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CONSTANT SIGN SOLUTIONS FOR SECOND-ORDER m-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. We will study the existence of constant sign solutions for the second-order m-point boundary-value problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

where $m \geq 3$, $\eta_i \in (0, 1)$ and $\alpha_i > 0$ for $i = 1, \dots, m-2$, with $\sum_{i=1}^{m-2} \alpha_i < 1$, we obtain that there exist at least a positive and a negative solution for the above problem. Our approach is based on unilateral global bifurcation theorem.

1. Introduction

In recent years, there has been considerable interests in the existence of nodal solutions of second-order m-point boundary value problems (BVPs) of the form

$$\begin{gathered} u''(t) + f(u(t)) = 0, \quad t \in (0, 1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{gathered} \tag{1.1}$$

see [1, 2, 6, 8, 9] and the references therein.

Ma and O'Regan [6] considered (1.1) under the assumption $f \in C^1(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$. They obtained the existence of nodal solutions for $f_0, f_\infty \in (0, \infty)$, where $f_0 = \lim_{u \to 0} \frac{f(u)}{u}$, $f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$.

In 2011, An [2] considered the problem

$$\begin{gathered} u''(t) + \lambda f(u(t)) = 0, \quad t \in (0, 1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{gathered} \tag{1.2}$$

under the assumption $f \in C^1(\mathbb{R} \backslash \{0\}, \mathbb{R}) \cap C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$. She investigated the global structure of nodal solutions of (1.2) in the case $f_0 = \infty$, $f_\infty \in [0, \infty]$, by using Rabinowit's global bifurcation theorem.

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From above results, we can see that the existence results are largely based on the assumption that f_0, f_∞ are constants and nonlinearity term is autonomous. It is interesting to know what will happen if f_0, f_∞ are functions and the nonlinear term is non-autonomous?

The above results rely largely on the direct computation of eigenvalues and eigenfunctions of the linear problem associated with (1.2), hence, it can not be extended to the more general problem. In view of the fact that the principle eigenvalue can be easily obtained by Krein-Rutman Theorem, in this paper, we obtain the existence of constant sign solution for

$$\begin{gathered} u''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{gathered} \tag{1.3}$$

by relating it to the principle eigenvalue of the associated linear problem. We make the following assumptions:

(H1) $\lambda_1 \le a(t) \equiv \lim_{|s|\to+\infty} \frac{f(t,s)}{s}$ uniformly on $[0,1]$, and the inequality is strict on some subset of positive measure in $(0,1)$; where λ_1 denotes the principle eigenvalue of

$$\begin{gathered} \psi''(t) + \lambda\psi(t) = 0, \quad t \in (0,1), \\ \psi(0) = 0, \quad \psi(1) = \sum_{i=1}^{m-2} \alpha_i \psi(\eta_i); \end{gathered} \tag{1.4}$$

(H2) $0 \le \lim_{|s|\to 0} \frac{f(t,s)}{s} \equiv c(t) \le \lambda_1$ uniformly on $[0,1]$, and all the inequalities are strict on some subset of positive measure in $(0,1)$;

(H3) $f(t,s)s > 0$ for all $t \in (0,1)$ and $s \ne 0$.

By applying the bifurcation theorem of López-Gómez [4, Theorem 6.4.3], we will establish the following results.

Theorem 1.1. *Suppose that $f(t,u)$ satisfies* (H1)–(H3). *Then* (1.3) *possesses at least one positive and one negative solution.*

Similar result is obtained under the following assumptions.

(H1') $\lambda_1 \ge a(t) \equiv \lim_{|s|\to+\infty} \frac{f(t,s)}{s} \ge 0$ uniformly on $[0,1]$, and all the inequalities are strict on some subset of positive measure in $(0,1)$, where λ_1 denotes the principle eigenvalue of (1.4);

(H2') $\lim_{|s|\to 0} \frac{f(t,s)}{s} \equiv c(t) \ge \lambda_1$ uniformly on $[0,1]$, and the inequality is strict on some subset of positive measure in $(0,1)$.

Theorem 1.2. *Suppose that $f(t,u)$ satisfies* (H1'), (H2'), (H3). *Then* (1.3) *possesses at least one positive and one negative solution.*

The existence of constant sign solutions of (1.3) is related to the eigenvalue problem

$$\begin{gathered} u''(t) + \mu f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{gathered} \tag{1.5}$$

where $\mu > 0$ is a parameter. Therefore, we will study the bifurcation phenomena for (1.5) with crossing nonlinearity. Moreover, the bifurcation point of (1.5) is related to the principle eigenvalues of the problem

$$\begin{gathered} u''(t) + \mu c(t)u(t) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{gathered} \tag{1.6}$$

it is well-known that there exists a principle eigenvalue $\mu_1(c(t))$ of (1.6) (see [10]).

The rest of the paper is organized as follows: in Section 2, we state some notations and preliminary results. In Section 3, we prove the main results.

2. Notation and preliminary results

To show the constant sign solutions of (1.5), we consider the operator equation

$$u = \mu T u. \tag{2.1}$$

This equations are usually called nonlinear eigenvalue problems. López-Gómez [4] studied a nonlinear eigenvalue problem of the form

$$u = \mu T u + H(\mu, u), \tag{2.2}$$

where $H(\mu, u) = o(\|u\|)$ as $\|u\| \to 0$ uniformly for μ on a bounded interval, and T is a linear completely continuous operator on a Banach space X. A solution of (2.2) is a pair $(\mu, u) \in \mathbb{R} \times X$, which satisfies (2.2). The closure of the set nontrivial solutions of (2.2) is denoted by $\mathcal{C}$. Let $\Sigma(T)$ denote the set of eigenvalues of linear operator T. López-Gómez [4] established the following results.

Lemma 2.1 ([4, Theorem 6.4.3]). *Assume $\Sigma(T)$ is discrete. Let $\mu_0 \in \Sigma(T)$ such that ind$(I - \mu T, \theta)$ changes sign as μ crosses μ_0, then each of the components $\mathcal{C}$ (denote the components of S emanating of (μ, θ) at (μ_0, θ)), satisfies $(\mu_0, \theta) \in \mathcal{C}$, and either*

- (i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times X$;*
- (ii) *there exist $\lambda_1 \in \Sigma(T) \setminus \{\lambda_0\}$ such that $(\lambda_1, \theta) \in \mathcal{C}$; or*
- (iii) *$\mathcal{C}$ contains a point*

$$(\iota, y) \in \mathbb{R} \times (V \setminus \{\theta\}),$$

where V is the complement of $\operatorname{span}\{\varphi_{\mu_0}\}$, φ_{μ_0} denotes the eigenfunction corresponding to eigenvalue μ_0.

Lemma 2.2 ([4, Theorem 6.5.1]). *Under the assumptions:*

- (A) *X is an ordered Banach space, whose positive cone, denoted by P, is normal and has a nonempty interior;*
- (B) *The family $\Upsilon(\mu)$ has the special form*

$$\Upsilon(\mu) = I_X - \mu T,$$

where T is a compact strongly positive operator, i.e., $T(P \setminus \{\theta\}) \subset$ int P;

- (C) *The solutions of $u = \mu T u + H(\mu, u)$ satisfy the strong maximum principle.*

Then the following assertions are true:

- (1) *$\operatorname{Spr}(T)$ is a simple eigenvalue of T, having a positive eigenfunction denoted by $\psi_0 > 0$, i.e., $\psi_0 \in \operatorname{int} P$, and there is no other eigenvalue of T with a positive eigenfunction;*

(2) *For every* $y \in$*int* P, *the equation*

$$u - \mu Tu = y$$

has exactly one positive solution if $\mu < \frac{1}{\mathrm{Spr}(T)}$, *whereas it does not admit a positive solution if* $\mu \geq \frac{1}{\mathrm{Spr}(T)}$.

Lemma 2.3 (cite[Theorem 2.5]b1)**.** *Assume* $T : X \to X$ *is a linear completely continuous operator, and 1 is not an eigenvalue of* T, *then*

$$\mathrm{ind}(I - T, \theta) = (-1)^{\beta},$$

where β *is the sum of the algebraic multiplicities of the eigenvalues of* T *large than 1, and* $\beta = 0$ *if* T *has no eigenvalue of this kind.*

Let Y be the space $C[0,1]$ with the norm $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Let

$$E = \{u \in C^1[0,1] : u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$$

with the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u'(t)|.$$

Define $L : D(L) \to Y$ by setting

$$Lu(t) := -u''(t), \quad t \in [0,1],\ u \in D(L),$$

where

$$D(L) = \{u \in C^2[0,1] : u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}.$$

Then $L^{-1} : Y \to E$ is compact.

Let $\mathbb{E} = \mathbb{R} \times E$ under the product topology. As in [7], we add the point $\{(\mu, \infty)|\ \mu \in \mathbb{R}\}$ to our space $\mathbb{E}$. For any $u \in C^1[0,1]$, if $u(x_0) = 0$, then x_0 is a simple zero of u if $u'(x_0) \neq 0$. For $\nu \in \{+,-\}$, define:

- S_1^ν is the set of functions such that
  - (i) $u(0) = 0$, $\nu u'(0) > 0$;
  - (ii) u has constant sign in $(0,1)$.
- T_1^ν is the set of functions such that
  - (i) $u(0) = 0$, $\nu u'(0) > 0$ and $u'(1) \neq 0$;
  - (ii) u' has exactly one simple zero point in $(0,1)$;
  - (iii) u has a zero strictly between each two consecutive zeros of u'.

Obviously, if $u \in T_1^\nu$, then $u \in S_1^\nu$. The sets T_1^ν are disjoint and open in E, (see [8, Remark 2.2]). Finally, let $\phi_1^\nu = \mathbb{R} \times T_1^\nu$.

Furthermore, let $\zeta \in C([0,1] \times \mathbb{R})$ be such that $f(t,u) = c(t)u + \zeta(t,u)$ with

$$\lim_{|u|\to 0} \frac{\zeta(t,u)}{u} = 0 \quad \text{uniformly on } [0,1].$$

Let

$$\bar{\zeta}(t,u) = \max_{0\leq|s|\leq u} |\zeta(t,s)| \quad \text{for } t \in [0,1].$$

Then $\bar{\zeta}$ is nondecreasing with respect to u and

$$\lim_{u\to 0^+} \frac{\bar{\zeta}(t,u)}{u} = 0. \tag{2.3}$$

From this equality, it follows that

$$\frac{\zeta(t,u)}{\|u\|} \le \frac{\bar{\zeta}(t,|u|)}{\|u\|} \le \frac{\bar{\zeta}(t,\|u\|_\infty)}{\|u\|} \le \frac{\bar{\zeta}(t,\|u\|)}{\|u\|} \to 0, \quad \text{as } \|u\| \to 0$$

uniformly for $t \in [0,1]$.

Let us study

$$Lu - \mu c(t)u = \mu\zeta(t,u) \tag{2.4}$$

as a bifurcation problem from the trivial solution $u \equiv 0$. Equation (2.4) can be converted to the equivalent equation

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)[\mu c(s)u(s) + \mu\zeta(s,u(s))]ds \\ &:= \mu L^{-1}[c(t)u(t)] + \mu L^{-1}[\zeta(t,u(t))], \end{aligned}$$

where $G(t,s)$ denotes the Green's function of $Lu = 0$.

We note that $\|L^{-1}[\zeta(t,u(t))]\| = o(\|u\|)$ for u near 0 in E. Since

$$\begin{aligned} \|L^{-1}[\zeta(t,u(t))]\| &= \max_{t\in[0,1]} |\int_0^1 G(t,s)\zeta(s,u(s))ds| + \max_{t\in[0,1]} |\int_0^1 G_t(t,s)\zeta(s,u(s))ds| \\ &\le C\|\zeta(t,u(t))\|_\infty. \end{aligned}$$

Lemma 2.4 ([8, Proposition 4.1])**.** *If $(\mu,u) \in \mathbb{E}$ is a non-trivial solution of* (2.4)*, then $u \in T_1^\nu$ for $\nu \in \{+,-\}$.*

Lemma 2.5. *For $\nu \in \{+,-\}$, there exists a continuum $\mathcal{C}_1^\nu \subset \mathbb{E}$ of solutions of* (2.4) *with the properties:*

- (i) $(\mu_1(c(t)),\theta) \in \mathcal{C}_1^\nu$;
- (ii) $\mathcal{C}_1^\nu \backslash \{(\mu_1(c(t)),\theta)\} \subset \mathbb{R} \times T_1^\nu$;
- (iii) $\mathcal{C}_1^\nu$ *is unbounded in* $\mathbb{E}$*, where* $\mu_1(c(t))$ *denotes the principle eigenvalue of* (1.6).

Proof. From above, we know that problem (2.4) is of the form considered in [4], and satisfies the general hypotheses imposed in that paper.

From [10], we know that the principle eigenvalues of (1.6) is simple. So for $\nu \in \{+,-\}$, combining Lemma 2.1 with Lemma 2.3, we know that there exists a continuum, $\mathcal{C}_1^\nu \subset \mathbb{E}$, of solutions of (2.4) such that:

- (a) $\mathcal{C}_1^\nu$ is unbounded and $(\mu_1(c(t)),\theta) \in \mathcal{C}_1^\nu, \mathcal{C}_1^\nu \backslash \{(\mu_1(c(t)),\theta)\} \subset \mathbb{E}$, or
- (b) $(\mu_j(c(t)),\theta) \in \mathcal{C}_1^\nu$, where $j \in \mathbb{N}, \mu_j(c(t))$ is another eigenvalue of (1.6) if possible, or
- (c) $\mathcal{C}_1^\nu$ contains a point

  $$(\iota, y) \in \mathbb{R} \times (V \backslash \{\theta\}),$$

  where V is the complement of span$\{\varphi_1\}$, φ_1 denotes the eigenfunction corresponding to principle eigenvalue $\mu_1(c(t))$.

We finally prove that the first choice (a) is the only possibility. In fact, all functions belong to the continuum set $\mathcal{C}_1^\nu$ are constant sign, this implies that it is impossible to exist $(\mu_j(c(t)),\theta) \in \mathcal{C}_1^\nu, j \in \mathbb{N}, j \ne 1$, where $\mu_j(c(t))$ is another eigenvalue of (1.6) if possible. If this happened, it will be contracted with the definition of S_1^ν.

Next, we will prove (c) is impossible, suppose (c) occurs, without loss of generality, suppose there exists a point $(\iota, y) \in \mathbb{R} \times (V\backslash\{\theta\}) \cap \mathcal{C}_1^+$. Define

$$P = \{u \in C^1[0,1] : u(t) \geq 0,\ t \in [0,1]\},$$

then P is a normal cone and has a nonempty interior, and $\mathcal{C}_1^+\backslash\{(\mu_1(c(t)), \theta)\} \subset \operatorname{int} P$.

Note that as the complement V of $\operatorname{Span}\{\varphi_1\}$ in E, we can take

$$V := R[c(t)I_E - \frac{1}{\mu_1(c(t))}L].$$

Thus, for this choice of V, if the component $\mathcal{C}_1^+$ contains a point

$$(\iota, y) \in \mathbb{R} \times (V\backslash\{\theta\}) \cap \mathcal{C}_1^+.$$

Then there exists $u \in E$ for which

$$c(t)u - \frac{1}{\mu_1(c(t))}Lu = y > 0, \quad \text{in } (0,1).$$

Thus, for each sufficiently large $\eta > 0$, we have that $c(t)u + \eta\varphi_1(t) > 0$ in $(0,1)$ and

$$c(t)u + \eta c(t)\varphi_1(t) - \frac{1}{\mu_1(c(t))}L(u + \eta\varphi_1) = y > 0 \quad \text{in } (0,1).$$

Hence, by Lemma 2.2, we have

$$\operatorname{Spr}(\frac{1}{\mu_1(c(t))}L) < 1,$$

which is impossible. since $Spr(L) = \mu_1(c(t))$. □

3. Proof of main results

Proof of Theorem 1.1. Theorem 1.2 is proved in similar manner. It is clear that any solution of (2.4) of the form $(1, u)$ yields a solution u of (1.3). We will show $\mathcal{C}_1^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

By $\mu_1(c(t))$ being strict decreasing with respect to $c(t)$ (see [5]), where $\mu_1(c(t))$ is the principle eigenvalue of (1.6), we have $\mu_1(c(t)) > \mu_1(\lambda_1) = 1$.

Let $(\mu_n, u_n) \in \mathcal{C}_1^\nu$ with $u_n \not\equiv 0$ satisfies

$$\mu_n + \|u_n\| \to +\infty.$$

We note that $\mu_n > 0$ for all $n \in \mathbb{N}$, since $(0, \theta)$ is the only solution of (2.4) for $\mu = 0$ and $\mathcal{C}_1^\nu \cap (\{0\} \times E) = \emptyset$.

Step 1: We show that if there exists a constant $M > 0$, such that $\mu_n \subset (0, M]$ for $n \in \mathbb{N}$ large enough, then $\mathcal{C}_1^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. In this case it follows that $\|u_n\| \to \infty$.

Let $\xi \in C([0,1] \times \mathbb{R})$ be such that

$$f(t, u) = a(t)u + \xi(t, u)$$

with

$$\lim_{|u| \to +\infty} \frac{\xi(t, u)}{u} = 0 \quad \text{uniformly on } [0,1]. \tag{3.1}$$

We divide the equation

$$Lu_n - \mu_n a(t)u_n = \mu_n \xi(t, u_n) \tag{3.2}$$

by $\|u_n\|$ and set $\bar{u}_n = \frac{u_n}{\|u_n\|}$. Since $\bar{u}_n$ is bounded in $C^2[0,1]$, after taking a subsequence if necessary, we have that $\bar{u}_n \to \bar{u}$ for some $\bar{u} \in E$ with $\|\bar{u}\| = 1$. By (3.1), using the similar proof of (2.3), we have that

$$\lim_{n\to+\infty} \frac{\xi(t, u_n(t))}{\|u_n\|} = 0 \quad \text{in} Y.$$

Thus, we obtain

$$-\bar{u}'' - \overline{\mu}(a(t))a(t)\bar{u} = 0,$$

where $\overline{\mu}(a(t)) = \lim_{n\to+\infty} \mu_n$.

It is clear that $\overline{u} \in \overline{\mathcal{C}_1^{\nu}} \subseteq \mathcal{C}_1^{\nu}$, since $\mathcal{C}_1^{\nu}$ is closed in $\mathbb{R} \times E$. Therefore, $\overline{\mu}(a(t))$ is the principle eigenvalue of (1.6) corresponding to weight function $a(t)$.

By the strict decreasing of $\overline{\mu}(a(t))$ with respect to $a(t)$ (see [5]), we have $\overline{\mu}(a(t)) < \overline{\mu}(\lambda_1) = 1$. Therefore, $\mathcal{C}_1^{\nu}$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

Step 2: We show that there exists a constant M such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough. On the contrary, we suppose that $\lim_{n\to+\infty} \mu_n = +\infty$. On the other hand, we note that

$$-u_n'' = \mu_n \frac{f(t, u_n)}{u_n} u_n.$$

We have $\mu_n \frac{f(t,u_n)}{u_n} > \lambda_1$ for n large enough and all $t \in (0, 1]$. We get u_n must change its sign in $(0, 1)$ for n large enough, which contradicts the fact that $u_n \in T_1^{\nu}$. Therefore,

$$\mu_n \le M$$

for some constant positive M and $n \in \mathbb{N}$ sufficiently large. □

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