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OPTIMAL CONTROL PROBLEMS FOR IMPULSIVE SYSTEMS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this article, the optimal control problem is considered when the state of the system is described by the impulsive differential equations with integral boundary conditions. Applying the Banach contraction principle the existence and uniqueness of the solution is proved for the corresponding boundary problem by the fixed admissible control. The first and second variation of the functional is calculated. Various necessary conditions of optimality of the first and second order are obtained by the help of the variation of the controls.

1. Introduction

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs [3, 4, 13, 21] and the references therein.

Many of the physical systems can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, boundary value problems with integral conditions constitute a very interesting and important class of problems. They include two point, three point, multi-point and nonlocal boundary value problems as special cases; see [1, 5, 7]. For boundary-value problems with nonlocal boundary conditions and comments on their importance, we refer the reader to the papers [6, 8, 12] and the references therein.

The optimal control problems with boundary conditions have been investigated by several authors (see,e.g., [15, 22, 18, 19, 20]). Note that optimal control problems with integral boundary condition are considered in [16, 17] and the first-order necessary conditions are obtained. In certain cases the first order optimality conditions are "left degenerate"; i.e., they are fulfilled trivially on a series of admissible controls. In this case it is necessary to obtain the second order optimality conditions.

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In the present paper, we investigate an optimal control problem in which the state of the system is described by differential equations with integral boundary conditions. Note that this problem is a natural generalization of the Cauchy problem. The matters of existence and uniqueness of solutions of the boundary value problem are investigated, first and second variations of the functional are calculated. Using the variations of the controls, various optimality conditions of the second order are obtained.

Consider the following impulsive system of differential equations with integral boundary condition

$$\frac{dx}{dt} = f(t, x, u(t)), \quad 0 < t < T, \tag{1.1}$$

$$x(0) + \int_0^T m(t)x(t)dt = C, \tag{1.2}$$

$$x(t_i^+) - x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \dots, p, \ 0 < t_1 < t_2 < \dots < t_p < T, \tag{1.3}$$

$$u(t) \in U, \quad t \in [0, T], \tag{1.4}$$

where $x(t) \in \mathbb{R}^n$; $f(t, x, u)$ and $I_i(x)$, $i = 1, 2, \dots, p$ are n-dimensional continuous vector functions. Suppose that f has the second order derivative with respect to (x, u) and I_i has the second order derivative with respect to x. $C \in \mathbb{R}^n$ is a given constant vector and $m(t)$ is $n \times n$ matrix function; u is a control parameter; $U \in \mathbb{R}^r$ is an open set.

It is required to minimize the functional

$$J(u) = \varphi(x(0), x(T)) + \int_0^T F(t, x, u)dt \tag{1.5}$$

on the solutions of boundary value problem (1.1)–(1.4).

Here, it is assumed that the scalar functions $\varphi(x, y)$ and $F(t, x, u)$ are continuous by their own arguments and have continuous and bounded partial derivatives with respect to x, y and u up to second order, inclusively. Under the condition of boundary value problem (1.1)–(1.3) corresponding to the fixed control parameter $u(\cdot) \in U$ we have the function $x(t) : [0, T] \to \mathbb{R}^n$ that is absolutely continuous on $[0, T]$, $t \neq t_i$, $i = 1, 2, \dots, p$ and continuous from left for $t = t_i$, for which there exists a finite right limit $x(t_i^+)$ for $i = 1, 2, \dots, p$. Denote the space of such functions by $PC([0, T], \mathbb{R}^n)$. It is obvious that such a space is Banach with the norm $\|x\|_{PC} = \operatorname{vrai}\max_{t\in[0,T]} |x(t)|$, where $|\cdot|$ is the norm in space $\mathbb{R}^n$.

The admissible process $\{u(t), x(t, u)\}$ being the solution of problem (1.1)-(1.5); i.e., delivering minimum to functional (1.5) under restrictions (1.1)-(1.4), is said to be an optimal process, and $u(t)$ is called an optimal control.

The organization of the present paper is as follows. First, we provide necessary background. Second, theorems on existence and uniqueness of a solution of problem (1.1)-(1.3) are established under some sufficient conditions on nonlinear terms. Third, the functional increment formula is presented. Fourth, variations of the functional are given. Fifth, Legendre-Klebsh conditions are obtained. Finally, the conclusion is given.

2. Existence of solutions to (1.1)-(1.3)

We will use the following assumptions:

(H1) $\|B\| < 1$ for the matrix B defined by the formula $B = \int_0^T m(t)dt$.

(H2) The functions $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ and $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \dots, p$ are continuous functions and there exist constants $K \geq 0$ and $l_i \geq 0$, $i = 1, 2, \dots, p$, such that
$$|f(t, x, u) - f(t, y, u)| \leq K|x - y|, \quad t \in [0, T],\ x, y \in \mathbb{R}^n,\ u \in \mathbb{R}^r,$$
$$|I_i(x) - I_i(y)| \leq l_i|x - y|, \quad x, y \in \mathbb{R}^n.$$

(H3)
$$L = (1 - \|B\|)^{-1}\big[KTN + \sum_{i=1}^{p} l_i\big] < 1,$$
where
$$N = \max_{0 \leq t,s \leq T} \|N(t, s)\|, \quad N(t, s) = \begin{cases} E + \int_0^s m(\tau)d\tau, & 0 \leq t \leq s, \\ -\int_s^T m(\tau)d\tau, & s \leq t \leq T. \end{cases}$$

Note that under condition (H1) the matrix $E + B$ is invertible and the estimate $\|(E + B)^{-1}\| < (1 - \|B\|)^{-1}$ holds.

Theorem 2.1. *Assume that condition* (H1) *is satisfied. Suppose that the functions* $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ *and* $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \dots, p$ *are continuous functions. Then the function* $x(\cdot) \in PC([0, T], \mathbb{R}^n)$ *is an absolutely continuous solution of boundary-value problem* (1.1)-(1.3) *if and only if it is a solution of the integral equation*
$$x(t) = (E+B)^{-1}C + \int_0^T K(t, \tau)f(\tau, x(\tau), u(\tau))d\tau + \sum_{0<t_k<T} Q(t, t_k)I_k(x(t_k)), \tag{2.1}$$
where
$$K(t, \tau) = (E + B)^{-1}N(t, \tau), Q(t, t_k) = \begin{cases} (E + B)^{-1}, & 0 \leq \tau_k \leq t, \\ -(E + B)^{-1}B, & t \leq \tau_k \leq T. \end{cases}$$

Proof. Assume that $x = x(t)$ is a solution of (1.1), then integrating this equation for $t \in (t_j, t_{j+1})$, we obtain
$$\begin{aligned}
&\int_0^t f(s, x(s), u(s))ds \\
&= \int_0^t x'(s)ds \\
&= [x(t_1) - x(0^+)] + [x(t_2) - x(t_1^+)] + \dots + [x(t) - x(t_j^+)] \\
&= -x(0) - [x(t_1^+) - x(t_1)] - [x(t_2^+) - x(t_2)] - \dots - [x(t_j^+) - x(t_j)] + x(t).
\end{aligned}$$
From what it follows that
$$x(t) = x(0) + \int_0^t f(s, x(s), u(s))ds + \sum_{0<t_j<t} (x(t_j^+) - x(t_j)), \tag{2.2}$$
where $x(0)$ is an arbitrary constant. Now, we obtain $x(0)$. Applying equality (2.2) and conditions (1.2)-(1.3), we obtain
$$(E + B)x(0) = C - \int_0^T m(t) \int_0^t f(\tau, x(\tau), u(\tau))d\tau dt - B \sum_{0<t_k<T} I_k(x(t_k)).$$

Since $\det(E+B)\neq 0$, we have

$$\begin{aligned} x(0) &= (E+B)^{-1}C-(E+B)^{-1}\int_0^T m(t)\int_0^t f(\tau,x(\tau),u(\tau))d\tau dt \\ &\quad -(E+B)^{-1}B\sum_{0<t_k<T} I_k(x(t_k)). \end{aligned} \tag{2.3}$$

Applying formulas (2.2) and (2.3), we obtain the integral equation (2.1). By direct verification we can show that the solution of integral equation (2.1) also satisfies equation (1.1) and nonlocal boundary condition (1.2). Also, it is easy to verify that it satisfies the condition (1.3). The proof is comlete. □

Define the operator $P: PC([0,T],\mathbb{R}^n)\to PC([0,T],\mathbb{R}^n)$, by

$$(Px)(t)=(E+B)^{-1}C+\int_0^T K(t,\tau)f(\tau,x(\tau),u(\tau))d\tau+\sum_{0<t_k<T}Q(t,t_k)I_k(x(t_k)). \tag{2.4}$$

Theorem 2.2. *Assume that conditions* (H1)–(H3) *are satisfied. Then for any* $C\in\mathbb{R}^n$ *and for each fixed admissible control, the boundary value problem* (1.1)-(1.3) *has a unique solution that satisfies the integral equation* (2.1).

Proof. Let $C\in\mathbb{R}^n$ and $u(\cdot)\in U$ be fixed, and let the mapping $P: PC([0,T],\mathbb{R}^n)\to PC([0,T],\mathbb{R}^n)$ be defined by (2.4). Clearly, the fixed points of the operator P are solution of the problem (1.1), (1.2) and (1.3). We will use the Banach contraction principle to prove that P has a fixed point. First, we will show that P is a contraction.

Let $v,w\in PC([0,T],\mathbb{R}^n)$. Then, for each $t\in[0,T]$ we have that

$$\begin{aligned} |(Pv)(t)-(Pw)(t)| &\le \int_0^T |K(t,s)||f(s,v(s),u(s))-f(s,w(s),u(s))|ds \\ &\quad + \sum_{i=1}^p |Q(t,t_i)||I_i(v(t_i))-I_i(w(t_i))| \\ &\le (1-\|B\|)^{-1}[KN\int_0^T |v(s)-w(s)|ds+\sum_{i=1}^p l_i|v(t_i)-w(t_i)|] \\ &\le (1-\|B\|)^{-1}[KNT+\sum_{i=1}^p l_i]\|v(\cdot)-w(\cdot)\|_{PC}. \end{aligned}$$

Therefore,

$$\|Pv-Pw\|_{PC}\le L\|v-w\|_{PC}.$$

Consequently, by assumption (H3) operator P is a contraction. As a consequence of Banach's fixed point theorem, we deduce that operator P has a fixed point which is a solution of problem (1.1)-(1.3). The proof is complete. □

The functional increment formula

Let $\{u, x=x(t,u)\}$ and $\{\tilde u=u+\Delta u,\ \tilde x=x+\Delta x=x(t,\tilde u)\}$ be two admissible processes. Applying (1.1)-(1.2), we obtain the boundary-value problem

$$\Delta\dot x=\Delta f(t,x,u),\quad t\in(0,T), \tag{2.5}$$

$$\Delta x(0) + \int_0^T m(t)\Delta x(t)dt = 0, \tag{2.6}$$

where $\Delta f(t,x,u) = f(t,\tilde{x},\tilde{u}) - f(t,x,u)$ denotes t he total increment of the function $f(t,x,u)$. Then we can represent the increment of the functional in the form

$$\Delta J(u) = J(\tilde{u}) - J(u) = \Delta\varphi(x(0),x(T)) + \int_0^T \Delta F(x,u,t)dt. \tag{2.7}$$

Let us introduce some non-trivial vector-function $\psi(t), \psi(t) \in \mathbb{R}^n$, and numerical vector $\lambda \in \mathbb{R}^n$. Then the increment of performance index (1.5) may be represented as

$$\begin{aligned}\Delta J(u) &= J(\tilde{u}) - J(u)\\ &= \Delta\varphi(x(0),x(T)) + \int_0^T \Delta F(x,u,t)dt + \int_0^T \langle \psi(t), \Delta\dot{x}(t) - \Delta f(t,x,u)\rangle dt\\ &\quad + \langle \lambda, \Delta x(0) + \int_0^T m(t)\Delta x(t)dt\rangle.\end{aligned}$$

After some standard computations usually used in deriving optimality conditions of the first and second orders for the increment of the functional, we obtain the formula

$$\begin{aligned}\Delta J(u) &= J(\tilde{u}) - J(u)\\ &= -\int_0^T \langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \Delta u(t)\rangle dt\\ &\quad - \int_0^T \langle \frac{\partial H(t,\psi,x,u)}{\partial x} + n'(t)\lambda + \dot{\psi}(t), \Delta x(t)\rangle dt\\ &\quad + \langle [\frac{\partial\varphi}{\partial x(0)} - \psi(0) + \lambda], \Delta x(0)\rangle + \langle [\frac{\partial\varphi}{\partial x(T)} + \psi(T)], \Delta x(T)\rangle\\ &\quad - \int_0^T \langle \frac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}\Delta u(t) + \frac{1}{2}\Delta x'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \Delta x(t)\rangle dt\\ &\quad - \frac{1}{2}\int_0^T \langle \Delta u(t)'\frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \Delta u(t)\rangle dt\\ &\quad + \frac{1}{2}\langle \Delta x(0)'\frac{\partial^2\varphi}{\partial x(0)^2} + \Delta x(T)'\frac{\partial^2\varphi}{\partial x(0)\partial x(T)}, \Delta x(0)\rangle\\ &\quad + \frac{1}{2}\langle \Delta x(0)'\frac{\partial^2\varphi}{\partial x(T)\partial x(0)} + \Delta x(T)'\frac{\partial^2\varphi}{\partial x(T)^2}, \Delta x(T)\rangle\\ &\quad + \sum_{i=1}^p \langle \psi(t_i^+) - \psi(t_i) + \frac{\partial I_i'(x(t_i))}{\partial x}[\frac{\partial I_i'(x(t_i))}{\partial x} + E]^{-1}\psi(t_i), \Delta x(t_i)\rangle + \eta_{\tilde{u}},\end{aligned} \tag{2.8}$$

where

$$\begin{gathered}H(t,\psi,x,u) = \langle \psi, f(t,x,u)\rangle - F(t,x,u),\\ \eta_{\tilde{u}} = -\int_0^T o_H(\|\Delta x(t)\|^2 + \|\Delta u(t)\|^2)dt\\ + o_\varphi(\|\Delta x(t_0)\|^2, \|\Delta x(t_1)\|^2) + \sum_{i=1}^p o_I(\|\Delta x(t_i)\|^2).\end{gathered}$$

Here, the vector function $\psi(t) \in \mathbb{R}^n$ and vector $\lambda \in \mathbb{R}^n$ is solution of the following adjoint problem (the stationary condition of the Lagrangian function by state)

$$\dot{\psi}(t) = -\frac{\partial H(t,\psi,x,u)}{\partial x} - m'(t)\lambda, \quad t \in (0,T), \tag{2.9}$$

$$\psi(t_i^+) - \psi(t_i) = -I'_{ix}(x(t_i))(I'_{ix}(x(t_i)) + E)^{-1}\psi(t_i), i = 1,2,\dots,p, \tag{2.10}$$

$$\frac{\partial \varphi}{\partial x(0)} - \psi(0) + \lambda = 0, \frac{\partial \varphi}{\partial x(T)} + \psi(T) = 0. \tag{2.11}$$

From this and (2.8) it follows that

$$\begin{aligned}
\Delta J(u) = &-\int_0^T \langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \Delta u(t)\rangle dt - \frac{1}{2}\int_0^T \langle \Delta u(t)' \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \Delta u(t)\rangle dt \\
&- \int_0^T \langle \Delta u(t)' \frac{\partial H^2(t,\psi,x,u)}{\partial x \partial u} + \frac{1}{2}\Delta x'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \Delta x(t)\rangle dt \\
&+ \frac{1}{2}\langle \Delta x(0)' \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x(T)' \frac{\partial^2 \varphi}{\partial x(0)\partial x(T)}, \Delta x(0)\rangle \\
&+ \frac{1}{2}\langle \Delta x(0)' \frac{\partial^2 \varphi}{\partial x(T)\partial x(0)} + \Delta x(T)' \frac{\partial^2 \varphi}{\partial x(T)^2}, \Delta x(T)\rangle + \eta_{\tilde{u}}.
\end{aligned} \tag{2.12}$$

3. Variations of the functional

Let $\Delta u(t) = \varepsilon \delta u(t)$, where $\varepsilon > 0$ is a rather small number and $\delta u(t)$ is some piecewise continuous function. Then the increment of the functional $\Delta J(u) = J(\tilde{u}) - J(u)$ for the fixed functions $u(t), \Delta u(t)$ is the function of the parameter ε. If the representation

$$\Delta J(u) = \varepsilon \delta J(u) + \frac{1}{2}\varepsilon^2 \delta^2 J(u) + o(\varepsilon^2) \tag{3.1}$$

is valid, then $\delta J(u)$ is called the first variation of the functional and $\delta^2 J(u)$ is called the second variation of the functional. Further, we obtain an obvious expression for the first and second variations. To achieve the object we have to select in $\Delta x(t)$ the principal term with respect to ε.

Assume that

$$\Delta x(t) = \varepsilon \delta x(t) + o(\varepsilon, t), \tag{3.2}$$

where $\delta x(t)$ is the variation of the trajectory. Such a representation exists and for the function $\delta x(t)$ one can obtain an equation in variations. Indeed, by definition of $\Delta x(t)$, we have

$$\begin{aligned}
\Delta x(t) = &(E+B)^{-1}C + \int_0^T K(t,\tau)\Delta f(\tau, x(\tau), u(\tau))d\tau \\
&+ \sum_{0<t_k<T} Q(t,t_k)\Delta I_k(x(t_k)).
\end{aligned} \tag{3.3}$$

Applying the Taylor formula to the integrand expression, we obtain

$$\begin{aligned}
&\varepsilon \delta x(t) + o(\varepsilon, t) \\
&= \int_0^T K(t,\tau)\Big\{\frac{\partial f(\tau,x,u)}{\partial x}[\varepsilon \delta x(\tau) + o(\varepsilon,\tau)] + \varepsilon \frac{\partial f(\tau,x,u)}{\partial u}\delta u + o_1(\varepsilon,\tau)\Big\} d\tau
\end{aligned}$$

$$+\sum_{i=1}^{p} Q(t,t_i)\big\{\frac{\partial I_i(x(t_i))}{\partial x}[\varepsilon\delta x(t_i)+o(\varepsilon,t_i)]\big\}.$$

Since this formula is true for any ε,

$$\begin{aligned}\delta x(t) = \int_0^T K(t,\tau)\{\frac{\partial f(\tau,x,u)}{\partial x}\delta x(t) + \frac{\partial f(\tau,x,u)}{\partial u}\delta u(t)\}d\tau \\ + \sum_{i=1}^{p} Q(t,t_i)\frac{\partial I_i(x(t_i))}{\partial x}\delta x(t_i).\end{aligned} \tag{3.4}$$

Equation (3.4) is called the equation in variations. Obviously, integral equation (3.4) is equivalent to the following nonlocal boundary-value problem

$$\delta\dot{x}(t) = \frac{\partial f(t,x,u)}{\partial x}\delta x(t) + \frac{\partial f(t,x,u)}{\partial u}\delta u(t), \tag{3.5}$$

$$\delta x(t_i^+) - \delta x(t_i) = I_{ix}(x(t_i))\delta x(t_i), \quad i=1,2,\dots,p, \tag{3.6}$$

$$\delta x(0) + \int_0^T m(t)\delta x(t)dt = 0. \tag{3.7}$$

By [21, p.52], any solution of differential equation (3.5) may be represented in the form

$$\delta x(t) = \Phi(t)\delta x(0) + \Phi(t)\int_0^t \Phi^{-1}(\tau)\frac{\partial f(\tau,x,u)}{\partial u}\delta u(\tau)d\tau, \tag{3.8}$$

where $\Phi(t)$ is a solution of the differential equation

$$\begin{gathered}\frac{d\Phi(t)}{dt} = \frac{\partial f(t,x,u)}{\partial x}\Phi(t), \quad \Phi(0)=E, \\ \Phi(t_i^+) - \Phi(t_i) = I_{ix}(x(t_i))\Phi(x(t_i)).\end{gathered}$$

Then for the solutions $\delta x(t)$ of the boundary-value problem we obtain the explicit formula

$$\delta x(t) = \int_0^T G(t,\tau)\frac{\partial f(\tau,x,u)}{\partial u}\delta(\tau)d\tau, \tag{3.9}$$

where

$$G(t,\tau) = \begin{cases}\Phi(t)(E+B_1)^{-1}[E+\int_0^s m(\tau)\Phi(\tau)d\tau]\Phi^{-1}(\tau), & 0\le\tau\le t,\\ -\Phi(t)(E+B_1)^{-1}\int_s^T m(\tau)\Phi(\tau)d\tau\Phi^{-1}(\tau), & t\le\tau\le T,\end{cases}$$

$$B_1 = \int_0^T m(t)\Phi(t)dt.$$

Now, using identity (3.2), formula (2.12) can be rewritten as

$$\begin{aligned}\Delta J(u) = -\varepsilon\int_0^T \langle\frac{\partial H(t,\psi,x,u)}{\partial u},\delta u(t)\rangle dt - \frac{\varepsilon^2}{2}\Big\{\int_0^T \langle\delta x'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x^2},\delta x(t)\rangle \\ + 2\langle\delta u'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u},\delta x(t)\rangle + \langle\delta u'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial u^2},\delta u(t)\rangle]dt \\ - \langle\delta x'(0)\frac{\partial^2\varphi}{\partial x(0)^2} + \Delta x'(T)\frac{\partial^2\varphi}{\partial x(0)\partial x(T)},\delta x(0)\rangle \\ - \langle\delta x'(0)\frac{\partial^2\varphi}{\partial x(T)\partial x(0)} + \delta x'(T)\frac{\partial^2\varphi}{\partial x(T)^2},\delta x(T)\rangle\} + o(\varepsilon^2).\end{aligned} \tag{3.10}$$

Applying (3.1), we obtain

$$\delta J(u) = -\int_0^T \langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \delta u(t)\rangle dt,$$

$$\begin{aligned} \delta^2 J(u) = -\int_0^T \Big[&\langle \delta x'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \delta x(t)\rangle \\ &+ 2\langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x \partial u}, \delta x(t)\rangle + \langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\rangle \Big] dt \\ &+ \langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x'(T) \frac{\partial^2 \varphi}{\partial x(0)\partial x(T)}, \delta x(0)\rangle \\ &+ \langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(T)\partial x(0)} + \delta x'(T) \frac{\partial^2 \varphi}{\partial x(T)^2}, \delta x(T)\rangle. \end{aligned} \tag{3.11}$$

4. Derivation of the Legendre-Klebsh Conditions

Applying (3.1), we obtain the following conditions

$$\delta J(u^0) = 0, \quad \delta^2 J(u^0) \geq 0 \tag{4.1}$$

on the optimal control $u^0(t)$. From the first condition of (4.1) it follows that

$$\int_0^T \langle \frac{\partial H(t,\psi^0,x^0,u^0)}{\partial u}, \delta u(t)\rangle dt = 0. \tag{4.2}$$

Hence, we can prove that

$$\frac{\partial H(t,\psi^0,x^0,u^0)}{\partial u} = 0, \quad t \in (0,T) \tag{4.3}$$

is satisfied along the optimal control (see [11, p. 54]). Equation (4.3) is called the Euler equation. From the second condition of (4.1) it follows that the following inequality

$$\begin{aligned} \delta^2 J(u) = -\int_0^T \Big[&\langle \delta x'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x^2}, \delta x(t)\rangle \\ &+ 2\langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial x \partial u}, \delta x(t)\rangle + \langle \delta u'(t) \frac{\partial^2 H(t,\psi,x,u)}{\partial u^2}, \delta u(t)\rangle \Big] dt \\ &+ \langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x'(T) \frac{\partial^2 \varphi}{\partial x(0)\partial x(T)}, \delta x(0)\rangle \\ &+ \langle \delta x'(0) \frac{\partial^2 \varphi}{\partial x(T)\partial x(0)} + \delta x'(T) \frac{\partial^2 \varphi}{\partial x(T)^2}, \delta x(T)\rangle \geq 0 \end{aligned} \tag{4.4}$$

holds along the optimal control. The inequality (4.4) is an implicit necessary optimality condition of first order. However, the practical value of conditions (4.4) in such a form is not applicable, since it requires bulky calculations. For obtaining effectively verifiable optimality conditions of second order, following [14, p. 16], we take into account (3.9) in (4.4) and introduce the matrix function

$$R(\tau,s) = -G'(0,\tau) \frac{\partial^2 \varphi}{\partial x(0)^2} G(0,s) - G'(T,\tau) \frac{\partial^2 \varphi}{\partial x(T)\partial x(0)} G(0,s)$$

$$- G'(0,\tau)\frac{\partial^2\varphi}{\partial x(0)\partial x(T)}G(T,s) - G'(T,\tau)\frac{\partial^2\varphi}{\partial x(T)^2}G(T,s) + \int_0^T G'(t,\tau)\frac{\partial^2 H}{\partial x^2}G(t,s)dt.$$

Then for the second variation of the functional, we obtain the final formula

$$\begin{aligned}\delta^2 J(u) = -\Big\{ &\int_0^T\int_0^T \langle \delta' u(\tau)\frac{\partial' f(\tau,x,u)}{\partial u}R(\tau,s)\frac{\partial f(s,x,u)}{\partial u}, \delta u(s)\rangle \, dt\, ds \\ &+ \int_0^T \langle \delta' u(\tau)\frac{\partial^2 H(t,\psi,\, x,u)}{\partial u^2}, \delta u(t)\rangle dt \\ &+ 2\int_0^T\int_0^T \langle \delta u'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}G(t,s)\frac{\partial f(s,x,u)}{\partial u}, \delta u(s)\rangle \, dt\, ds\Big\}.\end{aligned} \tag{4.5}$$

Theorem 4.1. *If the admissible control $u(t)$ satisfies condition* (4.3)*, then for its optimality in problem 1.1–*(1.4)*, the inequality*

$$\begin{aligned}\delta^2 J(u) = -\Big\{ &\int_0^T\int_0^T \langle \delta' u(\tau)\frac{\partial' f(\tau,x,u)}{\partial u}R(\tau,s)\frac{\partial f(s,x,u)}{\partial u}, \delta u(s)\rangle d\tau\, ds \\ &+ \int_0^T \langle \delta' u(\tau)\frac{\partial^2 H(t,\psi,\, x,u)}{\partial u^2}, \delta u(t)\rangle dt \\ &+ 2\int_0^T\int_0^T \langle \delta u'(t)\frac{\partial^2 H(t,\psi,x,u)}{\partial x\partial u}G(t,s)\frac{\partial f(s,x,u)}{\partial u}, \delta u(s)\rangle \, dt\, ds\Big\} \ge 0\end{aligned} \tag{4.6}$$

is satisfied for all $\delta u(t) \in L_\infty[0,T]$.

The analogous to the Legandre-Klebsh condition for the considered problem follows from condition (4.6).

Theorem 4.2. *Along the optimal process $(u(t), x(t))$ for all $\nu \in \mathbb{R}^r$ and $\theta \in [0,T]$*

$$\nu'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}\nu \le 0. \tag{4.7}$$

Proof. For the proof of estimate (4.7), we will construct the variation of the control by

$$\delta u(t) = \begin{cases} v & t\in[\theta,\theta+\varepsilon)\,, \\ 0 & t\notin[\theta,\theta+\varepsilon), \end{cases} \tag{4.8}$$

where $\varepsilon > 0$, v is some r-dimensional vector.

By (3.9) the corresponding variation of the trajectory is

$$\delta x(t) = a(t)\varepsilon + o(\varepsilon,t), \quad t\in(0,T), \tag{4.9}$$

where $a(t)$ is a continuous bounded function.

Substitute variation (4.8) in to (4.6) and select the principal term with respect to ε. Then

$$\begin{aligned}\delta^2 J(u) &= -\int_\theta^{\theta+\varepsilon} v'\frac{\partial^2 H(t,\psi(t),x(t),u(t))}{\partial u^2}v dt + o(\varepsilon) \\ &= -\varepsilon v'\frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}v + o_1(\varepsilon).\end{aligned}$$

Thus, considering the second condition of (4.1), we obtain the Legandre-Klebsh criterion (4.7). The proof is complete. □

The condition (4.7) is the second-order optimality condition. It is obvious that when the right-hand side of system (1.1) and function $F(t,x,u)$ are linear with respect to control parameters, condition (4.7) also degenerates; i.e., it is fulfilled trivially. Following [11, p. 27] and [14, p. 40], if for all $\theta \in (0,T)$, $\nu \in \mathbb{R}^r$,

$$\frac{\partial H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u} = 0, \quad \nu' \frac{\partial^2 H(\theta,\psi(\theta),x(\theta),u(\theta))}{\partial u^2}\nu = 0,$$

then the admissible control $u(t)$ is said be a singular control in the classical sense.

Theorem 4.3. *For optimality of the singular control $u(t)$ in the classical sense,*

$$\begin{aligned} &\nu'\Big\{ \int_0^T \int_0^T \langle \frac{\partial f(t,x,u)}{\partial u} R(t,s), \frac{\partial f(s,x,u)}{\partial u} \rangle \,dt\,ds \\ &+ 2\int_0^T \int_0^T \langle \frac{\partial^2 H(t,\psi,\,x,u)}{\partial x \partial u} G(t,s), \frac{\partial f(s,x,u)}{\partial u} \rangle \,dt\,ds \Big\}\nu \le 0 \end{aligned} \tag{4.10}$$

is satisfied for all $\nu \in \mathbb{R}^r$.

Condition (4.10) is a necessary condition of optimality of an integral type for singular controls in the classical sense. Choosing special variation in different way in formula (4.9), we can get various necessary optimality conditions.

Conclusion. In this work, the optimal control problem is considered when the state of the system is described by the impulsive differential equations with integral boundary conditions. Applying the Banach contraction principle the existence and uniqueness of the solution is proved for the corresponding boundary problem by the fixed admissible control. The first and second variation of the functional is calculated. Various necessary conditions of optimality of the first and second order are obtained by the help of the variation of the controls. These statements are formulated in [2] without proof. Of course, such type of existence and uniqueness results hold under the same sufficient conditions on nonlinear terms for the system of nonlinear impulsive differential equations (1.1), subject to multi-point nonlocal and integral boundary conditions

$$Ex(0) + \int_0^T m(t)x(t)dt + \sum_{j=1}^{J} B_j x(\lambda_j) = \int_0^T g(s,x(s))ds, \tag{4.11}$$

and impulsive conditions

$$x(t_i^+) - x(t_i) = I_i(x(t_i)), \quad i = 1,2,\dots,p,\ 0 < t_1 < t_2 < \dots < t_p < T, \tag{4.12}$$

where $B_j \in \mathbb{R}^{n\times n}$ are given matrices and

$$\|\int_0^T m(t)dt\| + \sum_{j=1}^{J} \|B_j\| < 1.$$

Here, $0 < \lambda_1 < \dots < \lambda_J \le T$. Moreover, method in monographs [9, 10] and the method present paper permit us investigate optimal control problem for infinite dimensional impulsive systems with integral boundary conditions.

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