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EXISTENCE OF PERIODIC SOLUTIONS FOR NON-AUTONOMOUS SECOND-ORDER HAMILTONIAN SYSTEMS

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Abstract. The purpose of this paper is to study the existence of periodic solutions for a class of non-autonomous second order Hamiltonian systems. New results are obtained by using the least action principle and the minimax methods, without the so-called Ahmad-Lazer-Paul type condition.

1. Introduction and main results

Consider the second-order Hamiltonian system

$$\begin{gathered} \ddot{u}(t) = \nabla F\big(t, u(t)\big), \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{gathered} \tag{1.1}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$, continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in \mathscr{L}^1(0, T; \mathbb{R}^+)$ such that
$$|F(t, x)| \le a(|x|)b(t), \quad |\nabla F(t, x)| \le a(|x|)b(t)$$
for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The corresponding functional $\varphi : H^1_T \to \mathbb{R}$,

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F\big(t, u(t)\big) dt$$

is continuously differentiable and weakly lower semi-continuous on H^1_T (see [4]), where H^1_T is the usual Sobolev space with the norm

$$\|u\| = \Big[\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\Big]^{1/2}.$$

It is well know that the solutions of problem (1.1) correspond to the critical points of φ.

Problem (1.1) has been extensively studied in the past thirty years; see for example the references in this article. Under some suitable solvability conditions,

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such as the coercivity condition (cf. [2]), the periodicity condition (cf. [5]), the convexity condition (cf. [6]), the subadditive condition (cf. [10]), the existence and multiplicity results are obtained. We note that in many contributions (for example, see [1, 3, 9, 12, 13, 14, 15]), the following condition was assumed:

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt = \infty \quad \text{or} \quad -\infty, \tag{1.2}$$

where α is a constant. In this article, instead of (1.2), we discuss the existence of periodic solutions of (1.1) under a weak condition that $\liminf_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt$ or $\limsup_{|x|\to\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt$ has appropriate lower or upper bound.

Our main results are as follows:

Theorem 1.1. *Suppose that $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(F1) *there exist $f, g \in \mathscr{L}^1(0,T;\mathbb{R}^+)$ and $\gamma \in [0,1)$ such that*

$$|\nabla F_1(t,x)| \le f(t)|x|^\gamma + g(t),$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(F2) *there exist constants $r > 0$ and $\alpha \in [0,2)$ such that*

$$(\nabla F_2(x) - \nabla F_2(y), x-y) \ge -r|x-y|^\alpha,$$

for all $x, y \in \mathbb{R}^N$;
item[(F3)]

$$\liminf_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F(t,x)\,dt \ge \frac{T^2}{8\pi^2} \int_0^T f^2(t)\,dt.$$

Then problem (1.1) *has at least one periodic solution which minimizes φ on H_T^1.*

Theorem 1.2. *Suppose that $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy assumptions* (A), (F1), (F2) *and the following conditions:*

(F4) *there exist $\delta \in [0,2)$ and $C > 0$ such that*

$$(\nabla F_2(x) - \nabla F_2(y), x-y) \le C|x-y|^\delta,$$

for all $x, y \in \mathbb{R}^N$;

(F5)

$$\limsup_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F(t,x)\,dt \le -\frac{3T^2}{8\pi^2} \int_0^T f^2(t)\,dt.$$

Then problem (1.1) *has at least one periodic solution which minimizes φ on H_T^1.*

Theorem 1.3. *Suppose that $F(t,x) = F_1(t,x) + F_2(x)$, where F_1 and F_2 satisfy assumptions* (A), (F1)*, and the following conditions:*

(F2') *there exists a constant $0 < r < 4\pi^2/T^2$, such that*

$$(\nabla F_2(x) - \nabla F_2(y), x-y) \ge -r|x-y|^2;$$

for all $x, y \in \mathbb{R}^N$;

(F3')

$$\liminf_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F(t,x)\,dt \ge \frac{T^2}{2(4\pi^2 - rT^2)} \int_0^T f^2(t)\,dt.$$

Then problem (1.1) *has at least one periodic solution which minimizes φ on H_T^1.*

Theorem 1.4. *Suppose that $F = F_1 + F_2$, where F_1 and F_2 satisfy assumptions* (A), (F1) *and the following conditions:*

(F6) *there exist $k \in \mathscr{L}^1(0,T;\mathbb{R}^+)$ and (λ,μ)-subconvex potential $G : \mathbb{R}^N \to \mathbb{R}$ with $\lambda > 1/2$ and $0 < \mu < 2\lambda^2$, such that*
$$(\nabla F_2(t,x), y) \geq -k(t)G(x-y),$$
for all $x, y \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(F7)
$$\limsup_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F_1(t,x)\,dt \leq -\frac{3T^2}{8\pi^2} \int_0^T f^2(t)\,dt,$$
$$\limsup_{|x|\to\infty} |x|^{-\beta} \int_0^T F_2(t,x)\,dt \leq -8\mu \max_{|s|\leq 1} G(s) \int_0^T k(t)\,dt,$$
where $\beta = \log_{2\lambda}(2\mu)$.

Then problem (1.1) *has at least one periodic solution which minimizes φ on H_T^1.*

Remark 1.5. Theorems 1.1–1.3 extend some existing results. On the one hand, we decomposed the potential F into F_1 and F_2. On the other hand, we weaken the so-called Ahmad-Lazer-Paul type condition (1.2) as conditions (F3), (F5) and (F3'). Note that [13, Theorem 2] and [3, Theorem 1] are the direct corollaries of Theorem 1.1 and Theorem 1.3 respectively. If $F_2 = 0$, [11, Theorems 1 and 2] are special cases of Theorem 1.1 and Theorem 1.2 respectively. Some examples of F are given in section 3, which are not covered in the references. Moreover, our Theorem 1.4 is a new result.

2. Proof of Theorems

For $u \in H_T^1$, let
$$\bar{u} = \frac{1}{T}\int_0^T u(t)\,dt, \quad \tilde{u}(t) = u(t) - \bar{u}.$$
The following inequalities are well known (cf. [4]):
$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12}\|\dot{u}\|_{L^2}^2 \quad \text{(Sobolev's inequality)},$$
$$\|\tilde{u}\|_{L^2}^2 \leq \frac{T^2}{4\pi^2}\|\dot{u}\|_{L^2}^2 \quad \text{(Wirtinger's inequality)}$$
For convenience, we denote
$$M_1 = \Big(\int_0^T f^2(t)\,dt\Big)^{1/2}, \quad M_2 = \int_0^T f(t)\,dt, \quad M_3 = \int_0^T g(t)\,dt.$$
Now we give the proofs of the main results.

Proof of Theorem 1.1. By (F3), we can choose an $a_1 > T^2/(4\pi^2)$ such that
$$\liminf_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F(t,x)\,dt > \frac{a_1}{2} M_1^2. \tag{2.1}$$

By (F1) and the Sobolev's inequality, for any $u \in H_T^1$,

$$\begin{aligned}
&\Big|\int_0^T [F_1(t,u(t)) - F_1(t,\bar{u})]\,dt\Big| \\
&= \Big|\int_0^T \int_0^1 \big(\nabla F_1\big(t,\bar{u}+s\tilde{u}(t)\big),\tilde{u}(t)\big)\,ds\,dt\Big| \\
&\le \int_0^T \int_0^1 f(t)|\bar{u}+s\tilde{u}(t)|^\gamma|\tilde{u}(t)|\,ds\,dt + \int_0^T \int_0^1 g(t)|\tilde{u}(t)|\,ds\,dt \\
&\le |\bar{u}|^\gamma\Big(\int_0^T f^2(t)\,dt\Big)^{1/2}\Big(\int_0^T |\tilde{u}(t)|^2\,dt\Big)^{1/2} \\
&\quad + \|\tilde{u}\|_\infty^{\gamma+1}\int_0^T f(t)\,dt + \|\tilde{u}\|_\infty\int_0^T g(t)\,dt \\
&\le \frac{1}{2a_1}\|\tilde{u}\|_{L^2}^2 + \frac{a_1}{2}M_1^2|\bar{u}|^{2\gamma} + M_2\|\tilde{u}\|_\infty^{\gamma+1} + M_3\|\tilde{u}\|_\infty \\
&\le \frac{T^2}{8\pi^2 a_1}\|\dot{u}\|_{L^2}^2 + \frac{a_1}{2}M_1^2|\bar{u}|^{2\gamma} + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}}M_2\|\dot{u}\|_{L^2}^{\gamma+1} + \big(\frac{T}{12}\big)^{1/2}M_3\|\dot{u}\|_{L^2}
\end{aligned} \tag{2.2}$$

Similarly, by (F2) and the Sobolev's inequality, for any $u \in H_T^1$,

$$\begin{aligned}
\int_0^T [F_2(u(t)) - F_2(\bar{u})]dt &= \int_0^T \int_0^1 \frac{1}{s}\left(\nabla F_2(\bar{u}+s\tilde{u}(t)) - \nabla F_2(\bar{u}), s\tilde{u}(t)\right)ds\,dt \\
&\ge -\int_0^T \int_0^1 rs^{\alpha-1}|\tilde{u}(t)|^\alpha\,ds\,dt \\
&\ge -\frac{rT}{\alpha}\|\tilde{u}\|_\infty^\alpha \\
&\ge -\frac{rT}{\alpha}\big(\frac{T}{12}\big)^{\alpha/2}\|\dot{u}\|_{L^2}^\alpha
\end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T [F_1(t,u(t)) - F_1(t,\bar{u})]\,dt \\
&\quad + \int_0^T [F_2(u(t)) - F_2(\bar{u})]\,dt + \int_0^T F(t,\bar{u})dt \\
&\ge \Big(\frac{1}{2} - \frac{T^2}{8\pi^2 a_1}\Big)\|\dot{u}\|_{L^2}^2 - \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}}M_2\|\dot{u}\|_{L^2}^{\gamma+1} - \big(\frac{T}{12}\big)^{1/2}M_3\|\dot{u}\|_{L^2} \\
&\quad - \frac{rT}{\alpha}\big(\frac{T}{12}\big)^{\alpha/2}\|\dot{u}\|_{L^2}^\alpha + |\bar{u}|^{2\gamma}\Big(|\bar{u}|^{-2\gamma}\int_0^T F(t,\bar{u})dt - \frac{a_1}{2}M_1^2\Big)
\end{aligned}$$

for all $u \in H_T^1$, which implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$, due to (2.1) and $\gamma < 1$.

By the least action principle (see [4, Theorem 1.1 and Corollary 1.1]), the proof is complete. □

Proof of Theorem 1.2. Step 1. We firstly show that φ satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence, that is, $\varphi'(u_n) \to 0$ as $n \to 0$ and $\{\varphi(u_n)\}$

is bounded. By (F5), we can choose an $a_2 > T^2/(4\pi^2)$ such that

$$\limsup_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F(t,x)\,dt < -\Big(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\Big)M_1^2. \tag{2.4}$$

In a way similar to the proof of Theorem 1.1, one has

$$\begin{aligned}
&\int_0^T (\nabla F_1(t,u_n(t)), \tilde{u}_n(t))\ dt \\
&\le \frac{T^2}{8\pi^2 a_2}\|\dot{u}_n\|_{L^2}^2 + \frac{a_2}{2}M_1^2|\bar{u}_n|^{2\gamma} + \big(\frac{T}{12}\big)^{\frac{(\gamma+1)}{2}} M_2\|\dot{u}_n\|_{L^2}^{\gamma+1} + \big(\frac{T}{12}\big)^{1/2} M_3\|\dot{u}_n\|_{L^2}
\end{aligned} \tag{2.5}$$

and

$$\int_0^T (\nabla F_2(u_n(t)), \tilde{u}_n(t))\,dt \ge -\frac{rT}{\alpha}\big(\frac{T}{12}\big)^{\alpha/2}\|\dot{u}_n\|_{L^2}^{\alpha}\int_0^T r(t)dt$$

for all n. Hence one has

$$\begin{aligned}
\|\tilde{u}_n\| &\ge (\varphi'(u_n), \tilde{u}_n) \\
&= \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt \\
&\ge \Big(1 - \frac{T^2}{8\pi^2 a_2}\Big)\|\dot{u}_n\|_{L^2}^2 - \frac{a_2}{2}M_1^2|\bar{u}_n|^{2\gamma} - \big(\frac{T}{12}\big)^{\frac{(\gamma+1)}{2}} M_2\|\dot{u}_n\|_{L^2}^{\gamma+1} \\
&\quad - \big(\frac{T}{12}\big)^{1/2} M_3\|\dot{u}_n\|_{L^2} - \frac{rT}{\alpha}\big(\frac{T}{12}\big)^{\alpha/2}\|\dot{u}_n\|_{L^2}^{\alpha}
\end{aligned} \tag{2.6}$$

for large n. It follows from Wirtinger's inequality that

$$\|\tilde{u}_n\| \le \frac{\big(T^2 + 4\pi^2\big)^{1/2}}{2\pi}\|\dot{u}_n\|_{L^2}. \tag{2.7}$$

By (2.6) and (2.7),

$$\begin{aligned}
\frac{a_2}{2}M_1^2|\bar{u}_n|^{2\gamma} &\ge \Big(1 - \frac{T^2}{8\pi^2 a_2}\Big)\|\dot{u}_n\|_{L^2}^2 - \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\|\dot{u}_n\|_{L^2}^{\gamma+1} - \big(\frac{T}{12}\big)^{\frac{1}{2}} M_3\|\dot{u}_n\|_{L^2} \\
&\quad - \frac{rT}{\alpha}\big(\frac{T}{12}\big)^{\frac{\alpha}{2}}\|\dot{u}_n\|_{L^2}^{\alpha} - \frac{\big(T^2 + 4\pi^2\big)^{1/2}}{2\pi}\|\dot{u}_n\|_{L^2} \\
&\ge \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + C_1,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
C_1 = \min_{s\in[0,+\infty)} \Big\{ &\frac{4\pi^2 a_2 - T^2}{8\pi^2 a_2}s^2 - \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2 s^{\gamma+1} - \big[\frac{rT}{\alpha}\big(\frac{T}{12}\big)^{\alpha/2}\big]s^{\alpha} \\
&- \big[\big(\frac{T}{12}\big)^{1/2} M_3 + \frac{\big(T^2 + 4\pi^2\big)^{1/2}}{2\pi}\big]s\Big\}.
\end{aligned}$$

Note that $a_2 > T^2/(4\pi^2)$ implies $-\infty < C_1 < 0$. Hence, it follows from (2.8) that

$$\|\dot{u}_n\|_{L^2}^2 \le a_2 M_1^2|\bar{u}_n|^{2\gamma} - 2C_1, \tag{2.9}$$

and then

$$\|\dot{u}_n\|_{L^2} \le \sqrt{a_2}M_1|\bar{u}_n|^{\gamma} + C_2, \tag{2.10}$$

where $0 < C_2 < +\infty$. In a way similar to the proof of Theorem 1.1, we have

$$
\begin{aligned}
&\Big|\int_0^T [F_1(t,u(t)) - F_1(t,\bar{u})]dt\Big| \\
&\le M_1|\bar{u}|^{\gamma}\|\tilde{u}\|_{L^2} + M_2\|\tilde{u}\|_{\infty}^{\gamma+1} + M_3\|\tilde{u}\|_{\infty} \\
&\le \frac{\pi}{\sqrt{a_2}T}\,\|\tilde{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2}T}{4\pi}M_1^2|\bar{u}_n|^{2\gamma} + M_2\,\|\tilde{u}_n\|_{\infty}^{\gamma+1} + M_3\|\tilde{u}_n\|_{\infty} \\
&\le \frac{T}{4\pi\sqrt{a_2}}\,\|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2}T}{4\pi}M_1^2|\bar{u}_n|^{2\gamma} + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\,\|\dot{u}_n\|_{L^2}^{\gamma+1} \\
&\quad + \big(\frac{T}{12}\big)^{1/2} M_3\|\dot{u}_n\|_{L^2}.
\end{aligned}
\tag{2.11}
$$

By (F4), we obtain

$$
\begin{aligned}
&\int_0^T [F_2(u_n(t)) - F_2(\bar{u}_n)]\,dt \\
&= \int_0^T\int_0^1 \frac{1}{s}\,(\nabla F_2(\bar{u}_n + s\tilde{u}_n(t)) - \nabla F_2(\bar{u}_n), s\tilde{u}_n(t))\,ds\,dt \\
&\le \int_0^T\int_0^1 Cs^{\delta-1}|\tilde{u}_n(t)|^{\delta}\,ds\,dt \le \frac{CT}{\delta}\,\|\tilde{u}_n\|_{\infty}^{\delta} \\
&\le \frac{CT}{\delta}\big(\frac{T}{12}\big)^{\delta/2}\,\|\dot{u}_n\|_{L^2}^{\delta}\,.
\end{aligned}
$$

It follows from the boundedness of $\{\varphi(u_n)\}$ and (2.9)-2.11 that

$$
\begin{aligned}
C_3 &\le \varphi(u_n) \\
&= \frac{1}{2}\,\|\dot{u}_n\|_{L^2}^2 + \int_0^T [F_1\,(t,u_n(t)) - F_1\,(t,\bar{u}_n)]\,dt + \int_0^T [F_2\,(u_n(t)) - F_2\,(\bar{u}_n)]\,dt \\
&\quad + \int_0^T F(t,\bar{u}_n)dt \\
&\le \Big(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\Big)\,\|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_2}T}{4\pi}M_1^2|\bar{u}_n|^{2\gamma} + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\,\|\dot{u}_n\|_{L^2}^{\gamma+1} \\
&\quad + \big(\frac{T}{12}\big)^{1/2} M_3\|\dot{u}_n\|_{L^2} + \frac{CT}{\delta}\big(\frac{T}{12}\big)^{\delta/2}\,\|\dot{u}_n\|_{L^2}^{\delta} + \int_0^T F(t,\bar{u}_n)dt \\
&\le \Big(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_2}}\Big)\Big(a_2M_1^2|\bar{u}_n|^{2\gamma} - 2C_1\Big) + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2(\sqrt{a_2}M_1|\bar{u}_n|^{\gamma} + C_2)^{\gamma+1} \\
&\quad + \big(\frac{T}{12}\big)^{1/2} M_3\,(\sqrt{a_2}M_1|\bar{u}_n|^{\gamma} + C_2) + \frac{CT}{\delta}\big(\frac{T}{12}\big)^{\delta/2}(\sqrt{a_2}M_1|\bar{u}_n|^{\gamma} + C_2)^{\delta} \\
&\quad + \frac{\sqrt{a_2}T}{4\pi}M_1^2|\bar{u}_n|^{2\gamma} + \int_0^T F(t,\bar{u}_n)dt \\
&\le \Big(\frac{a_2}{2} + \frac{\sqrt{a_2}T}{2\pi}\Big)M_1^2|\bar{u}_n|^{2\gamma} + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\Big(2^{\gamma}(\sqrt{a_2}M_1)^{\gamma+1}|\bar{u}_n|^{\gamma(\gamma+1)} + 2^{\gamma}C_2^{\gamma+1}\Big) \\
&\quad + \frac{CT}{\delta}\big(\frac{T}{12}\big)^{\delta/2}\Big(2^{\delta-1}(\sqrt{a_2}M_1)^{\delta}|\bar{u}_n|^{\gamma\delta} + 2^{\delta-1}C_2^{\delta}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(\frac{T}{12}\Big)^{1/2} M_3\,(\sqrt{a_2}M_1|\bar{u}_n|^{\gamma}+C_2) - \Big(1+\frac{T}{2\pi\sqrt{a_2}}\Big)C_1 + \int_0^T F\,(t,\bar{u}_n)\,dt \\
&= |\bar{u}_n|^{2\gamma}\Big[|\bar{u}_n|^{-2\gamma}\int_0^T F(t,\bar{u}_n)dt + \Big(\frac{a_2}{2}+\frac{\sqrt{a_2}T}{2\pi}\Big)M_1^2 \\
&+ \Big(\frac{a_2T}{12}\Big)^{\frac{\gamma+1}{2}} 2^{\gamma} M_1^{\gamma+1}M_2|\bar{u}_n|^{\gamma(\gamma-1)} + \Big(\frac{a_2T}{12}\Big)^{1/2} M_1M_3|\bar{u}_n|^{-\gamma} \\
&+ \frac{CT}{\delta}\Big(\frac{a_2T}{12}\Big)^{\delta/2} 2^{\delta-1}M_1^{\delta}|\bar{u}_n|^{\gamma(\delta-2)}\Big] + \Big(\frac{T}{12}\Big)^{\frac{\gamma+1}{2}} 2^{\gamma}M_2C_2^{\gamma+1} \\
&+ \frac{CT}{\delta}\Big(\frac{T}{12}\Big)^{\delta/2} 2^{\delta-1}C_2^{\delta} + \Big(\frac{T}{12}\Big)^{1/2} M_3C_2 - \Big(1+\frac{T}{2\pi\sqrt{a_2}}\Big)C_1
\end{aligned}
$$

for large n. The above inequality and (2.4) imply that $\{|\bar{u}|\}$ is bounded. Hence $\{u_n\}$ is bounded by (2.9). Arguing as in the proof of Proposition 4.1 of [4], we conclude that (PS) condition is satisfied.

Step 2. Let $\widetilde{H}_T^1 = \{u \in H_T^1 : \bar{u} = 0\}$. We show that for $u \in \widetilde{H}_T^1$,

$$
\varphi(u) \to +\infty \quad (\|u\| \to \infty). \tag{2.12}
$$

In fact, by (F1) and Sobolev's inequality, one has

$$
\begin{aligned}
\Big|\int_0^T [F_1\,(t,u(t)) - F_1\,(t,0)]dt\Big| &= \Big|\int_0^T \int_0^1 (\nabla F_1(t,su(t)),u(t))\,ds\,dt\Big| \\
&\le \int_0^T f(t)|u(t)|^{\gamma+1}dt + \int_0^T g(t)\,|u(t)|\,dt \\
&\le \Big(\frac{T}{12}\Big)^{\frac{\alpha+1}{2}} M_2\|\dot{u}\|_{L^2}^{\alpha+1} + \Big(\frac{T}{12}\Big)^{1/2} M_3\|\dot{u}\|_{L^2}
\end{aligned}
$$

for all $u \in \widetilde{H}_T^1$. It follows from (F2) that

$$
\begin{aligned}
\int_0^T [F_2\,(u(t)) - F_2\,(0)]dt &= \int_0^T \int_0^1 (\nabla F_2(su(t)) - \nabla F_2\,(0)\,,u(t))\,ds\,dt \\
&\ge -\int_0^T \int_0^1 rs^{\alpha-1}|u(t)|^{\alpha}\,ds\,dt \\
&\ge -\frac{rT}{\alpha}\,\|u\|_\infty^{\alpha} \\
&\ge -\frac{rT}{\alpha}\Big(\frac{T}{12}\Big)^{\alpha/2}\,\|u\|_{L^2}^{\alpha}\,.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T [F\,(t,u(t)) - F\,(t,0)]dt + \int_0^T F(t,0)dt \\
&\ge \frac{1}{2}\|\dot{u}\|_{L^2}^2 - \Big(\frac{T}{12}\Big)^{\frac{\alpha+1}{2}} M_2\|\dot{u}\|_{L^2}^{\alpha+1} - \Big(\frac{T}{12}\Big)^{1/2} M_3\|\dot{u}\|_{L^2} \\
&\quad - \frac{rT}{\alpha}\Big(\frac{T}{12}\Big)^{\alpha/2}\,\|u\|_{L^2}^{\alpha} + \int_0^T F(t,0)dt.
\end{aligned}
$$

By Wirtinger's inequality, $\|u\| \to \infty$ if and only if $\|\dot{u}\|_{L^2} \to \infty$ in $\widetilde{H}_T^1$. Hence (2.12) is satisfied.

Step 3. By (F5), we can easily see that $\int_0^T F(t,x)dt \to -\infty$ as $|x| \to \infty$ for all $x \in \mathbb{R}^N$. Thus, for all $u \in (\widetilde{H}_T^1)^\perp = \mathbb{R}^N$,

$$\varphi(u) = \int_0^T F(t,u)dt \to -\infty \quad \text{as } |u| \to \infty.$$

Now, the proof is completed by saddle point theorem (cf. [7, Theorem 4.6]) □

Proof of Theorem 1.3. By (F3'), we can choose an $a_3 > \frac{T^2}{4\pi^2 - rT^2}$ such that

$$\liminf_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F(t,x)\,dt > \frac{a_3}{2} M_1^2. \tag{2.13}$$

The condition (F2') and the Sobolev's inequality imply that

$$\begin{aligned}\int_0^T [F_2(u(t) - F_2(\bar{u}))]\,dt &= \int_0^T \int_0^1 \frac{1}{s}\left(\nabla F_2(\bar{u} + s\tilde{u}(t)) - \nabla F_2(\bar{u}), s\tilde{u}(t)\right) ds\,dt \\ &\geq -\int_0^T \int_0^1 rs|\tilde{u}(t)|^2\,ds\,dt - \frac{rT^2}{8\pi^2}\|\dot{u}\|_{L^2}^2.\end{aligned}$$

It follows immediately from the similar method of the proof of Theorem 1.1 that

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T F(t,u(t))dt \\ &\geq \Big(\frac{1}{2} - \frac{T^2}{8\pi^2 a_3} - \frac{rT^2}{8\pi^2}\Big)\|\dot{u}\|_{L^2}^2 - \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\|\dot{u}\|_{L^2}^{\gamma+1} \\ &\quad - \big(\frac{T}{12}\big)^{1/2} M_3\|\dot{u}\|_{L^2} + |\bar{u}|^{2\gamma}\Big(|\bar{u}|^{-2\gamma}\int_0^T F(t,\bar{u})dt - \frac{a_3}{2}M_1^2\Big)\end{aligned}$$

for all $u \in H_T^1$, which implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$ by (2.13), due to the facts that $\gamma < 1$, $r < \frac{4\pi^2}{T^2}$ and $\|u\| \to \infty$ if and only if

$$\left(|\bar{u}|^2 + \|\dot{u}\|_{L^2}^2\right)^{1/2} \to \infty.$$

By the least action principle, Theorem 1.3 holds. □

Proof of Theorem 1.4. We firstly show that φ satisfies the (PS) condition. Suppose that $\{u_n\}$ satisfies $\varphi'(u_n) \to 0$ as $n \to 0$ and $\{\varphi(u_n)\}$ is bounded. By (F7), we can choose an $a_4 > T^2/(4\pi^2$ such that

$$\limsup_{|x|\to\infty} |x|^{-2\gamma} \int_0^T F_1(t,x)\,dt < -\big(\frac{a_4}{2} + \frac{\sqrt{a_4}T}{2\pi}\big)M_1^2. \tag{2.14}$$

By the (λ,μ)-subconvexity of $G(x)$, we have

$$G(x) \leq \Big(2\mu|x|^\beta + 1\Big)\,G_0 \tag{2.15}$$

for all $x \in \mathbb{R}^N$, and a.e. $t \in [0,T]$, where $G_0 = \max_{|s|\leq 1} G(s)$, $\beta = \log_{2\lambda}(2\mu) < 2$. Then

$$\begin{aligned}\int_0^T \left(\nabla F_2(t,u_n(t)), \tilde{u}_n(t)\right) dt &\geq -\int_0^T k(t)G(\bar{u}_n)dt \\ &\geq -\int_0^T k(t)\Big(2\mu|\bar{u}_n|^\beta + 1\Big)G_0 dt \\ &= -2\mu M_4|\bar{u}_n|^\beta - M_4,\end{aligned} \tag{2.16}$$

where $M_4 = G_0 \int_0^T k(t)dt$. It follows from (2.5) and (2.16) that for large n,

$$\begin{aligned} \|\tilde{u}_n\| &\geq (\varphi(u_n), \tilde{u}_n) \\ &= \|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t))dt \\ &\geq \Big(1 - \frac{T^2}{8\pi^2 a_4}\Big) \|\dot{u}_n\|_{L^2}^2 - \frac{a_4}{2} M_1^2 |\bar{u}_n|^{2\gamma} - \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2 \|\dot{u}_n\|_{L^2}^{\gamma+1} \\ &\quad - \big(\frac{T}{12}\big)^{1/2} M_3 \|\dot{u}_n\|_{L^2} - 2\mu M_4 |\bar{u}_n|^\beta - M_4. \end{aligned} \tag{2.17}$$

Then (2.7) and (2.17) imply that

$$\begin{aligned} \frac{a_4}{2} M_1^2 |\bar{u}_n|^{2\gamma} + 2\mu M_4 |\bar{u}_n|^\beta &\geq \Big(1 - \frac{T^2}{8\pi^2 a_4}\Big) \|\dot{u}_n\|_{L^2}^2 - \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2 \|\dot{u}_n\|_{L^2}^{\gamma+1} \\ &\quad - \Big(\big(\frac{T}{12}\big)^{1/2} M_3 + \frac{\big(T^2 + 4\pi^2\big)^{1/2}}{2\pi}\Big) \|\dot{u}_n\|_{L^2} - M_4 \\ &\geq \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 + C_4, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} C_4 = \min_{s \in [0, +\infty)} \Big\{ &\frac{8\pi^2 a_4 - T^2}{8\pi^2 a_4} s^2 - \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2 s^{\gamma+1} - M_4 \\ &- \Big[\big(\frac{T}{12}\big)^{1/2} M_3 + \frac{\big(T^2 + 4\pi^2\big)^{1/2}}{2\pi}\Big] s \Big\}. \end{aligned}$$

Note that $-\infty < C_4 < 0$ due to $a_4 > \frac{T^2}{4\pi^2}$. By (2.18), one has

$$\|\dot{u}_n\|_{L^2}^2 \leq a_4 M_1^2 |\bar{u}_n|^{2\gamma} + 4\mu M_4 |\bar{u}_n|^\beta - 2C_4, \tag{2.19}$$

and then

$$\|\dot{u}_n\|_{L^2} \leq \frac{\sqrt{2a_4}}{2} M_1 |\bar{u}_n|^\gamma + \sqrt{2\mu M_4} |\bar{u}_n|^{\beta/2} + C_5, \tag{2.20}$$

where $C_5 > 0$. It follows from (F6) and (2.15) that

$$\begin{aligned} &\int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] \, dt \\ &= -\int_0^T \int_0^1 (\nabla F_2(t, \bar{u}_n + s\tilde{u}_n(t)), -\tilde{u}_n(t)) \, ds \, dt \\ &\leq \int_0^T \int_0^1 k(t) G(\bar{u}_n + (s+1)\tilde{u}_n(t)) \, ds \, dt \\ &\leq \int_0^T \int_0^1 k(t) \Big(2\mu |\bar{u}_n + (s+1)\tilde{u}_n(t)|^\beta + 1\Big) G_0 \, ds \, dt \\ &\leq 4\mu \int_0^T k(t) \Big(|\bar{u}_n|^\beta + 2^\beta |\tilde{u}_n(t)|^\beta\Big) G_0 dt + G_0 \int_0^T k(t) dt \\ &\leq 2^{\beta+2} \mu M_4 \|\tilde{u}_n\|_\infty^\beta + 4\mu M_4 |\bar{u}_n|^\beta + M_4 \\ &\leq \big(\frac{T}{12}\big)^{\beta/2} 2^{\beta+2} \mu M_4 \|\dot{u}_n\|_{L^2}^\beta + 4\mu M_4 |\bar{u}_n|^\beta + M_4 \end{aligned} \tag{2.21}$$

for all $u \in H_T^1$. By the boundedness of $\{\varphi(u_n)\}$ and the inequalities (2.11), (2.19)-(2.21), one has

$$
\begin{aligned}
C_6 &\le \varphi(u_n) \\
&= \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + \int_0^T [F_1(t,u_n(t)) - F_1(t,\bar{u}_n)]\,dt \\
&\quad + \int_0^T [F_2(t,u_n(t)) - F_2(t,\bar{u}_n)]\,dt + \int_0^T F(t,\bar{u}_n)dt \\
&\le \Big(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_4}}\Big)\|\dot{u}_n\|_{L^2}^2 + \frac{\sqrt{a_4}T}{4\pi}M_1^2|\bar{u}_n|^{2\gamma} + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\|\dot{u}_n\|_{L^2}^{\gamma+1} \\
&\quad + \big(\frac{T}{12}\big)^{1/2} M_3\|\dot{u}_n\|_{L^2} + \big(\frac{T}{12}\big)^{\beta/2} 2^{\beta+2}\mu M_4\|\dot{u}_n\|_{L^2}^{\beta} + 4\mu M_4|\bar{u}_n|^{\beta} + M_4 \\
&\quad + \int_0^T F(t,\bar{u}_n)dt \\
&\le \Big(\frac{1}{2} + \frac{T}{4\pi\sqrt{a_4}}\Big)\Big(a_4M_1^2|\bar{u}_n|^{2\gamma} + 4\mu M_4|\bar{u}_n|^{\beta} - 2C_4\Big) + \frac{\sqrt{a_4}T}{4\pi}M_1^2|\bar{u}_n|^{2\gamma} \\
&\quad + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\Big(\sqrt{a_4}M_1|\bar{u}_n|^{\gamma} + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5\Big)^{\gamma+1} \\
&\quad + \big(\frac{T}{12}\big)^{1/2} pM_3\Big(\sqrt{a_4}M_1|\bar{u}_n|^{\gamma} + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5\Big) \\
&\quad + \big(\frac{T}{12}\big)^{\beta/2} 2^{\beta+2}\mu M_4\Big(\sqrt{a_4}M_1|\bar{u}_n|^{\gamma} + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5\Big)^{\beta} \\
&\quad + \mu M_4|\bar{u}_n|^{\beta} + M_4 + \int_0^T F(t,\bar{u}_n)dt \\
&\le \Big(\frac{a_4}{2} + \frac{\sqrt{a_4}T}{2\pi}\Big)M_1^2|\bar{u}_n|^{2\gamma} + \Big(6 + \frac{T}{\pi\sqrt{a_4}}\Big)\mu M_4|\bar{u}_n|^{\beta} - \Big(1 + \frac{T}{2\pi\sqrt{a_4}}\Big)C_4 \\
&\quad + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} M_2\Big(2^{\gamma}a_4^{\frac{\gamma+1}{2}}M_1^{\gamma+1}|\bar{u}_n|^{\gamma(\gamma+1)} + 2^{3\gamma+1}\mu^{\frac{\gamma+1}{2}}M_4^{\frac{\gamma+1}{2}}|\bar{u}_n|^{\frac{\beta(\gamma+1)}{2}} \\
&\quad + 2^{2\gamma}C_5^{\gamma+1}\Big) + \big(\frac{T}{12}\big)^{\frac{\beta}{2}} 2^{\beta+2}\mu M_4\Big(2^{\beta-1}a_4^{\frac{\beta}{2}}M_1^{\beta}|\bar{u}_n|^{\gamma\beta} + 2^{3\beta-2}\mu^{\frac{\beta}{2}}M_4^{\frac{\beta}{2}}|\bar{u}_n|^{\frac{\beta^2}{2}} \\
&\quad + 2^{2(\beta-1)}C_5^{\beta}\Big) + \Big(\frac{T}{12}\Big)^{1/2} M_3\Big(\sqrt{a_4}M_1|\bar{u}_n|^{\gamma} + 2\sqrt{\mu M_4}|\bar{u}_n|^{\beta/2} + C_5\Big) \\
&\quad + M_4 + \int_0^T F(t,\bar{u}_n)dt \\
&= |\bar{u}_n|^{2\gamma}\Big[|\bar{u}_n|^{-2\gamma}\int_0^T F_1(t,\bar{u}_n)dt + \Big(\frac{a_4}{2} + \frac{\sqrt{a_4}T}{2\pi}\Big)M_1^2 \\
&\quad + \big(\frac{T}{12}\big)^{1/2}\sqrt{a_4}M_1M_3|\bar{u}_n|^{-\gamma} + \big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}} 2^{\gamma-\frac{1}{2}}\sqrt{a_4}M_1M_2|\bar{u}_n|^{\gamma(\gamma-1)} \\
&\quad + \big(\frac{T}{12}\big)^{\beta/2} 2^{2\beta+1}\mu a_4^{\frac{\beta}{2}}M_1^{\beta}M_4|\bar{u}_n|^{\gamma(\beta-2)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+|\bar{u}_n|^{\beta}\Big[|\bar{u}_n|^{-\beta}\int_0^T F_2(t,\bar{u}_n)dt+\Big(6+\frac{T}{\pi\sqrt{a_4}}\Big)\mu M_4\\
&+\big(\frac{T}{12}\big)^{\beta/2}2^{4\beta}\mu^{\frac{\beta+2}{2}}M_4^{\frac{\beta+2}{2}}|\bar{u}_n|^{\frac{1}{2}\beta^2-2}\\
&+\big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}}M_2 2^{3\gamma+1}\mu^{\frac{\gamma+1}{2}}M_2M_4^{\frac{\gamma+1}{2}}|\bar{u}_n|^{\frac{\beta(\gamma-1)}{2}}+\big(\frac{T}{12}\big)^{1/2}2M_3\sqrt{\mu M_4}|\bar{u}_n|^{-\beta/2}\Big]\\
&-\Big(1+\frac{T}{2\pi\sqrt{a_4}}\Big)C_4+\big(\frac{T}{12}\big)^{\frac{\gamma+1}{2}}2^{2\gamma}M_2C_5^{\gamma+1}+\big(\frac{T}{12}\big)^{1/2}M_3C_5\\
&+\big(\frac{T}{12}\big)^{\beta/2}2^{3\beta}\mu M_4C_5^{\beta}+M_4
\end{aligned}
$$

for large n. The above inequality and (2.14) imply that $\{|\bar{u}|\}$ is bounded. Hence $\{u_n\}$ is bounded by (2.19). By using the usual method, the (PS) condition holds.

Similar to the proof of Theorem 1.2, we can verify that functional satisfies the other conditions of the saddle point theorem. We omit the details. □

3. Examples

In this section, we give some examples of F to illustrate that our results are new.

Example 3.1. Let $F = F_1 + F_2$, with

$$
\begin{gathered}
F_1(t,x)=\sin\big(\frac{2\pi t}{T}\big)|x|^{7/4}+(0.6T-t)|x|^{3/2}+(h(t),x)\,,\\
F_2(x)=C(x)-\frac{3r}{4}|x|^{4/3},
\end{gathered}
$$

where $h\in\mathscr{L}^1(0,T;\mathbb{R}^N)$, $r>0$, $C(x)=\frac{3r}{4}(|x_1|^4+|x_2|^{4/3}+\dots+|x_N|^{4/3})$.

By Young's inequality, it is easy to see that

$$
\begin{aligned}
|\nabla F_1(t,x)| &\le \frac{7}{4}\Big|\sin\Big(\frac{2\pi t}{T}\Big)\Big||x|^{3/4}+\frac{3}{2}|0.6T-t||x|^{1/2}+|h(t)|\\
&\le \frac{7}{4}\Big(\Big|\sin\Big(\frac{2\pi t}{T}\Big)\Big|+\varepsilon\Big)|x|^{3/4}+\frac{T^3}{\varepsilon^2}+|h(t)|
\end{aligned}
$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, where $\varepsilon>0$. And

$$
(\nabla F_2(x)-\nabla F_2(y),x-y)\ge -r|x-y|^{4/3}
$$

for all $x,y\in\mathbb{R}^N$. Thus, (F1), (F2) hold with $\gamma=3/4$, $\alpha=4/3$ and

$$
f(t)=\frac{7}{4}(|\sin(\frac{2\pi t}{T})|+\varepsilon),\quad g(t)=\frac{T^3}{\varepsilon^2}+|h(t)|.
$$

However, F does not satisfy (1.2). In fact

$$
\begin{aligned}
&|x|^{-2\gamma}\int_0^T F(t,x)dt\\
&=|x|^{-3/2}\int_0^T\big[\sin\big(\frac{2\pi t}{T}\big)|x|^{7/4}+(0.6T-t)|x|^{3/2}+\big(C(x)-\frac{3r}{4}|x|^{4/3}\big)+(h(t),x)\big]dt\\
&=0.1T^2+\frac{T(C(x)-\frac{3r}{4}|x|^{4/3})}{|x|^{3/2}}+\Big(\int_0^T h(t)dt,|x|^{-3/2}x\Big)
\end{aligned}
$$

On the other hand, we have

$$\frac{T^2}{8\pi^2}\int_0^T f^2(t)dt = \frac{49T^3}{128\pi^2}\Big(\frac{1}{2}+\frac{4\varepsilon}{\pi}+\varepsilon^2\Big)$$

If $T < \frac{128\pi^2}{245}$, we choose $\varepsilon > 0$ sufficient small such that

$$\liminf_{|x|\to\infty} |x|^{-2\gamma}\int_0^T F(t,x)dt = 0.1T^2 > \frac{T^2}{8\pi^2}\int_0^T f^2(t)dt$$

which implies that (F3) holds. Then $F = F_1 + F_2$ is not convex, not γ-subadditive, not periodic, not a.e. uniformly coercive, and ∇F is not sublinear. Thus, F is not covered by results in the references.

Example 3.2. Let $F = F_1 + F_2$, with

$$F_1(t,x) = (0.5T - t)|x|^{7/4} + (0.4T - t)|x|^{3/2} + (h(t), x)\,,$$
$$F_2(x) = -\frac{4r}{5}|x|^{5/4},$$

where $h \in \mathscr{L}^1(0,T;\mathbb{R}^N)$, $r > 0$.

Similar to Example 3.1, we can see that all conditions of Theorem 1.2 hold but F is not covered by results in the references.

Example 3.3. Let $F = F_1 + F_2$, with

$$F_1(t,x) = (0.5T - t)|x|^{7/4} + (0.6T - t)|x|^{3/2} + (h(t), x)\,,$$
$$F_2(x) = C(x) - \frac{r}{2}|x|^2,$$

where $h \in \mathscr{L}^1(0,T;\mathbb{R}^N)$, $C(x) = \frac{r}{2}(|x_1|^4 + |x_2|^2 + \dots + |x_N|^2)$, $0 < r < \frac{4\pi^2}{T^2}$.

In a way similar to Example 3.1, it is easy to see that condition (F1) and (F2') are satisfied with $\gamma = 3/4$. However, F does not satisfies (1.2). In fact,

$$\begin{aligned}
&|x|^{-2\gamma}\int_0^T F(t,x)dt \\
&= |x|^{-2/3}\int_0^T \big[(0.5T - t)|x|^{7/4} + (0.6T - t)|x|^{3/2} + \Big(C(x) - \frac{r}{2}|x|^2\Big) + (h(t), x)\big]dt \\
&= 0.1T^2 + \frac{T\left(C(x) - \frac{r}{2}|x|^2\right)}{|x|^{3/2}} + \Big(\int_0^T h(t)dt, x|x|^{-3/2}\Big) \\
&= 0.1T^2 + \frac{rT(|x_1|^4 - |x_1|^2)}{2|x|^{3/2}} + \Big(\int_0^T h(t)dt, x|x|^{-3/2}\Big).
\end{aligned}$$

We can choose $\varepsilon > 0$ small enough and some suitable T such that

$$\liminf_{|x|\to\infty} |x|^{-2\gamma}\int_0^T F(t,x)dt = 0.1T^2 > \frac{T^2}{2(4\pi^2 - rT^2)}\int_0^T f^2(t,x)dt,$$

which implies that (F3') holds. F is also not covered by results in the references.

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