fréchet derivative of $U(t)$ at zero is $T(t)$.*

Proof. It suffices to prove that for any $\phi \in \mathscr{B}_\alpha$, $t > 0$ and $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|U(t)\phi - T(t)\phi\|_{\mathscr{B}_\alpha} \le \epsilon \|\phi\|_{\mathscr{B}_\alpha}, \quad \text{for } \|\phi\|_{\mathscr{B}_\alpha} < \delta. \tag{4.5}$$

In fact, we have

$$\begin{aligned}
&\|U(t)\phi - T(t)\phi\|_{\mathscr{B}_\alpha} \\
&\le K(t) \sup_{0\le s\le t} \|(U(s)(\phi))(0) - (T(s)(\phi))(0)\|_\alpha \\
&\le K_t \sup_{0\le s\le t} \Big\{ \|S(s)[F(\phi) - L_1(\phi)]\|_\alpha \\
&\quad + \|F(U(s)\phi) - F(T(s)\phi) + F(T(s)\phi) - L_1(T(s)\phi)\|_\alpha \\
&\quad + \Big\| \int_0^s AS(s-\tau)[F(U(\tau)\phi) - F(T(\tau)\phi) + F(T(\tau)\phi) - L_1(T(\tau)\phi)]d\tau \Big\|_\alpha \\
&\quad + \Big\| \int_0^s S(s-\tau)[G(U(\tau)\phi)) - G(T(\tau)\phi)) + G(T(\tau)\phi)) - L_2((\tau)\phi))]d\tau \Big\|_\alpha \Big\}.
\end{aligned}$$

Take $t_0 > 0$ such that $1 - K_{t_0}(l_3\|A^{-\beta}\| + l_3C_{1-\beta}\Gamma(\beta)(\mu-\omega)^{-\beta} + l_4C_\alpha\Gamma(1-\alpha)(\mu-\omega)^{\alpha-1}) > 0$, and by virtue of the continuous differentiability of $A^\beta F$ and G at 0 and from Theorem 4.1 we infer that, for any $\epsilon > 0$, there is a $\delta_0 > 0$ such that, for each $0 < t < t_0$ and any $\|\phi\|_{\mathscr{B}_\alpha} < \delta_0$,

$$\begin{gathered}
\|S(s)[F(\phi) - L_1(\phi)]\|_\alpha \le \epsilon\|\phi\|_{\mathscr{B}_\alpha}, \\
\big\|F(T(s)\phi) - L_1(T(s)\phi)\big\|_\alpha \le \epsilon\|\phi\|_{\mathscr{B}_\alpha}, \\
\Big\| \int_0^s AS(s-\tau)[F(T(\tau)\phi) - L_1(T(\tau)\phi)]d\tau \Big\|_\alpha \le \epsilon\|\phi\|_{\mathscr{B}_\alpha}, \\
\Big\| \int_0^s S(s-\tau)[G(T(\tau)\phi)) - L_2((\tau)\phi))]d\tau \Big\|_\alpha \le \epsilon\|\phi\|_{\mathscr{B}_\alpha}.
\end{gathered}$$

Hence,

$$\begin{aligned}
&e^{-\mu t}\|U(t)\phi - T(t)\phi\|_{\mathscr{B}_\alpha} \\
&\le 4\epsilon K_t\|\phi\|_{\mathscr{B}_\alpha} + K_t\|A^{-\beta}\|l_3 e^{-\mu t} \sup_{0\le s\le t} \|U(s)\phi - T(s)\phi\|_{\mathscr{B}_\alpha} \\
&\quad + K_t l_3 \sup_{0\le s\le t} \Big[\int_0^s C_{1-\beta}(s-\tau)^{\beta-1} e^{\omega(s-\tau)} e^{-\mu t} \|U(\tau)\phi - T(\tau)\phi\|_{\mathscr{B}_\alpha} d\tau \\
&\quad + K_t l_4 \sup_{0\le s\le t} \int_0^s C_\alpha(s-\tau)^{-\alpha} e^{\omega(s-\tau)} e^{-\mu t} \|U(\tau)\phi - T(\tau)\phi\tau\|_{\mathscr{B}_\alpha} d\tau \Big] \\
&\le 4\epsilon K_t\|\phi\|_{\mathscr{B}_\alpha} + K_t l_3\|A^{-\beta}\| \sup_{0\le s\le t} e^{-\mu s}\|U(s)\phi - T(s)\phi\|_{\mathscr{B}_\alpha} \\
&\quad + K_t l_3 C_{1-\beta}\Gamma(\beta)(\mu-\omega)^{-\beta} \sup_{0\le s\le t} e^{-\mu s}\|U(s)\phi - T(s)\phi\|_{\mathscr{B}_\alpha} \\
&\quad + K_t l_4 C_\alpha\Gamma(1-\alpha)(\mu-\omega)^{\alpha-1} \sup_{0\le s\le t} e^{-\mu s}\|U(\tau)\phi - T(\tau)\phi\|_{\mathscr{B}_\alpha}.
\end{aligned}$$

So, using (4.2) again we obtain that (4.5) is true for all $0 < t \le t_0$, and then as in the Proof of Theorem 4.1 we can conclude that (4.5) holds for all $t > 0$. □

As a consequence of the above two results we obtain the following theorem.

Theorem 4.3. *Under the assumptions of Theorems 4.1 and 4.2, if the zero equilibrium of $(T(t))_{t\geq 0}$ is exponentially stable, then the zero equilibrium of $(U(t))_{t\geq 0}$ is locally exponentially stable in the sense that there exist $\mu, \mu > 0$ and $k \geq 1$ such that, for $t \geq 0$ and any $\phi \in \mathscr{B}_\alpha$ with $\|\phi\|_{\mathscr{B}_\alpha} < \mu$,*

$$\|U(t)\phi\|_{\mathscr{B}_\alpha} \leq ke^{-\mu t}\|\phi\|_{\mathscr{B}_\alpha}.$$

Proof. Based on Theorems 4.1,4.2 and 2.2 this theorem can be proved by using the similar method as that in [11] and [3], and we omit the proof here. □

5. An Example

To apply Theorems 3.2 and 3.7, we consider the system

$$\begin{gathered}
\frac{\partial}{\partial t}\Big[z(t,x) + \int_{-\infty}^{t}\int_{0}^{\pi} b(s-t,x,y)\Big[z(s,y) + \frac{\partial}{\partial y}z(s,y)\Big]\,dy\,ds\Big] \\
= \frac{\partial^2}{\partial x^2}z(t,x) + h\Big(z(\cdot,x), \frac{\partial}{\partial x}z(\cdot,x)\Big), \quad 0 \leq t \leq a,\ 0 \leq x \leq \pi, \qquad (5.1) \\
z(t,0) = z(t,\pi) = 0, \\
z(\theta,x) = \phi(\theta,x), \quad \theta \leq 0, \quad 0 \leq x \leq \pi,
\end{gathered}$$

where the functions b and h will be described below.

Let $X = L^2([0,\pi])$ and operator A be defined by

$$Af = -f''$$

with the domain

$$D(A) = H_0^2([0,\pi]) = \{f(\cdot) \in X : f', f'' \in X,\ f(0) = f(\pi) = 0\}.$$

then $-A$ generates a strongly continuous semigroup $(S(\cdot))_{t\geq 0}$ which is analytic, compact and self-adjoint. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are $-n^2, \in N$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Then the following properties hold:

(a) If $f \in D(A)$, then

$$Af = \sum_{n=1}^{\infty} n^2\langle f, z_n\rangle z_n.$$

(b) For every $f \in X$,

$$S(t)f = \sum_{n=1}^{\infty} e^{-n^2 t}\langle f, z_n\rangle z_n,$$

$$A^{-1/2}f = \sum_{n=1}^{\infty} \frac{1}{n}\langle f, z_n\rangle z_n.$$

In particular, $\|S(t)\| \leq e^{-t}$, $\|A^{-1/2}\| = 1$.

(c) The operator $A^{1/2}$ is given by

$$A^{1/2}f = \sum_{n=1}^{\infty} n\langle f, z_n\rangle z_n.$$

on the space $D(A^{1/2}) = \{f(\cdot) \in X,\ \sum_{n=1}^{\infty} n\langle f, z_n\rangle z_n \in X\}$.

Here we take $\alpha = \beta = 1/2$ and the phase space $\mathscr{B} = \mathscr{C}_g$, where the space $\mathscr{C}_g$ is defined as: let g be a continuous function on $(-\infty, 0]$ with $g(0) = 1$, $\lim_{\theta\to-\infty} g(\theta) = \infty$, and g is decreasing on $(-\infty, 0]$, then

$$\mathscr{C}_g = \{\phi \in C((-\infty, 0]; X) : \sup_{s\le 0} \frac{\|\phi(s)\|}{g(s)} < \infty\},$$

and the norm is defined by, for $\phi \in \mathscr{C}_g$,

$$|\phi|_g = \sup_{s\le 0} \frac{\|\phi(s)\|}{g(s)}.$$

It is known that $\mathscr{C}_g$ satisfies the axioms $(A1), (A2)$, and (B1), see [18]. Further, the subspace $\mathscr{C}_{g,\frac{1}{2}}$ is defined by

$$\mathscr{C}_{g,\frac{1}{2}} = \{\phi \in C((-\infty, 0]; X_{\frac{1}{2}}) : \sup_{s\le 0} \frac{\|A^{\frac{1}{2}}\phi(s)\|}{g(s)} < \infty\},$$

endowed with the norm $|\phi|_{g,\frac{1}{2}} = \sup_{s\le 0} \frac{\|A^{\frac{1}{2}}\phi(s)\|}{g(s)}$. Clearly, $\mathscr{C}_{g\frac{1}{2}}$ satisfies correspondingly the axioms (A1'),(A2'), and (B1'), and we may choose a proper g such that $H, K(\cdot), M(\cdot) \le 1$ (also see [9]).

We assume that the following conditions hold:

(i) The function $b(\cdot,\cdot,\cdot) \in C^2$ with $b(\cdot,\cdot,0) = b(\cdot,\cdot,\pi) \equiv 0$, and

$$c := \{\int_0^\pi [\int_{-\infty}^0 g(\theta)(\int_0^\pi (\frac{\partial^2}{\partial x^2} b(\theta, y, x))^2 dy)^{1/2} d\theta]^2 dx\}^{1/2} < \infty.$$

(ii) The function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in the two variables.

(iii) The function ϕ defined by $\phi(\theta)(x) = \phi(\theta, x)$ belongs to $\mathscr{C}_{g,\frac{1}{2}}$.

Now define the abstract functions $F, G : \mathscr{C}_{g,\frac{1}{2}} \to X_{\frac{1}{2}}$ by

$$F(\phi) = \int_{-\infty}^0 \int_0^\pi b(\theta, y, x)[\phi(\theta)(y) + \phi(\theta)'(y)]\,dy\,d\theta,$$
$$G(\phi) = h(\phi(\cdot)(x), \phi(\cdot)'(x)).$$

Then the system (5.1) is rewritten as the abstract form (1.1), and condition (i) implies that $R(F) \subset D(A)$, since

$$\begin{aligned}
\langle F(\phi), z_n\rangle &= -\frac{1}{n}\Big\langle \int_{-\infty}^0 \int_0^\pi \frac{\partial}{\partial x} b(\theta, y, x)[\phi(\theta)(y) + \phi(\theta)'(y)]dyd\theta, \bar{z}_n(x)\Big\rangle \\
&= \frac{1}{n^2}\Big\langle \int_{-\infty}^0 \int_0^\pi \frac{\partial^2}{\partial x^2} b(\theta, y, x)[\phi(\theta)(y) + \phi(\theta)'(y)]dyd\theta, z_n(x)\Big\rangle,
\end{aligned}$$

where $\bar{z}_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx)$, $n = 1, 2, \dots$. Observe that, for any $\theta \in (-\infty, 0]$,

$$\begin{aligned}
\|\phi_2(\theta)(x) - \phi_1(\theta)(x)\|^2 &= \sum_{n=1}^\infty \langle \phi_2 - \phi_1, z_n\rangle^2 \\
&\le \sum_{n=1}^\infty n^2\langle \phi_2 - \phi_1, z_n\rangle^2 \\
&\le \|\phi_2(\theta)(x) - \phi_1(\theta)(x)\|_{\frac{1}{2}}{}^2,
\end{aligned}$$

and

$$\begin{aligned}\|\phi_2(\theta)'(x)-\phi_1(\theta)'(x)\|^2 &= \sum_{n=1}^{\infty}\langle \phi_2'-\phi_1', z_n\rangle^2\\ &= \sum_{n=1}^{\infty}\langle \phi_2-\phi_1, z_n'\rangle^2\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^2\langle \phi_2-\phi_1, z_n\rangle\langle \phi_2-\phi_1, z_m\rangle\langle -z_n'', z_m'\rangle\\ &\le \|\phi_2(\theta)(x)-\phi_1(\theta)(x)\|_{\frac12}{}^2,\end{aligned}$$

we see that

$$\begin{gathered}|\phi_2(\cdot)-\phi_1(\cdot)|_g \le |\phi_2(\cdot)-\phi_1(\cdot)|_{g,\frac12},\\ |\phi_2(\cdot)'-\phi_1(\cdot)'|_g \le |\phi_2(\cdot)-\phi_1(\cdot)|_{g,\frac12}.\end{gathered}$$

Thus, conditions (i) and (ii) ensure that $A^{\frac12}F(\cdot)$ satisfies the Lipschitz continuous on $\mathscr{C}_{g,\frac12}$, F and G verify assumption (H_1) and (H2) respectively. Consequently, by Theorem 3.2 the system (5.1) has a mild solution on $(-\infty, b_\phi]$ for some $b_\phi > 0$.

Furthermore, if take $\mathscr{B} = \mathscr{C}^0_{g,\frac12}$, where

$$\mathscr{C}^0_{g,\frac12} = \{\phi \in \mathscr{C}_{g,\frac12}; X_{\frac12}) : \lim_{s\le 0}\frac{\|A^{\frac12}\phi(s)\|}{g(s)} = 0\},$$

so that Axiom (C1) is satisfied (see [18]) and assume that h is Hölder continuous in two variables, then condition (H1') and (H2') are satisfied. Therefore, if $\phi(\cdot, x)$ is uniformly Hölder continuous and $\phi(0, x) \in D(A)$, then the system (5.1) has a strict solution on $(-\infty, b_\phi]$.

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