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EXISTENCE OF SOLUTIONS TO SINGULAR FOURTH-ORDER ELLIPTIC EQUATIONS

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Abstract. Using a method developed by Ambrosetti et al [1, 2] we prove the existence of weak non trivial solutions to fourth-order elliptic equations with singularities and with critical Sobolev growth.

1. Introduction

Fourth-order elliptic equations have been widely studied, because of their importance in the analysis on manifolds particularly those involving the Paneitz-Branson operators; see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 16]. Different techniques have been used for solving fourth-order equations, as example the variational method which was developed by Yamabe to solve the problem of the prescribed scalar curvature. Let (M, g) a compact smooth Riemannian manifold of dimension $n \geq 5$ with a metric g. We denote by $H_2^2(M)$ the standard Sobolev space which is the completed of the space $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{2,2} = \sum_{k=0}^{k=2} \|\nabla^k \varphi\|_2.$$

$H_2^2(M)$ will be endowed with the suitable equivalent norm

$$\|u\|_{H_2^2(M)} = \Big(\int_M ((\Delta_g u)^2 + |\nabla_g u|^2 + u^2) dv_g \Big)^{1/2}.$$

In 1979, Vaugon [17] proved the existence of a positive value λ and a non trivial solution $u \in C^4(M)$ to the equation

$$\Delta_g^2 u - \operatorname{div}_g(a(x)\nabla_g u) + b(x)u = \lambda f(t, x)$$

where a, b are smooth functions on M and $f(t, x)$ is odd and increasing function in t fulfilling the inequality

$$|f(t, x)| < a + b|t|^{\frac{n+4}{n-4}}.$$

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Edminds, Fortunato and Jannelli [14] showed that all the solutions in $\mathbb{R}^n$ to the equation

$$\Delta^2 u = u^{\frac{n+4}{n-4}}$$

are positive, symmetric, radial and decreasing functions of the form

$$u_\epsilon(x) = \frac{((n-4)n(n^2-4)\epsilon^4)^{\frac{n-4}{8}}}{(r^2+\epsilon^2)^{\frac{n-4}{2}}}.$$

In 1995, Van Der Vorst [15] obtained the same results for the problem

$$\begin{gathered} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} \quad \text{in } \Omega, \\ \Delta u = u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a bounded domain of $\mathbb{R}^n$.

In 1996, Bernis, Garcia-Azorero and Peral [9] obtained the existence at least of two positive solutions to the problem

$$\begin{gathered} \Delta^2 u - \lambda u|u|^{q-2} = u|u|^{\frac{8}{n-4}} \text{ in } \Omega, \\ \Delta u = u = 0 \text{ on } \partial\Omega, \end{gathered}$$

where Ω is bounded domain of $\mathbb{R}^n$, $1 < q < 2$ and $\lambda > 0$ in some interval. In 2001, Caraffa [12] obtained the existence of a non trivial solution of class $C^{4,\alpha}$, $\alpha \in (0,1)$ for the equation

$$\Delta_g^2 u - \nabla^\alpha(a(x)\nabla_\alpha u) + b(x)u = \lambda f(x)|u|^{N-2}u$$

with $\lambda > 0$, first for f a constant and next for a positive function f on M.

Recently the first author [4] showed the existence of at least two distinct non trivial solutions in the subcritical case and a non trivial solution in the critical case for the equation

$$\Delta_g^2 u - \nabla^\alpha(a(x)\nabla_\alpha u) + b(x)u = f(x)|u|^{N-2}u$$

where f is a changing sign smooth function and a and b are smooth functions. In [6] the same author proved the existence of at least two non trivial solutions to

$$\Delta_g^2 u - \nabla^\alpha(a(x)\nabla_\alpha u) + b(x)u = f(x)|u|^{N-2}u + |u|^{q-2}u + \varepsilon g(x)$$

where a, b, f, g are smooth functions on M with $f > 0$, $2 < q < N$, $\lambda > 0$ and $\epsilon > 0$ small enough. Let S_g denote the scalar curvature of M. In 2011, the authors proved the following result

Theorem 1.1 ([8])**.** *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 6$ and a, b, f smooth functions on M, $\lambda \in (0,\lambda_*)$ for some specified $\lambda_* > 0$, $1 < q < 2$ such that*

1. *$f(x) > 0$ on M.*
2. *At the point x_0 where f attains its maximum, we suppose that for $n = 6$, $S_g(x_0) + 3a(x_0) > 0$, and for $n > 6$*

$$\Big(\frac{(n^2+4n-20)}{2(n+2)(n-6)}S_g(x_0) + \frac{(n-1)}{(n+2)(n-6)}a(x_0) - \frac{1}{8}\frac{\Delta f(x_0)}{f(x_0)}\Big) > 0.$$

Then the equation

$$\Delta_g^2 u + \mathrm{div}_g(a(x)\nabla_g u) + b(x)u = \lambda|u|^{q-2}u + f(x)|u|^{N-2}u$$

admits a non trivial solution of class $C^{4,\alpha}(M)$, $\alpha \in (0,1)$.

Recently Madani [14] studied the Yamabe problem with singularities when the metric g admits a finite number of points with singularities and is smooth outside these points. More precisely, let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$, we denote by T^*M the cotangent space of M. The space $H_2^p(M, T^*M \otimes T^*M)$ is the set of sections s (2-covariant tensors) such that in normal coordinates the components s_{ij} of s are in H_2^p the complement of the space $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\varphi\|_{2,p} = \sum_{k=0}^{k=2} \|\nabla^k \varphi\|_p$.

Solving the singular Yamabe problem is equivalent to finding a positive solution $u \in H_2^p(M)$ of the equation

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = k|u|^{N-2}u, \tag{1.1}$$

where S_g is the scalar curvature of the g and k is a real constant. The Christoffels symbols belong to $H_1^p(M)$, the Riemannian curvature tensor, the Ricci tensor Ric_g and scalar curvature S_g are in $L^p(M)$, hence equation 1.1 is the singular Yamabe equation.

Under the assumptions that g is a metric in the Sobolev space $H_2^p(M, T^*M \otimes T^*M)$ with $p > n/2$ and that there exist a point $P \in M$ and $\delta > 0$ such that g is smooth in the ball $B_p(\delta)$, Madani [14] proved the existence of a metric $\overline{g} = u^{N-2} g$ conformal to g such that $u \in H_2^p(M)$, $u > 0$ and the scalar curvature $S_{\overline{g}}$ of $\overline{g}$ is constant if (M,g) is not conformal to the round sphere.

The author in [7] considered fourth-order elliptic equations, with singularities, of the form

$$\Delta^2 u - \nabla^i (a(x) \nabla_i u) + b(x) u = f|u|^{N-2}u \tag{1.2}$$

where the functions a and b are in $L^s(M)$, $s > \frac{n}{2}$ and in $L^p(M)$, $p > \frac{n}{4}$ respectively, $N = \frac{2n}{n-4}$ is the Sobolev critical exponent in the embedding $H_2^2(M) \hookrightarrow L^N(M)$. He established the following results. Let (M,g) be a compact n-dimensional Riemannian manifold, $n \geq 6$, $a \in L^s(M)$, $b \in L^p(M)$, with $s > \frac{n}{2}$, $p > \frac{n}{4}$, $f \in C^\infty(M)$ a positive function and $x_0 \in M$ such that $f(x_0) = \max_{x \in M} f(x)$.

Theorem 1.2. *For $n \geq 10$, or $n = 8, 9$ and $2 < p < 5$, $\frac{9}{4} < s < 11$ or $n = 7$, $\frac{7}{2} < s < 9$ and $\frac{7}{4} < p < 9$ we suppose that*

$$\frac{n^2 + 4n - 20}{6(n-6)(n^2-4)} S_g(x_0) - \frac{n-4}{2n(n-2)} \frac{\Delta f(x_0)}{f(x_0)} > 0.$$

For $n = 6$ and $\frac{3}{2} < p < 2$, $3 < s < 4$, we assume that

$$S_g(x_0) > 0.$$

Then (1.2) has a non trivial weak solution u in $H_2^2(M)$. Moreover if $a \in H_1^s(M)$, then $u \in C^{0,\beta}(M)$, for some $\beta \in (0, 1 - \frac{n}{4p})$.

In this article, we extend results obtained in Theorem 1.1 to the case of singular elliptic fourth order, more precisely we are concerned with the following problem: Let (M,g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $a \in L^r(M)$, $b \in L^s(M)$ where $r > \frac{n}{2}$, $s > \frac{n}{4}$ and f a positive C^∞-function on M; we look for non trivial solution of the equation

$$\Delta_g^2 u + \operatorname{div}_g(a(x) \nabla_g u) + b(x) u = \lambda |u|^{q-2} u + f(x) |u|^{N-2} u \tag{1.3}$$

where $1 < q < 2$ and $N = \frac{2n}{n-4}$ is the critical Sobolev exponent and $\lambda > 0$ a real number.

In case the $\lambda = 0$ and
$$a = \frac{4}{n-2} R_{ic_g} - \frac{(n-2)^2+4}{2(n-1)(n-2)} S_g.g, \quad b = \frac{n-4}{2} Q_g^n,$$
where
$$Q_g^n = \frac{1}{2(n-1)} \Delta S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|^2$$
suppose that g is a metric in the Sobolev space $H_4^p(M, T^*M \otimes T^*M)$ with $p > \frac{n}{4}$, then the Ricci Ric_g curvature and the scalar curvature S_g are in the Sobolev spaces $H_2^p(M, T^*M \otimes T^*M)$ and $H_2^p(M)$ respectively, hence $b \in L^s(M)$ with $s > \frac{n}{4}$ and by Sobolev embedding $a \in L^r(M)$ with $r > \frac{n}{2}$. In this latter case the equation
$$\Delta_g^2 u + \operatorname{div}_g(a(x)\nabla_g u) + b(x)u = f(x)|u|^{N-2}u \tag{1.4}$$
is called singular Q-curvature equation.For more general coefficients $a \in L^r(M)$ with $r > \frac{n}{2}$ and $b \in L^s(M)$ with $s > \frac{n}{4}$, the equation (1.4) is called singular Q-curvature type equation. To solve equation (1.3), we use a method developed in [1] and [2] which resumes to study the variations of functional associated to equation 1.3 on the manifold M_λ defined in section 2. Serious difficulties appear compared with the smooth case: considering the equation (4.3) in section 4, we need a Hardy-Sobolev inequality and Releich-Kondrakov embedding on a manifolds. In the case of the singular Yamabe equation theses latters were established in [14] and in the case of singular Q-curvature type equations by the first author in [7]. In the sharp cases (see section 5) the Hardy Sobolev inequality and the Releich-Kondrakov embedding are no more valid so we need an additional assumption with some tricks combined with the Lebesgue dominated convergence theorem.

Denote by P_g the operator defined in the weak sense on $H_2^2(M)$ by $P_g(u) = \Delta^2 u + \operatorname{div}(a\nabla u) + bu$. P_g is called coercive if there exits $\Lambda > 0$ such that for any $u \in H_2^2(M)$
$$\int_M u P_g(u) dv_g \geq \Lambda \|u\|^2_{H_2^2(M)}.$$
Our main result reads as follows.

Theorem 1.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$ and f a positive function. Suppose that P_g is coercive and at a point x_0 where f attains its maximum the following two conditions hold:*
$$\begin{gathered} \frac{\Delta f(x_0)}{f(x_0)} < \Big(\frac{n(n^2+4n-20)}{3(n+2)(n-4)(n-6)} \frac{1}{(1+\|a\|_r + \|b\|_s)^{n/4}} \\ - \frac{n-2}{3(n-1)} \Big) S_g(x_0) \quad \text{when } n > 6, \\ S_g(x_0) > 0 \quad \text{when } n = 6. \end{gathered} \tag{1.5}$$
Then there is $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the equation (1.3) has a non trivial weak solution.

For fixed $R \in M$, we define the function ρ on M by
$$\rho(Q) = \begin{cases} d(R, Q) & \text{if } d(R, Q) < \delta(M) \\ \delta(M) & \text{if } d(R, Q) \geq \delta(M) \end{cases} \tag{1.6}$$
where $\delta(M)$ denotes the injectivity radius of M.

For real numbers σ and μ, consider the following equation, in the distribution sense,

$$\Delta^2 u - \nabla^i(\frac{a}{\rho^\sigma}\nabla_i u) + \frac{bu}{\rho^\mu} = \lambda|u|^{q-2}u + f(x)|u|^{N-2}u \tag{1.7}$$

where the functions a and b are smooth on M.

Corollary 1.4. *Let $0 < \sigma < \frac{n}{r} < 2$ and $0 < \mu < \frac{n}{s} < 4$. Suppose that*

$$\frac{\Delta f(x_0)}{f(x_0)} < \frac{1}{3}\Big(\frac{(n-1)n(n^2+4n-20)}{(n^2-4)(n-4)(n-6)}\frac{1}{(1+\|a\|_r+\|b\|_s)^{n/4}} - 1\Big)S_g(x_0)$$

$$\text{when } n > 6,$$

$$S_g(x_0) > 0 \quad \text{when } n = 6.$$

Then there is $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, the (1.7) possesses a weak non trivial solution $u_{\sigma,\mu} \in M_\lambda$.

In the sharp case $\sigma = 2$ and $\mu = 4$, letting $K(n, 2, \gamma)$ be the best constant in the Hardy-Sobolev inequality given by Theorem 4.1 we obtain the following result.

Theorem 1.5. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $(u_{\sigma_m,\mu_m})_m$ be a sequence in M_λ such that*

$$J_{\lambda,\sigma,\mu}(u_{\sigma_m,\mu_m}) \leq c_{\sigma,\mu}$$

$$\nabla J_\lambda(u_{\sigma,\mu}) - \mu_{\sigma,\mu}\nabla\Phi_\lambda(u_{\sigma,\mu}) \to 0$$

Suppose that

$$c_{\sigma,\mu} < \frac{2}{nK_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}$$

and

$$1 + a^- \max(K(n,2,\sigma), A(\varepsilon,\sigma)) + b^- \max(K(n,2,\mu), A(\varepsilon,\mu)) > 0$$

then the equation

$$\Delta^2 u - \nabla^\mu(\frac{a}{\rho^2}\nabla_\mu u) + \frac{bu}{\rho^4} = f|u|^{N-2}u + \lambda|u|^{q-2}u$$

in the distribution has a weak non trivial solution.

Our paper is organized as follows: in a first section we show that the manifold of constraints is non empty, in the second one we establish a generic existence result to equation 1.3. The third section deals with applications to particular equations which could arise from conformal geometry. In the fourth section and under supplementary assumption we obtain non trivial solution in the critical case. The last section is devoted to tests functions which verify geometric assumptions and by the same way complete the proofs of our claimed theorems in the introduction.

2. The manifold M_λ of constraints is non empty

In this section, we consider on $H_2^2(M)$ the functional

$$J_\lambda(u) = \frac{1}{2}\int_M (|\Delta_g u|^2 - a(x)|\nabla_g u|^2 + b(x)u^2)dv_g - \frac{\lambda}{q}\int_M |u|^q dv_g - \frac{1}{N}\int_M f(x)|u|^N dv_g$$

associated to Equation 1.3. First, we put

$$\Phi_\lambda(u) = \langle\nabla J_\lambda(u),\ u\rangle$$

hence

$$\Phi_\lambda(u)=\int_M((\Delta_g u)^2-a(x)|\nabla_g u|^2+b(x)u^2)dv_g-\lambda\int_M|u|^q dv_g-\int_M f(x)|u|^N dv_g.$$

We let

$$M_\lambda=\{u\in H_2^2(M):\ \Phi_\lambda(u)=0 \text{ and } \|u\|\geq\tau>0\}.$$

Proposition 2.1. *The norm*

$$\|u\|=(\int_M|\Delta_g u|^2-a(x)|\nabla_g u|^2+b(x)u^2dv_g)^{1/2}$$

is equivalent to the usual norm on $H_2^2(M)$ if and only if P_g is coercive.

Proof. If P_g is coercive there is $\Lambda>0$ such that for any $u\in H_2^2(M)$,

$$\int_M P_g(u)udv_g\geq\Lambda\|u\|^2_{H_2^2(M)}$$

and since $a\in L^r(M)$ and $b\in L^s(M)$ where $r>\frac{n}{2}$ and $s>\frac{n}{4}$, by Hölder's inequality we obtain

$$\int_M uP_g(u)dv_g\leq\|\Delta_g u\|_2^2+\|a\|_{\frac{n}{2}}\|\nabla_g u\|^2_{2^*}+\|b\|_{\frac{n}{4}}\|u\|_N^2$$

where $2^*=2n/(n-2)$.

The Sobolev's inequalities lead to: for any $\eta>0$,

$$\|\nabla_g u\|^2_{2^*}\leq\max((1+\eta)K(n,1)^2,A_\eta)\int_M(|\nabla_g^2 u|^2+|\nabla_g u|^2)dv_g$$

where $K(n,1)$ denotes the best Sobolev's constant in the embedding $H_1^2(\mathbb{R}^n)\hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, and for any $\epsilon>0$,

$$\|u\|_N^2\leq\max((1+\varepsilon)K_0,B_\varepsilon)\|u\|^2_{H_2^2(M)}$$

where in this latter inequality K_0 is the best Sobolev's constant in the embedding $H_1^2(M)\hookrightarrow L^{\frac{2n}{n-2}}(M)$ and B_ϵ the corresponding (see [3]). Now by the well known formula (see [3, page 115])

$$\int_M|\nabla_g^2 u|^2dv_g=\int_M(|\Delta_g u|^2-R_{ij}\nabla^i u\nabla^j u)dv_g$$

where R_{ij} denote the components of the Ricci curvature, there is a constant $\beta>0$ such that

$$\int_M|\nabla_g^2 u|^2dv_g\leq\int_M|\Delta_g u|^2+\beta|\nabla_g u|^2dv_g$$

so we obtain

$$\|\nabla_g u\|^2_{2^*}\leq(\beta+1)\max((1+\eta)K(n,1)^2,A_\eta)\int_M(|\Delta_g u|^2+|\nabla_g u|^2+u^2)dv_g$$

and we infer that

$$\begin{aligned}\int_M P_g(u)udv_g\leq\|u\|^2_{H_2^2(M)}&+(\beta+1)\|a\|_{\frac{n}{2}}\max((1+\eta)K(n,1)^2,A_\eta)\|u\|^2_{H_2^2(M)}\\&+\|b\|_{\frac{n}{4}}\max((1+\varepsilon)K_0,B_\varepsilon)\|u\|^2_{H_2^2(M)}.\end{aligned}$$

Hence

$$\int_M uP_g(u)dv_g$$

$$\leq \underbrace{\max(1, \|b\|_{\frac{n}{4}} \max((1+\varepsilon)K_0, B_\varepsilon), (\beta+1)\|a\|_{\frac{n}{2}} \max((1+\varepsilon)K(n,1)^2, A_\varepsilon))}_{>0}$$
$$\times \|u\|^2_{H_2^2(M)}.$$

□

Lemma 2.2. *The set M_λ is non empty provided that $\lambda \in (0, \lambda_0)$ where*

$$\lambda_0 = \frac{(2^{q-2} - 2^{q-N})\Lambda^{\frac{N-q}{N-2}}}{V(M)^{(1-\frac{q}{N})}(\max_{x\in M} f(x))^{\frac{2-q}{N-2}}(\max((1+\varepsilon)K(n,2), A_\varepsilon))^{\frac{N-q}{N-2}}}.$$

Proof. The proof of this lemma is the same as in [8], but we give it here for convenience. Let $t > 0$ and $u \in H_2^2(M) - \{0\}$. Evaluating Φ_λ at tu, we obtain

$$\Phi_\lambda(tu) = t^2\|u\|^2 - \lambda t^q \|u\|_q^q - t^N \int_M f(x)|u|^N dv_g.$$

Put

$$\alpha(t) = \|u\|^2 - t^{N-2}\int_M f(x)|u|^N dv(g),$$
$$\beta(t) = \lambda t^{q-2}\|u\|_q^q;$$

by Sobolev's inequality, we obtain

$$\alpha(t) \geq \|u\|^2 - \max_{x\in M} f(x)(\max((1+\varepsilon)K_0, A_\varepsilon))^{N/2}\|u\|^N_{H_2^2(M)} t^{N-2}.$$

By the coercivity of the operator $P_g = \Delta_g^2 - \operatorname{div}_g(a\nabla_g) + b$ there is a constant $\Lambda > 0$ such that

$$\alpha(t) \geq \|u\|^2 - \Lambda^{-N/2}\max_{x\in M} f(x)(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{N}{2}}\|u\|^N t^{N-2}.$$

Letting

$$\alpha_1(t) = \|u\|^2 - \Lambda^{-N/2}\max_{x\in M} f(x)(\max((1+\varepsilon)K_0, A_\varepsilon))^{N/2}\|u\|^N t^{N-2}$$

Hölder and Sobolev inequalities lead to

$$\beta(t) \leq \lambda V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u\|^q_{H_2^2(M)} t^{q-2}$$

and the coercivity of P_g assures the existence of a constant $\Lambda > 0$ such that

$$\beta(t) \leq \lambda\Lambda^{-q/2}V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u\|^q t^{q-2}.$$

Put

$$\beta_1(t) = \lambda\Lambda^{-q/2}V(M)^{(1-\frac{q}{N})}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u\|^q t^{q-2}.$$

Let t_0 such $\alpha_1(t_0) = 0$; i.e.,

$$t_0 = \frac{\Lambda^{\frac{N}{2(N-2)}}}{\|u\|(\max_{x\in M} f(x))^{\frac{1}{N-2}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{N}{2(N-2)}}}$$

Now since $\alpha_1(t)$ is a decreasing and a concave function and $\beta_1(t)$ is a decreasing and convex function, then

$$\min_{t\in(0,\frac{t_0}{2}]} \alpha_1(t) = \alpha_1(\frac{t_0}{2}) = \|u\|^2(1 - 2^{2-N}) > 0,$$

$$\min_{t\in(0,\frac{t_0}{2}]} \beta_1(t) = \beta_1(\frac{t_0}{2}) > 0,$$

where

$$\beta_1(\frac{t_0}{2}) = \frac{2^{2-q}\lambda V(M)^{(1-\frac{q}{N})}\Lambda^{\frac{q-N}{N-2}}\|u\|^2}{(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{q-N}{N-2}}(\max_{x\in M} f(x))^{\frac{q-2}{N-2}}}.$$

Consequently $\Phi_\lambda(tu) = 0$ with $t \in (0, \frac{t_0}{2}]$ has a solution if

$$\min_{t\in(0,\frac{t_0}{2}]} \alpha_1(t) \geq \max_{t\in(0,\frac{t_0}{2}]} \beta_1(t);$$

that is to say

$$0 < \lambda < \frac{(2^{q-2} - 2^{q-N})(\max_{x\in M} f(x))^{\frac{q-2}{N-2}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{q-N}{N-2}}}{\Lambda^{\frac{N-q}{N-2}}V(M)^{(1-\frac{q}{N})}} = \lambda_0$$

Let $t_1 \in (0, \frac{t_0}{2}]$ such that $\Phi_\lambda(t_1u) = 0$. If we take $u \in H_2^2(M)$ such that $\|u\| \geq \frac{\rho}{t_1}$ and $v = t_1u$ we obtain $\Phi_\lambda(v) = 0$ and $\|v\| = t_1\|u\| \geq \rho$; i.e., $v \in M_\lambda$ provided that $\lambda \in (0, \lambda_0)$. □

3. Existence of non trivial solutions in M_λ

The following lemmas whose proofs are similar modulo minor modifications as in [8] give the geometric conditions to the functional J_λ.

Lemma 3.1. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. For all $u \in M_\lambda$ and all $\lambda \in (0, \min(\lambda_0, \lambda_1))$ there is $A > 0$ such that $J_\lambda(u) \geq A > 0$ where*

$$\lambda_1 = \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{q/2}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2), A_\varepsilon))^{q/2}\tau^{q-2}}.$$

Lemma 3.2. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. The following assertions are true:*

(i) *$\langle\nabla\Phi_\lambda(u), u\rangle < 0$ for all $u \in M_\lambda$ and for all $\lambda \in (0, \min(\lambda_0, \lambda_1))$.*
(ii) *The critical points of J_λ are points of M_λ.*

Now, we show that J_λ satisfies the Palais-Smale condition on M_λ provided that $\lambda > 0$ is sufficiently small. The result is given by the following lemma whose proof is different from the one in the case of smooth coefficients.

Lemma 3.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Let $(u_m)_m$ be a sequence in M_λ such that*

$$J_\lambda(u_m) \leq c$$
$$\nabla J_\lambda(u_m) - \mu_m\nabla\Phi_\lambda(u_m) \to 0.$$

Suppose that

$$c < \frac{2}{nK_0^{n/4}(f(x_0))^{(n-4)/4}}$$

then there is a subsequence $(u_m)_m$ converging strongly in $H_2^2(M)$.

Proof. Let $(u_m)_m \subset M_\lambda$ and

$$J_\lambda(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q dv_g\,.$$

As in the proof of Lemma 3.2, we have

$$J_\lambda(u_m) \geq \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\Lambda^{-q/2}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\|u_m\|^q,$$

$$
\begin{aligned}
J_\lambda(u_m) &\geq \|u_m\|^2\Big(\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-q/2}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\tau^{q-2}\Big)\\
&> 0\,.
\end{aligned}
$$

Since $0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda^{q/2}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2),A_\varepsilon))^{q/2}\tau^{q-2}}$ and $J_\lambda(u_m) \leq c$, we obtain

$$
\begin{aligned}
c &\geq J_\lambda(u_m)\\
&\geq \Big[\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-\frac{q}{2}}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{\frac{q}{2}}\tau^{q-2}\Big]\|u_m\|^2 > 0
\end{aligned}
$$

so

$$
\|u_m\|^2 \leq \frac{c}{\frac{N-2}{2N} - \lambda\frac{N-q}{Nq}\Lambda^{-q/2}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K_0, A_\varepsilon))^{q/2}\tau^{q-2}} < +\infty.
$$

Then $(u_m)_m$ is a bounded in $H_2^2(M)$. By the compactness of the embedding $H_2^2(M) \subset H_p^k(M)$ ($k = 0,1$; $p < N$) we obtain a subsequence still denoted $(u_m)_m$ such that

$$
\begin{gathered}
u_m \to u \quad \text{weakly in } H_2^2(M),\\
u_m \to u \quad \text{strongly in } L^p(M) \text{ where } p < N,\\
\nabla u_m \to \nabla u \quad \text{strongly in } L^p(M) \text{ where } p < 2^* = \frac{2n}{n-2}\\
u_m \to u \quad \text{a.e. in } M.
\end{gathered}
$$

On the other hand since $\frac{2s}{s-1} < N = \frac{2n}{n-4}$, we obtain

$$
\begin{aligned}
\Big|\int_M b(x)|u_m-u|^2dv_g\Big| &\leq \|b\|_s\|u_m-u\|^2_{\frac{2s}{s-1}}\\
&\leq \|b\|_s((K_0+\epsilon)\|\Delta(u_m-u)\|_2^2 + A_\epsilon\|u_m-u\|_2^2).
\end{aligned}
$$

Now taking into account

$$
K_0 = \frac{16}{n(n^2-4)(n-4)\omega_n^{n/4}} < 1 \tag{3.1}
$$

we obtain

$$
\int_M b(x)(u_m-u)^2dv_g \leq \|b\|_s\|\Delta(u_m-u)\|_2^2 + o(1).
$$

By the same process as above, we obtain

$$
\int_M a(x)|\nabla(u_m-u)|^2dv_g \leq \|a\|_r\|\Delta(u_m-u)\|_2^2 + o(1).
$$

By Brezis-Lieb lemma, we write

$$
\int_M(\Delta_g u_m)^2dv_g = \int_M(\Delta_g u)^2dv_g + \int_M(\Delta_g(u_m-u))^2dv_g + o(1)
$$

and

$$
\int_M f(x)|u_m|^N dv_g = \int_M f(x)|u|^N dv_g + \int_M f(x)|u_m-u|^N dv_g + o(1).
$$

Now we claim that $\mu_m \to 0$ as $m \to +\infty$ Testing with u_m we obtain

$$
\langle \nabla J_\lambda(u_m) - \mu_m\nabla\Phi_\lambda(u_m), u_m\rangle = o(1);
$$

then

$$\langle \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m), u_m\rangle = \underbrace{\langle \nabla J_\lambda(u_m), u_m\rangle}_{=0} - \mu_m \langle \nabla \Phi_\lambda(u_m), u_m\rangle = o(1);$$

hence

$$\mu_m \langle \nabla \Phi_\lambda(u_m), u_m\rangle = o(1).$$

By Lemma 3.2, we obtain $\limsup_m \langle \nabla \Phi_\lambda(u_m), u_m\rangle < 0$ so $\mu_m \to 0$ as $m \to +\infty$.

Our last claim is that $u_m \to u$ strongly in $H_2^2(M)$, indeed

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2}\int_M (\Delta_g(u_m - u))^2 dv_g - \frac{1}{N}\int_M f(x)|u_m - u|^N dv_g + o(1).$$

Since $u_m - u \to 0$ weakly in $H_2^2(M)$, testing with $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$, we have $\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u\rangle = o(1)$ and

$$\begin{aligned} &\langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u\rangle \\ &= \int_M (\Delta_g(u_m - u))^2 dv_g - \int_M f(x)|u_m - u|^N dv_g = o(1); \end{aligned} \tag{3.2}$$

then

$$\int_M (\Delta_g(u_m - u))^2 dv_g = \int_M f(x)|u_m - u|^N dv_g + o(1),$$

and taking account of (3.2) we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{1}{2}\int_M (\Delta_g(u_m - u))^2 dv_g - \frac{1}{N}\int_M (\Delta_g(u_m - u))^2 dv_g + o(1);$$

i. e.,

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{N}\int_M (\Delta_g(u_m - u))^2 dv_g + o(1).$$

Independently, by the Sobolev's inequality, we have

$$\|u_m - u\|_N^2 \le (1+\varepsilon) K_0 \int_M (\Delta_g(u_m - u))^2 dv_g + o(1). \tag{3.3}$$

Since

$$\int_M f(x)|u_m - u|^N dv_g \le \max_{x\in M} f(x) \|u_m - u\|_N^N$$

we infer by (3.3) that

$$\int_M f(x)|u_m - u|^N dv_g \le (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N + o(1)$$

and using equality (3.2),

$$o(1) \ge \|\Delta_g(u_m - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N$$

and

$$\begin{aligned} &\|\Delta_g(u_m - u)\|_2^2 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^N \\ &= \|\Delta_g(u_m - u)\|_2^2 (1 - (1+\varepsilon)^{\frac{n}{n-4}} \max_{x\in M} f(x) K_0^{\frac{n}{n-4}} \|\Delta_g(u_m - u)\|_2^{N-2}) \end{aligned}$$

so if

$$\limsup_{m\to+\infty} \|\Delta_g(u_m - u)\|_2^2 < \frac{1}{K_0^{n/4} (\max_{x\in M} f(x))^{\frac{n}{4}-1}} \tag{3.4}$$

then $u_m \to u$ strongly in $H_2^2(M)$. The condition (3.4) is fulfilled since by Lemma 3.1 $J_\lambda(u) > 0$ on M_λ with λ is as in Lemma 3.1 and by hypothesis,

$$c \geq J_\lambda(u_m) > (J_\lambda(u_m) - J_\lambda(u)) = \frac{2}{n}\int_M (\Delta_g(u_m - u))^2 dv_g$$

and

$$c < \frac{2}{nK_0^{n/4}(\max_{x\in M} f(x))^{\frac{n}{4}-1}}.$$

It is obvious that

$$\Phi_\lambda(u) = 0 \quad \text{and } \|u\| \geq \tau$$

i.e. $u \in M_\lambda$. □

Now we show the existence of a sequence in M_λ satisfying the conditions of Palais-Smale.

Lemma 3.4. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, then there is a couple $(u_m, \mu_m) \in M_\lambda \times R$ such that $\nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m) \to 0$ strongly in $(H_2^2(M))^*$ and $J_\lambda(u_m)$ is bounded provide that $\lambda \in (0, \lambda_*)$ with $\lambda_* = \{\min(\lambda_0, \lambda_1), 0\}$.*

Proof. Since J_λ is Gateau differentiable and by Lemma 3.1 bounded below on M_λ it follows from Ekeland's principle that there is a couple $(u_m, \mu_m) \in M_\lambda \times R$ such that $\nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m) \to 0$ strongly in $(H_2^2(M))'$ and $J_\lambda(u_m)$ is bounded i.e. $(u_m, \mu_m)_m$ is a Palais-Smale sequence on M_λ. □

Now we are in position to establish the following generic existence result.

Theorem 3.5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$ and f a positive function. Suppose that P_g is coercive and*

$$c < \frac{2}{nK_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}. \tag{3.5}$$

Then there is $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the equation (1.3) has a non trivial weak solution.

Proof. By Lemma 3.3 and 3.4 there is $u \in H_2^2(M)$ such that

$$J_\lambda(u) = \min_{\varphi \in M_\lambda} J_\lambda(\varphi).$$

By Lagrange multiplicative theorem there is a real number μ such that for any $\varphi \in H_2^2(M)$,

$$\langle \nabla J_\lambda(u), \varphi \rangle = \mu \langle \nabla \Phi_\lambda(u), \varphi \rangle \tag{3.6}$$

and letting $\varphi = u$ in the equation (3.6), we obtain

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u), u \rangle = \mu \langle \nabla \Phi_\lambda(u), u \rangle.$$

By Lemma 3.2 we obtain that $\mu = 0$ and by equation (3.6), we infer that for any $\varphi \in H_2^2(M)$

$$\langle \nabla J_\lambda(u), \varphi \rangle = 0$$

hence u is weak non trivial solution to equation (1.3) and since by Lemma 3.2, u is a critical points of J_λ. We conclude that $u \in M_\lambda$. □

4. Applications

Let $P \in M$, we define a function on M by

$$\rho_P(Q) = \begin{cases} d(P,Q) & \text{if } d(P,Q) < \delta(M) \\ \delta(M) & \text{if } d(P,Q) \geq \delta(M) \end{cases} \tag{4.1}$$

where $\delta(M)$ is the injectivity radius of M. For brevity we denote this function by ρ. The weighted $L^p(M, \rho^\gamma)$ space will be the set of measurable functions u on M such that $\rho^\gamma |u|^p$ are integrable where $p \geq 1$. We endow $L^p(M, \rho^\gamma)$ with the norm

$$\|u\|_{p,\rho} = (\int_M \rho^\gamma |u|^p dv_g)^{1/p}.$$

In this section we need the Hardy-Sobolev inequality and the Releich-Kondrakov embedding whose proofs are given in [7].

Theorem 4.1. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$ and p, q , γ are real numbers such that $\frac{\gamma}{p} = \frac{n}{q} - \frac{n}{p} - 2$ and $2 \leq p \leq \frac{2n}{n-4}$. For any $\epsilon > 0$, there is $A(\epsilon, q, \gamma)$ such that for any $u \in H_2^2(M)$,*

$$\|u\|^2_{p,\rho^\gamma} \leq (1+\epsilon) K(n, 2, \gamma)^2 \|\Delta_g u\|_2^2 + A(\epsilon, q, \gamma) \|u\|_2^2 \tag{4.2}$$

where $K(n, 2, \gamma)$ is the optimal constant.

In the case $\gamma = 0$, $K(n, 2, 0) = K(n, 2) = K_0^{-1/2}$ is the best constant in the Sobolev's embedding of $H_2^2(M)$ in $L^N(M)$ where $N = \frac{2n}{n-4}$.

Theorem 4.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$ and p, q, γ are real numbers satisfying $1 \leq q \leq p \leq \frac{nq}{n-2q}$, $\gamma < 0$ and $l = 1, 2$.*

If $\frac{\gamma}{p} = n\ (\frac{1}{q} - \frac{1}{p}) - l$ then the inclusion $H_l^q(M) \subset L^p(M, \rho^\gamma)$ is continuous. If $\frac{\gamma}{p} > n\ (\frac{1}{q} - \frac{1}{p}) - l$ then inclusion $H_l^q(M) \subset L^p(M, \rho^\gamma)$ is compact.

We consider the equation

$$\Delta_g^2 u + \operatorname{div}_g \Big(\frac{a(x)}{\rho^\sigma} \nabla_g u \Big) + \frac{b(x)}{\rho^\mu} u = \lambda |u|^{q-2} u + f(x) |u|^{N-2} u \tag{4.3}$$

where a and b are smooth functions and ρ denotes the distance function defined by (4.1), $\lambda > 0$ in some interval $(0, \lambda_*)$, $1 < q < 2$, σ, μ will be precise later and we associate to (4.3) on $H_2^2(M)$ the functional

$$\begin{aligned} J_\lambda(u) = \frac{1}{2} \int_M ((\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2) dv_g \\ - \frac{\lambda}{q} \int_M |u|^q dv_g - \frac{1}{N} \int_M f(x) |u|^N dv_g. \end{aligned}$$

If we put

$$\Phi_\lambda(u) = \langle \nabla J_\lambda(u), u \rangle$$

we obtain

$$\Phi_\lambda(u) = \int_M (\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2 dv_g - \lambda \int_M |u|^q dv_g - \int_M f(x) |u|^N dv_g.$$

Theorem 4.3. *Let $0 < \sigma < \frac{n}{s} < 2$ and $0 < \mu < \frac{n}{p} < 4$. Suppose that*

$$\sup_{u \in H_2^2(M)} J_{\lambda,\sigma,\mu}(u) < \frac{2}{n\ K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}$$

then there is $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, equation (4.3) possesses a weak non trivial solution $u_{\sigma,\mu} \in M_\lambda$.

Proof. Let $\tilde{a} = \frac{a(x)}{\rho^\sigma}$ and $\tilde{b} = \frac{b(x)}{\rho^\mu}$, so if $\sigma \in (0, \min(2, \frac{n}{s}))$ and $\mu \in (0, \min(4, \frac{n}{p}))$, obliviously $\tilde{a} \in L^s(M)$, $\tilde{b} \in L^p(M)$, where $s > \frac{n}{2}$ and $p > \frac{n}{4}$. Theorem 4.3 is a consequence of Theorem 3.5. □

5. The critical cases $\sigma = 2$ and $\mu = 4$

In the cases $\sigma = 2$ and $\mu = 4$ the Hardy-Sobolev inequality proved in case of manifolds by the first author in [7] and is formulated in Theorem 4.1 is no longer valid, so we consider the subcritical cases $0 < \sigma < 2$ and $0 < \mu < 4$ and we tend σ to 2and μ to 4. This can be done successfully by adding an appropriate assumption and by using the Lebesgue dominated converging theorem.

By section four, for any $\sigma \in (0, \min(2, \frac{n}{s}))$ and $\mu \in (0, \min(4, \frac{n}{p}))$, there is a solution $u_{\sigma,\mu} \in M_\lambda$ of equation (1.3). Now we are going to show that the sequence $(u_{\sigma,\mu})_{\sigma,\mu}$ is bounded in $H_2^2(M)$. Evaluating $J_{\lambda,\sigma,\mu}$ at $u_{\sigma,\mu}$

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{1}{2}\|u_{\sigma,\mu}\|^2 - \frac{1}{N}\int_M f(x)|u_{\sigma,\mu}|^N dv_g - \frac{1}{q}\lambda \int_M |u_{\sigma,\mu}|^q dv_g$$

and taking account of $u_{\sigma,\mu} \in M_\lambda$, we infer that

$$J_{\lambda,\sigma,\mu}(u_{\sigma,\mu}) = \frac{N-2}{2N}\|u_{\sigma,\mu}\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_{\sigma,\mu}|^q dv_g.$$

For a smooth function a on M, denotes by $a^- = \min(0, \min_{x\in M}(a(x))$. Let $K(n, 2, \sigma)$ the best constant and $A(\varepsilon, \sigma)$ the corresponding constant in the Hardy-Sobolev inequality given in Theorem 4.1.

Theorem 5.1. *Let (M, g) be a Riemannian compact manifold of dimension $n \geq 5$. Let $(u_m)_m = (u_{\sigma_m,\mu_m})_m$ be a sequence in M_λ such that*

$$J_{\lambda,\sigma,\mu}(u_m) \leq c_{\sigma,\mu}$$
$$\nabla J_\lambda(u_m) - \mu_{\sigma,\mu}\nabla\Phi_\lambda(u_m) \to 0.$$

Suppose that

$$c_{\sigma,\mu} < \frac{2}{n\ K(n,2)^{n/4}(\max_{x\in M} f(x))^{(n-4)/4}}$$

and

$$1 + a^- \max(K(n, 2, \sigma), A(\varepsilon, \sigma)) + b^- \max(K(n, 2, \mu), A(\varepsilon, \mu)) > 0.$$

Then the equation

$$\Delta^2 u - \nabla^\mu(\frac{a}{\rho^2}\nabla_\mu u) + \frac{bu}{\rho^4} = f|u|^{N-2}u + \lambda|u|^{q-2}u$$

has a non trivial solution in the sense of distributions.

Proof. Let $(u_m)_m \subset M_{\lambda,\sigma,\mu}$,

$$J_{\lambda,\sigma,\mu}(u_m) = \frac{N-2}{2N}\|u_m\|^2 - \lambda\frac{N-q}{Nq}\int_M |u_m|^q dv_g$$

As in proof of Theorem 3.5, we obtain

$$\begin{aligned} J_{\lambda,\sigma,\mu}(u_m) \geq \|u_m\|^2\Big(&\frac{N-2}{2N} \\ &- \lambda\frac{N-q}{Nq}\Lambda_{\sigma,\mu}^{-q/2}V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2),A_\varepsilon))^{q/2}\tau^{q-2}\Big) > 0 \end{aligned}$$

where

$$0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)}\Lambda_{\sigma,\mu}^{q/2}}{V(M)^{1-\frac{q}{N}}(\max((1+\varepsilon)K(n,2),A_\varepsilon))^{q/2}\tau^{q-2}}.$$

First we claim that

$$\lim_{(\sigma,\mu)\to(2^-,4^-)} \inf \Lambda_{\sigma,\mu} > 0.$$

Indeed, if $\nu_{1,\sigma,\mu}$ denotes the first nonzero eigenvalue of the operator

$$P_g = \Delta_g^2 - \operatorname{div}(\frac{a}{\rho^\sigma}\nabla_g) + \frac{b}{\rho^\mu},$$

then clearly $\Lambda_{\sigma,\mu} \geq \nu_{1,\sigma,\mu}$. Suppose on the contrary that $\lim_{(\sigma,\mu)\to(2^-,4^-)} \inf \Lambda_{\sigma,\mu} = 0$, then $\liminf_{(\sigma,\mu)\to(2^-,4^-)} \nu_{1,\sigma,\mu} = 0$. Independently, if $u_{\sigma,\mu}$ is the corresponding eigenfunction to $\nu_{1,\sigma,\mu}$ we have

$$\begin{aligned} \nu_{1,\sigma,\mu} &= \|\Delta u_{\sigma,\mu}\|_2^2 + \int_M \frac{a|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma}dv_g + \int_M \frac{bu_{\sigma,\mu}^2}{\rho^\mu}dv_g \\ &\geq \|\Delta u_{\sigma,\mu}\|_2^2 + a^-\int \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma}dv_g + b^-\int_M \frac{u_{\sigma,\mu}^2}{\rho^\mu}dv_g \end{aligned} \tag{5.1}$$

where $a^- = \min(0, \min_{x\in M} a(x))$ and $b^- = \min(0, \min_{x\in M} b(x))$. The Hardy-Sobolev's inequality given by Theorem 4.1 leads to

$$\int_M \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma}dv_g \leq C(\|\nabla|\nabla u_{\sigma,\mu}|\|^2 + \|\nabla u_{\sigma,\mu}\|^2),$$

and since

$$\|\nabla|\nabla u_{\sigma,\mu}|\|^2 \leq \|\nabla^2 u_{\sigma,\mu}\|^2 \leq \|\Delta u_{\sigma,\mu}\|^2 + \beta\|\nabla u_{\sigma,\mu}\|^2$$

where $\beta > 0$ is a constant and it is well known that for any $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$\|\nabla u_{\sigma,\mu}\|^2 \leq \varepsilon\|\Delta u_{\sigma,\mu}\|^2 + c\|u_{\sigma,\mu}\|^2.$$

Hence

$$\int_M \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma}dv_g \leq C(1+\varepsilon)\|\Delta u_{\sigma,\mu}\|^2 + A(\varepsilon)\|u_{\sigma,\mu}\|^2 \tag{5.2}$$

Now if $K(n,2,\sigma)$ denotes the best constant in inequality (5.2) we obtain that for any $\varepsilon > 0$,

$$\int_M \frac{|\nabla u_{\sigma,\mu}|^2}{\rho^\sigma}dv_g \leq (K(n,2,\sigma)^2 + \varepsilon)\|\Delta u_{\sigma,\mu}\|^2 + A(\varepsilon,\sigma)\|u_{\sigma,\mu}\|^2. \tag{5.3}$$

By inequalities (4.2), (5.1) and (5.3), we have

$$\begin{aligned} \nu_{1,\sigma,\mu} \geq (1 &+ a^-\max(K(n,2,\sigma), A(\varepsilon,\sigma)) \\ &+ b^-\max(K(n,2,\mu), A(\varepsilon,\mu)))(\|\Delta u_{\sigma,\mu}\|^2 + \|u_{\sigma,\mu}\|^2) \end{aligned}$$

So if

$$1 + a^- \max(K(n,2,\sigma), A(\varepsilon,\sigma)) + b^- \max(K(n,2,\mu), A(\varepsilon,\mu)) > 0$$

then we obtain $\lim_{\sigma,\mu}(u_{\sigma,\mu}) = 0$ and $\|u_{\sigma,\mu}\| = 1$ a contradiction. The reflexivity of $H_2^2(M)$ and the compactness of the embedding $H_2^2(M) \subset H_p^k(M)$ ($k = 0,1$; $p < N$), imply that up to a subsequence, we have

$$\begin{gathered} u_m \to u \quad \text{weakly in } H_2^2(M), \\ u_m \to u \quad \text{strongly in } L^p(M),\ p < N, \\ \nabla u_m \to \nabla u \quad \text{strongly in } L^p(M),\ p < 2^* = \frac{2n}{n-2}, \\ u_m \to u \quad \text{a. e. in } M. \end{gathered}$$

The Brézis-Lieb lemma allows us to write

$$\int_M (\Delta_g u_m)^2 dv_g = \int_M (\Delta_g u)^2 dv_g + \int_M (\Delta_g (u_m - u))^2 dv_g + o(1)$$

and

$$\int_M f(x)|u_m|^N dv_g = \int_M f(x)|u|^N dv_g + \int_M f(x)|u_m - u|^N dv_g + o(1).$$

Now by the boundedness of the sequence $(u_m)_m$, we have that $u_m \to u$ weakly in $H_2^2(M)$, $\nabla u_m \to \nabla u$ weakly in $L^2(M,\rho^{-2})$ and $u_m \to u$ weakly in $L^2(M,\rho^{-4})$; i.e., for any $\varphi \in L^2(M)$,

$$\int_M \frac{a(x)}{\rho^2} \nabla u_m \nabla \varphi dv_g = \int_M \frac{a(x)}{\rho^2} \nabla u \nabla \varphi dv_g + o(1)$$

and

$$\int_M \frac{b(x)}{\rho^4} u_m \varphi dv_g = \int_M \frac{b(x)}{\rho^4} u \varphi dv_g + o(1).$$

For every $\phi \in H_2^2(M)$ we have

$$\begin{aligned} &\int_M \Big(\Delta_g^2 u_m + \operatorname{div}_g \big(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m\big) + \frac{b(x)}{\rho^{\delta_m}} u_m\Big)\phi dv_g \\ &= \int_M (\lambda |u_m|^{q-2} u_m + f(x)|u_m|^{N-2} u_m)\phi dv_g. \end{aligned} \tag{5.4}$$

By the weak convergence in $H_2^2(M)$, we have immediately that

$$\int_M \phi \Delta_g^2 u_m dv_g = \int_M \phi \Delta_g^2 u dv_g + o(1)$$

and

$$\begin{aligned} &\int_M \big(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m - \frac{a(x)}{\rho^2} \nabla_g u\big)\phi dv_g \\ &= \int_M \big(\frac{a(x)}{\rho^{\sigma_m}} \nabla_g u_m + \frac{a(x)}{\rho^2}(\nabla_g u_m - \nabla_g u_m) - \frac{a(x)}{\rho^2} \nabla_g u\big)\phi dv_g \end{aligned}$$

Then

$$\begin{aligned}
&|\int_M \Big(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u\Big)\phi dv_g| \\
&\leq |\int_M \Big(\frac{a(x)}{\rho^{\sigma_m}}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u_m\Big)\phi dv_g| + |\int_M (\frac{a(x)}{\rho^2}\nabla_g u_m - \frac{a(x)}{\rho^2}\nabla_g u)\phi dv_g| \\
&\leq \int_M |a(x)\phi\nabla_g u_m||\frac{1}{\rho^{\sigma_m}} - \frac{1}{\rho^2}|dv_g + |\int_M \frac{a(x)}{\rho^2}\nabla_g(u_m - u)\phi dv_g|.
\end{aligned} \tag{5.5}$$

The weak convergence in $L^2(M,\rho^{-2})$ and the Lebesgue's dominated convergence theorem imply that the second right hand side of (5.5) goes to 0. For the third term of the left hand side of (5.3), we write

$$\int_M \Big(\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u\Big)\phi dv_g = \int_M \Big(\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u_m + \frac{b(x)}{\rho^4}u_m - \frac{b(x)}{\rho^4}u\Big)\phi dv_g$$

and

$$\begin{aligned}
&|\int_M (\frac{b(x)}{\rho^{\delta_m}}u_m - \frac{b(x)}{\rho^4}u)\phi dv_g| \\
&\leq \int_M |b(x)\phi u_m||\frac{1}{\rho^{\delta_m}} - \frac{1}{\rho^4}|dv_g + |\int_M \frac{b(x)}{\rho^4}(u_m - u)\phi dv_g|.
\end{aligned} \tag{5.6}$$

Here also the weak convergence in $L^2(M,\rho^{-4})$ and the Lebesgue's dominated convergence allows us to affirm that the left hand side of (5.6) converges to 0.

It remains to show that $\mu_m \to 0$ as $m \to +\infty$ and $u_m \to u$ strongly in $H_2^2(M)$ but this is the same as in the proof of Theorem 3.5 which implies also $u \in M_\lambda$. □

6. Test Functions

In this section, we give the proof of the main theorem to do so, we consider a normal geodesic coordinate system centered at x_0. Denote by $S_{x_0}(\rho)$ the geodesic sphere centered at x_0 and of radius ρ ($\rho < d$ which is the injectivity radius). Let $d\Omega$ be the volume element of the $n-1$-dimensional Euclidean unit sphere S^{n-1} and put

$$G(\rho) = \frac{1}{\omega_{n-1}}\int_{S(\rho)} \sqrt{|g(x)|}d\Omega$$

where ω_{n-1} is the volume of S^{n-1} and $|g(x)|$ the determinant of the Riemannian metric g. The Taylor's expansion of $G(\rho)$ in a neighborhood of x_0 is given by

$$G(\rho) = 1 - \frac{S_g(x_0)}{6n}\rho^2 + o(\rho^2)$$

where $S_g(x_0)$ denotes the scalar curvature of M at x_0. Let $B(x_0,\delta)$ be the geodesic ball centered at x_0 and of radius δ such that $0 < 2\delta < d$ and denote by η a smooth function on M such that

$$\eta(x) = \begin{cases} 1 & \text{on } B(x_0,\delta) \\ 0 & \text{on } M - B(x_0,2\delta). \end{cases}$$

Consider the radial function

$$u_\epsilon(x) = (\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)})^{\frac{n-4}{8}} \frac{\eta(\rho)}{((\rho\theta)^2 + \epsilon^2)^{\frac{n-4}{2}}}$$

with

$$\theta = (1 + \|a\|_r + \|b\|_s)^{1/n}$$

where $\rho = d(x_0, x)$ is the distance from x_0 to x and $f(x_0) = \max_{x\in M} f(x)$. For further computations we need the following integrals: for any real positive numbers p, g such that $p - q > 1$ we put

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt\,.$$

The following relations are immediate

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q, \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q.$$

6.1. **Application to compact Riemannian manifolds of dimension $n > 6$.**

Theorem 6.1. *Let (M, g) be a compact Riemannian manifold of dimension $n > 6$. Suppose that at a point x_0 where f attains its maximum the following condition*

$$\frac{\Delta f(x_0)}{f(x_0)} < \frac{1}{3}\Big(\frac{(n-1)n(n^2+4n-20)}{(n^2-4)(n-4)(n-6)} \frac{1}{(1+\|a\|_r+\|b\|_s)^{n/4}} - 1\Big) S_g(x_0)$$

holds. Then (1.2) has a non trivial solution with energy

$$J_\lambda(u) < \frac{1}{K_0^{n/4} (\max_{x\in M} f(x))^{\frac{n}{4}-1}}.$$

Proof. The proof of Theorem 6.1 reduces to show that the condition (3.5) of Theorem 3.5 is satisfied and since by Lemma 2.2 there is a $t_0 > 0$ such that $t_0 u_\epsilon \in M_\lambda$ for sufficiently small λ, so it suffices to show that

$$\sup_{t>0} J_\lambda(tu_\epsilon) < \frac{1}{K_0^{\frac{n}{4}} (\max_{x\in M} f(x))^{\frac{n}{4}-1}}.$$

To compute the term $\int_M f(x)|u_\epsilon(x)|^N dv_g$, we need the following Taylor's expansion of f at the point x_0

$$f(x) = f(x_0) + \frac{\partial^2 f(x_0)}{2\partial y^i \partial y^j} y^i y^j + o(\rho^2)$$

and also that of the Riemannian measure

$$dv_g = 1 - \frac{1}{6} R_{ij}(x_0) y^i y^j + o(\rho^2)$$

where $R_{ij}(x_0)$ denotes the Ricci tensor at x_0. The expression of $\int_M f(x)|u_\epsilon(x)|^N dv_g$ is well known (see for example [11]) and is given in case $n > 6$ by

$$\int_M f(x)|u_\epsilon(x)|^N dv_g = \frac{\theta^{-n}}{K_0^{n/4} (f(x_0))^{\frac{n-4}{4}}} \Big(1 - \big(\frac{\Delta f(x_0)}{2(n-2) f(x_0)} + \frac{S_g(x_0)}{6(n-2)}\big)\epsilon^2 + o(\epsilon^2)\Big)$$

where K_0 is given by (3.1) and $\omega_n = 2^{n-1} I_n^{\frac{n}{2}-1} \omega_{n-1}$ and ω_n is the volume of S^n, the standard unit sphere of R^{n+1} endowed with its round metric.

Now the restriction of $|\frac{\partial u_\epsilon}{\partial \rho}|$ to the geodesic ball $B(x_0, \delta)$ is computed as follows

$$|\frac{\partial u_\epsilon}{\partial \rho}|_{B(x_0,\delta)} = |\nabla u_\epsilon| = \theta^{-2}(n-4)\big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\big)^{\frac{n-4}{8}} \frac{\rho}{((\frac{\rho}{\theta})^2 + \epsilon^2)^{\frac{n-2}{2}}}$$

and Since $a \in L^r(M)$ with $r > n/2$ we have

$$\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g \le \theta^{-4}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}\|a\|_r\omega_{n-1}^{1-\frac{1}{r}} \times\Big(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{((\frac{\rho}{\theta})^2+\epsilon^2)^{\frac{(n-2)r}{r-1}}}\Big(\int_{S(\rho)}\sqrt{|g(x)|}d\Omega\Big)d\rho\Big)^{\frac{r-1}{r}}$$

Since

$$\int_{S(\rho)}\sqrt{|g(x)|}d\Omega = \omega_{n-1}\Big(1-\frac{S_g(x_0)}{6n}\rho^2+o(\rho^2)\Big)$$

we obtain

$$\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g \le \theta^{-4}(n-4)^2(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)})^{\frac{n-4}{4}}\|a\|_r\omega_{n-1}^{1-\frac{1}{r}} \times\Big(\int_0^\delta \frac{\rho^{\frac{2r}{r-1}+n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-2)r}{r-1}}}d\rho\Big(1-\frac{S_g(x_0)}{6n}\rho^2+o(\rho^2)\Big)\Big)^{\frac{r-1}{r}}$$

and by the following change of variable

$$t=(\frac{\rho\theta}{\epsilon})^2 \quad \text{i.e. } \rho=\frac{\epsilon}{\theta}\sqrt{t}$$

we obtain

$$\begin{aligned}&\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g\\ &\le \theta^{-n\frac{r}{r-1}}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}\|a\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}}\\ &\quad\times\Big(\int_0^{(\frac{\delta\theta}{\epsilon})^2}\frac{t^{\frac{n-2}{2}+\frac{r}{r-1}}}{(t+1)^{\frac{(n-2)r}{r-1}}}dt-\frac{S_g(x_0)}{6n}\theta^{-2}\epsilon^2\int_0^{(\frac{\delta\theta}{\epsilon})^2}\frac{t^{\frac{n}{2}+\frac{r}{r-1}}}{(t+1)^{\frac{(n-2)r}{r-1}}}dt+o(\epsilon^2)\Big)^{\frac{r-1}{r}}.\end{aligned}$$

Letting $\epsilon\to 0$ we obtain

$$\begin{aligned}&\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g\\ &\le 2^{-1+\frac{1}{r}}\theta^{-n(1-\frac{1}{r})}(n-4)^2(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)})^{\frac{n-4}{4}}\|a\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}}\\ &\quad\times(I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}}-\theta^{-2}\frac{S_g(x_0)}{6n}I_{\frac{(n-2)r}{r-1}}^{\frac{n}{2}+\frac{r}{r-1}}\epsilon^2+o(\epsilon^2))^{\frac{r-1}{r}}.\end{aligned}$$

Then

$$\begin{aligned}&\int_{B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g\\ &\le 2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}\|a\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{\epsilon^{-(n-4)+2-\frac{n}{r}}}\\ &\quad\times I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}\cdot\frac{r-1}{r}}\Big[1-\frac{r-1}{r}\theta^2\frac{S_g(x_0)}{6n}I_{\frac{(n-2)r}{r-1}}^{\frac{n}{2}+\frac{r}{r-1}}I_{\frac{(n-2)r}{r-1}}^{-\frac{n-2}{2}-\frac{r}{r-1}}\epsilon^2+o(\epsilon^2)\Big].\end{aligned}$$

It remains to compute the integral $\int_{B(x_0,2\delta)-B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g$.

First we remark that

$$\Big|\int_{(\frac{\delta\theta}{\epsilon})^2}^{(\frac{2\delta\theta}{\epsilon})^2} h(t)\frac{t^q}{(t+1)^p}dt\Big| \le C(\frac{1}{\epsilon})^{2(q-p+1)} = C\epsilon^{2(p-q-1)}$$

and since $p-q=n-4\ge 3$, we obtain

$$\int_{(\frac{\delta\theta}{\epsilon})^2}^{(\frac{2\delta\theta}{\epsilon})^2} h(t)\frac{i^q}{(t+1)^p}dt = o(\epsilon^2)$$

and then

$$\int_{B(x_0,2\delta)-B(x_0,\delta)} a(x)|\nabla u_\epsilon|^2 dv_g = o(\epsilon^2). \tag{6.1}$$

Finally we obtain

$$\begin{aligned}
&\int_M a(x)|\nabla u_\epsilon|^2 dv_g \\
&\le 2^{-1+\frac{1}{r}}\theta^{-n\frac{r}{r-1}}(n-4)^2\Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}\|a\|_r\omega_{n-1}^{1-\frac{1}{r}}\epsilon^{-(n-4)+2-\frac{n}{r}} \\
&\quad\times\Big(I_{\frac{(n-2)r}{r-1}}^{1+\frac{n-2}{2}\cdot\frac{r-1}{r}} + o(\epsilon^2)\Big).
\end{aligned}$$

Letting

$$A = K_0^{n/4}\frac{(n-4)^{\frac{n}{4}+1}\times(\omega_{n-1})^{\frac{r-1}{r}}}{2^{\frac{r-1}{r}}}(n(n^2-4))^{\frac{n-4}{4}}\Big(I_{\frac{(n-2)r}{r-1}}^{\frac{n-2}{2}+\frac{r}{r-1}}\Big)^{\frac{r-1}{r}} \tag{6.2}$$

we obtain

$$\int_M a(x)|\nabla u_\epsilon|^2 dv_g \le \epsilon^{2-\frac{n}{r}}\theta^{-n\frac{r}{r-1}}\frac{A}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\|a\|_r(1+o(\epsilon^2)).$$

Now we compute

$$\int_M b(x)u_\epsilon^2 dv_g = \int_{B(x_0,\delta)} b(x)u_\epsilon^2 dv_g + \int_{B(x_0,2\delta)-B(x_0,\delta)} b(x)u_\epsilon^2 dv_g$$

and since $b\in L^s(M)$ with $s>\frac{n}{4}$, we have

$$\int_M b(x)u_\epsilon^2 dv_g \le \|b\|_s\|u_\epsilon\|^2_{\frac{2s}{s-1}}.$$

Independently,

$$\begin{aligned}
\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} &= \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}} \\
&\quad\times\Big(\int_0^\delta \frac{\rho^{n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-4)s}{(s-1)}}}\Big(\int_{S(r)}\sqrt{|g(x)|}d\Omega\Big)dr\Big)^{\frac{s-1}{s}}
\end{aligned}$$

and

$$\int_{S(r)}\sqrt{|g(x)|}d\Omega = \omega_{n-1}\Big(1-\frac{S_g(x_0)}{6n}\rho^2+o(\rho^2)\Big).$$

Consequently,

$$\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} = \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)}\Big)^{\frac{n-4}{4}}$$

$$\omega_{n-1}^{\frac{s-1}{s}} \quad \times \Big(\int_0^{\delta} \frac{\rho^{n-1}}{((\rho\theta)^2+\epsilon^2)^{\frac{(n-4)s}{(s-1)}}} \Big(1 - \frac{S_g(x_0)}{6n}\rho^2 + o(\rho^2)\Big) d\rho \Big)^{\frac{s-1}{s}}.$$

And putting $t = (\rho\theta/\epsilon)^2$, we obtain

$$\begin{aligned} \|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} &= \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)} \Big)^{\frac{n-4}{4}} (\omega_{n-1})^{\frac{s-1}{s}} \epsilon^{-n+4+4-\frac{n}{s}} \\ &\times \Big(\frac{\epsilon^n \theta^{-n}}{2} \int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n}{2}-1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}} dt \\ &- \frac{\theta^{-n-2} S_g(x_0)}{12n} \epsilon^{n+2} \int_0^{(\frac{\delta\theta}{\epsilon})^2} \frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}} dt + o(\epsilon^{n+2}) \Big)^{\frac{s-1}{s}}. \end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\begin{aligned} \|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} &= \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)} \Big)^{\frac{n-4}{4}} (\omega_{n-1})^{\frac{s-1}{s}} \epsilon^{-n+4+4-\frac{n}{s}} \\ &\times \theta^{-n\frac{s}{s-1}} (\frac{\epsilon^n}{2})^{\frac{s-1}{s}} \Big(\int_0^{+\infty} \frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}} dt \\ &- \frac{S_g(x_0)}{12n} \epsilon^2 \theta^{-2} \int_0^{+\infty} \frac{t^{\frac{n}{2}+1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}} dt + o(\epsilon^2) \Big)^{\frac{s-1}{s}}. \end{aligned}$$

Hence

$$\begin{aligned} &\|u_\epsilon\|^2_{\frac{2s}{s-1},B(x_0,\delta)} \\ &= \Big(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_0)} \Big)^{\frac{n-4}{4}} (\omega_{n-1})^{\frac{s-1}{s}} \epsilon^{-n+4+4-\frac{n}{s}} \theta^{-n\frac{s}{s-1}} (\frac{\epsilon^n}{2})^{\frac{s-1}{s}} \\ &\times \Big(\int_0^{+\infty} \frac{t^{\frac{n}{2}}}{(t+1)^{\frac{(n-4)s}{(s-1)}}} dt - \theta^{-2} \frac{S_g(x_0)}{12n} \epsilon^2 \int_0^{+\infty} \frac{t^{\frac{n}{2}+1}}{(t+1)^{\frac{(n-4)s}{(s-1)}}} dt + o(\epsilon^2) \Big)^{\frac{s-1}{s}}, \end{aligned}$$

or

$$\begin{aligned} \|u_\epsilon\|^2_{\frac{2s}{s-1}} &= \Big(\frac{(n-4)n(n^2-4)}{f(x_0)} \Big)^{\frac{n-4}{4}} \Big(\frac{\omega_{n-1}}{2} \Big)^{\frac{s-1}{s}} \epsilon^{4-\frac{n}{s}} \theta^{-n\frac{s}{s-1}} \\ &\times \Big[(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}})^{\frac{s-1}{s}} - \frac{\theta^{-2}(s-1) S_g(x_0)}{12n\ s} (I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}})^{-\frac{1}{s}} I^{\frac{n}{2}+1}_{\frac{(n-4)s}{(s-1)}} \epsilon^2 + o(\epsilon^2) \Big] \end{aligned}$$

Finally, by the same method as in equality (6.1), we obtain

$$\begin{aligned} &\int_M b(x) u_\epsilon^2 dv_g \\ &\le \|b\|_s (\frac{(n-4)n(n^2-4)}{f(x_0)})^{\frac{n-4}{4}} (\frac{\omega_{n-1}}{2})^{\frac{s-1}{s}} \epsilon^{4-\frac{n}{s}} \theta^{-n\frac{s}{s-1}} \Big(\Big(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}} \Big)^{\frac{s-1}{s}} + o(\epsilon^2) \Big). \end{aligned}$$

Putting

$$B = K_0^{n/4} ((n-4)n(n^2-4))^{\frac{n-4}{4}} (\frac{\omega_{n-1}}{2})^{\frac{s-1}{s}} \Big(I^{\frac{n}{2}}_{\frac{(n-4)s}{(s-1)}} \Big)^{\frac{s-1}{s}} \tag{6.3}$$

we obtain

$$\int_M b(x) u_\epsilon^2 dv_g \le \epsilon^{4-\frac{n}{s}} \theta^{-n\frac{s}{s-1}} \frac{\|b\|_s B}{K_0^{\frac{n}{4}} (f(x_0))^{\frac{n-4}{4}}} (1 + o(\epsilon^2)).$$

The computation of $\int_M (\Delta u_\epsilon)^2 dv_g$ is well known see for example ([11]) and is given by

$$\int_M (\Delta u_\epsilon)^2 dv_g = \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big(1 - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\epsilon^2 + o(\epsilon^2)\Big).$$

Summarizing, we obtain

$$\begin{aligned}
&\int_M (\Delta u_\epsilon)^2 - a(x)|\nabla u_\epsilon|^2 + b(x)u_\epsilon^2 dv_g \\
&\leq \frac{\theta^{-n}}{K_0^{n/4} f(x_0)^{\frac{n-4}{4}}}\Big(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s \\
&\quad - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\epsilon^2 + o(\epsilon^2)\Big).
\end{aligned}$$

Now, we have

$$\begin{aligned}
J_\lambda(tu_\epsilon) &\leq J_0(tu_\epsilon) = \frac{t^2}{2}\|u_\epsilon\|^2 - \frac{t^N}{N}\int_M f(x)|u_\epsilon(x)|^N dv_g \\
&\leq \frac{\theta^{-n}}{K_0^{n/4} f(x_0)^{\frac{n-4}{4}}}\Big\{\frac{1}{2}t^2(1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s) - \frac{t^N}{N} \\
&\quad + \Big[\Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)} + \frac{S_g(x_0)}{6(n-1)}\Big)\frac{t^N}{N} - \frac{1}{2}t^2\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\Big]\epsilon^2\Big\} \\
&\quad + o(\epsilon^2)
\end{aligned}$$

and letting ϵ be small enough so that

$$1 + \epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}} A\|a\|_r + \epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}} B\|b\|_s \leq (1 + \|a\|_r + \|b\|_s)^{\frac{4}{n}}$$

and since the function $\varphi(t) = \alpha\frac{t^2}{2} - \frac{t^N}{N}$, with $\alpha > 0$ and $t > 0$, attains its maximum at $t_0 = \alpha^{\frac{1}{N-2}}$ and

$$\varphi(t_0) = \frac{2}{n}\alpha^{n/4}.$$

Consequently,

$$\begin{aligned}
J_\lambda(tu_\epsilon) &\leq \frac{2\theta^{-n}}{nK_0^{n/4} f(x_0)^{\frac{n-4}{4}}}\Big\{1 + \|a\|_r + \|b\|_s + \Big[\Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)} + \frac{S_g(x_0)}{6(n-1)}\Big)\frac{t_0^N}{N} \\
&\quad - \frac{1}{2}t_0^2\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)\Big]\epsilon^2\Big\} + o(\epsilon^2).
\end{aligned}$$

Taking into account the value of θ and putting

$$R(t) = \Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)} + \frac{S_g(x_0)}{6(n-1)}\Big)\frac{t^N}{N} - \frac{1}{2}\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_0)t^2$$

we obtain

$$\sup_{t\geq 0} J_\lambda(tu_\epsilon) < \frac{2}{nK_0^{n/4}(\max_{x\in M} f(x))^{\frac{n}{4}-1}}$$

provided that $R(t_0) < 0$; i.e.,

$$\frac{\Delta f(x_0)}{f(x_0)} < \Big(\frac{n(n^2+4n-20)}{3(n+2)(n-4)(n-6)}\frac{1}{(1 + \|a\|_r + \|b\|_s)^{n/4}} - \frac{n-2}{3(n-1)}\Big) S_g(x_0).$$

Which completes the proof. $\square$

6.1.1. *Application to compact Riemannian manifolds of dimension $n=6$.*

Theorem 6.2. *In case $n=6$, we suppose that at a point x_0 where f attains its maximum $S_g(x_0)>0$. Then the equation (1.2) has a non trivial solution.*

Proof. The same calculations as in case $n>6$ gives us

$$\int_M f(x)|u_\epsilon(x)|^N dv_g = \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big(1-\Big(\frac{\Delta f(x_0)}{2(n-2)f(x_0)}+\frac{S_g(x_0)}{6(n-2)}\Big)\epsilon^2+o(\epsilon^2)\Big).$$

Also, we have

$$\int_M a(x)|\nabla u_\epsilon|^2 dv_g \le \frac{\|a\|_r A}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\epsilon^{2-\frac{n}{r}}\theta^{-\frac{r}{r-1}}(1+o(\epsilon^2))$$

and

$$\int_M b(x)u_\epsilon^2 dv_g \le \frac{\|b\|_s B}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\epsilon^{4-\frac{n}{s}}\theta^{-\frac{s}{s-1}}+(1+o(\epsilon^2)).$$

where A and B are given by (6.2) and (6.3) respectively for $n=6$. The computations of the term $\int_M(\Delta u_\epsilon)^2 dv_g$ are well known (see for example [11])

$$\begin{aligned}
&\int_M(\Delta u_\epsilon)^2 dv(g)\\
&=\theta^{-n}(n-4)^2\Big(\frac{(n-4)n(n^2-4)}{f(x_0)}\Big)^{\frac{n-4}{4}}\frac{\omega_{n-1}}{2}\\
&\quad\times\Big(\frac{n(n+2)(n-2)}{(n-4)}I_n^{\frac{n}{2}-1}-\frac{2}{n}\theta^{-2}S_g(x_0)\epsilon^2\log(\frac{1}{\epsilon^2})+O(\epsilon^2)\Big).
\end{aligned}$$

$$\int_M(\Delta u_\epsilon)^2 dv_g = \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big(1-\frac{2(n-4)}{n^2(n^2-4)I_n^{\frac{n}{2}-1}}S_g(x_0)\epsilon^2\log(\frac{1}{\epsilon^2})+O(\epsilon^2)\Big).$$

Now summarizing and letting ϵ so that

$$1+\epsilon^{2-\frac{n}{r}}\theta^{-\frac{n}{r-1}}A\|b\|_s+\epsilon^{4-\frac{n}{s}}\theta^{-\frac{n}{s-1}}B\|a\|_r \le (1+\|a\|_r+\|b\|_s)^{\frac{4}{n}}$$

we obtain

$$\begin{aligned}
J_\lambda(u_\epsilon) &\le \frac{1}{2}\|u_\epsilon\|^2-\frac{1}{N}\int_M f(x)|u_\epsilon(x)|^N dv_g\\
&\le \frac{\theta^{-n}}{K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}\Big[\frac{t^2}{2}(1+\|a\|_r+\|b\|_s)^{1-\frac{4}{n}}-\frac{t^N}{N}\\
&\quad-\frac{n-4}{n^2(n^2-4)I_n^{\frac{n}{2}-1}}\theta^{-2}S_g(x_0)t^2\epsilon^2\log(\frac{1}{\epsilon^2})\Big]+O(\epsilon^2).
\end{aligned}$$

The same arguments as in the case $n>6$ allow us to infer that

$$\max_{t\ge 0}J_\lambda(tu_\epsilon) < \frac{2}{n\ K_0^{n/4}(f(x_0))^{\frac{n-4}{4}}}$$

if $S_g(x_0)>0$. Which completes the proof. □

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