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EXISTENCE OF SOLUTIONS TO IMPULSIVE FRACTIONAL PARTIAL NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL INCLUSIONS WITH STATE-DEPENDENT DELAY

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Abstract. We study the existence of mild solutions for a class of impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay. We assume that the undelayed part generates a solution operator and transform it into an integral equation. Sufficient conditions for the existence of solutions are derived by using the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan and properties of the solution operator. An example is given to illustrate the theory.

1. Introduction

The study of impulsive functional differential or integro-differential systems is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. Now impulsive partial neutral functional differential or integro-differential systems have become an important object of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. With regard to this matter, we refer the reader to [11, 12, 19, 20, 33]. Besides impulsive effects, stochastic effects likewise exist in real systems. Therefore, impulsive stochastic differential equations describing these dynamical systems subject to both impulse and stochastic changes have attracted considerable attention. Particularly, the papers [5, 22, 27] considered the existence of mild solutions for some impulsive neutral stochastic functional differential and integro-differential equations with infinite delay in Hilbert spaces. As the generalization of classic impulsive differential equations, impulsive stochastic differential inclusions in Hilbert spaces have attracted the researchers great interest. Among them, Ren et al [30] established the controllability of impulsive neutral stochastic functional differential inclusions with infinite delay in an abstract space by means of the fixed point theorem for discontinuous multi-valued operators due to Dhage.

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On the other hand, fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc.. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; see the monograph of Kilbas et al [23] and the papers [1, 3, 7, 24, 25] and the references therein. The existence of solutions for fractional semilinear differential or integro-differential equations is one of the theoretical fields that investigated by many authors [2, 16, 32]. Several papers [4, 15] devoted to the existence of mild solutions for abstract fractional functional differential and integro-differential equations with state-dependent delay in Banach spaces by using fixed point techniques. Recently, the existence, uniqueness and other quantitative and qualitative properties of solutions to various impulsive semilinear fractional differential and integrodifferential systems have been extensively studied in Banach spaces. For example, Balachandran et al [6], Chauhan et al [8], Debbouche and Baleanu [14], Mophou [28], Shu et al [31]. However, the deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. In this paper, we consider the existence of a class of impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay of the form

$$dD(t,x_t) \in \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD(s,x_s)\,ds\,dt + F(t,x_{\rho(t,x_t)})\,dw(t), \tag{1.1}$$
$$t \in J=[0,b], t \neq t_k, k=1,\dots,m,$$
$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$
$$\Delta x(t_k) = I_k(x_{t_k}), \quad k=1,\dots,m, \tag{1.3}$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, $1<\alpha<2$, $A: D(A)\subset H \to H$ is a linear densely defined operator of sectorial type on H. The time history $x_t:(-\infty,0]\to H$ given by $x_t(\theta)=x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; Let K be another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t): t\geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q>0$ defined on a complete probability space $(\Omega,\mathcal{F},P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. We are also employing the same notation $\|\cdot\|$ for the norm $L(K,H)$, where $L(K,H)$ denotes the space of all bounded linear operators from K into H. The initial data $\{\varphi(t): -\infty<t\leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment. $F, G, D(t,\varphi)=\varphi(0)+G(t,\varphi), \varphi\in\mathcal{B}, \rho, I_k (k=1,\dots,m)$, are given functions to be specified later. Moreover, let $0<t_1<\dots<t_m<b$, are prefixed points and the symbol $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively.

We notice that the convolution integral in (1.1) is known as the Riemann-Liouville fractional integral (see [9, 10]). In [10], the authors established the existence of S-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay. To the best of our knowledge, the existence of mild solutions for the impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay in Hilbert spaces has not been investigated yet. Motivated by this consideration, in this paper we will study this interesting problem, which are natural generalizations of the concept of mild solution for impulsive fractional evolution equations well known in the theory of infinite dimensional deterministic systems. Specifically, sufficient conditions for the existence are given by means of the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan combined with the solution operator. The known results appeared in [6, 8, 14, 28, 31] are generalized to the fractional stochastic multi-valued settings and the case of infinite delay.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give our main results. In Section 4, an example is given to illustrate our results. In the last section, concluding remarks are given.

2. Preliminaries

In this section, we introduce some basic definitions, notation and lemmas which are used throughout this paper.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_b = \mathcal{F}$.

Let $L(K, H)$ denote the space of all bounded linear operators from K into H equipped with the usual operator norm $\|\cdot\|_{L(K,H)}$. For $\psi \in L(K, H)$ we define

$$\|\psi\|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2.$$

If $\|\psi\|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operators ψ. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\psi\|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology.

The collection of all strongly measurable, square integrable, H-valued random variables, denoted by $L_2(\Omega, H)$ is a Banach space equipped with norm$\|x(\cdot)\|_{L_2} = (E\|x(\cdot, w)\|^2)^{\frac{1}{2}}$, where the expectation, E is defined by $Ex = \int_\Omega x(w) dP$. Let $C(J, L_2(\Omega, H))$ be the Banach space of all continuous maps from J into $L_2(\Omega, H)$ satisfying the condition $\sup_{0\leq t\leq b} E\|x(t)\|^2 < \infty$. Let $L_2^0(\Omega, H)$ denote the family of all $\mathcal{F}_0$-measurable, H-valued random variables $x(0)$.

Definition 2.1 ([13]). We call $S \subset \Omega$ a P-null set if there is $B \in \mathcal{F}$ such that $S \subseteq B$ and $P(B) = 0$.

Definition 2.2 ([13]). A stochastic process $\{x(t) : t \geq 0\}$ in a real separable Hilbert space H is a Wiener process if for each $t \geq 0$,

- (i) $x(t)$ has continuous sample paths and independent increments.
- (ii) $x(t) \in L^2(\Omega, H)$ and $E(x(t)) = 0$.
- (iii) $\mathrm{Cov}(w(t) - w(s)) = (t - s)Q$, where $Q \in L(K, H)$ is a nonnegative nuclear operator.

Definition 2.3 ([13])**.** Brownian motion is a continuous adapted real-valued process $x(t), t \geq 0$ such that

(i) $x(0) = 0$.
(ii) $x(t) - x(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t$.
(iii) $x(t) - x(s)$ is $N(0, t-s)$-distributed for all $0 \leq s \leq t$.

Definition 2.4 ([13])**.** Normal filtration $\{\mathcal{F}_t : 0 \leq t \leq b\}$ is a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$.

Definition 2.5 ([13])**.** The process x is $\mathcal{F}_0$-adapted if each $x(0)$ is measurable with respect to $\mathcal{F}_0$.

We say that a function $x : [\mu, \tau] \to H$ is a normalized piecewise continuous function on $[\mu, \tau]$ if x is piecewise continuous and left continuous on $(\mu, \tau]$. We denote by $\mathcal{PC}([\mu, \tau], H)$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable processes from $[\mu, \tau]$ into H. In particular, we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, H-valued stochastic processes $\{x(t) : t \in [0, b]\}$ such that x is continuous at $t \neq t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for $k = 1, 2..., m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm

$$\|x\|_{\mathcal{PC}} = (\sup_{0 \leq t \leq b} E\|x(t)\|^2)^{\frac{1}{2}}.$$

Then, we have the following conclusion.

Lemma 2.6. *The set $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\mathcal{PC}$, and fix any $\varepsilon > 0$. There is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $p \in \mathbb{N}$

$$\|x_{n+p} - x_n\|_{\mathcal{PC}} = (\sup_{0 \leq t \leq b} E\|x_{n+p}(t) - x_n(t)\|^2)^{\frac{1}{2}} < \varepsilon$$

for each $t \in [0, b]$. From the above inequality it follows that the sequence $x_n(t)$ is a Cauchy sequence in $L^2(\Omega, H)$; moreover, by the completeness of $L^2(\Omega, H)$ with respect to $\|\cdot\|_{L_2}$, for its limit $x(t) := \lim x_n(t)$, we obtain

$$E\|x_n(t) - x(t)\|^2 < \varepsilon^2$$

for all $n > n_0$. Consequently, $\|x_n - x\|_{\mathcal{PC}} \to 0$ as $n \to \infty$. Next, we need to show that $x \in \mathcal{PC}$. In fact, we verify that x is continuous. By

$$x(t + \Delta t) - x(t) = x(t + \Delta t) - x_n(t + \Delta t) + x_n(t + \Delta t) - x_n(t) + x_n(t) - x(t),$$

it follows that

$$\begin{aligned} E\|x(t + \Delta t) - x(t)\|^2 &\leq 3E\|x(t + \Delta t) - x_n(t + \Delta t)\|^2 \\ &\quad + 3E\|x_n(t + \Delta t) - x_n(t)\|^2 + 3E\|x_n(t) - x(t)\|^2. \end{aligned}$$

Using the uniform convergence of x_n to x with respect to $\|\cdot\|_{L_2}$ and the continuity of x_n, the continuity of x follows. The proof is complete. □

To simplify notation, we put $t_0 = 0, t_{m+1} = b$ and for $x \in \mathcal{PC}$, we denote by $\hat{x}_k \in C([t_k, t_{k+1}]; L_2(\Omega, H))$, $k = 0, 1, \dots, m$, the function given by

$$\hat{x}_k(t) := \begin{cases} x(t) & \text{for } t \in (t_k, t_{k+1}], \\ x(t_k^+) & \text{for } t = t_k. \end{cases}$$

Moreover, for $B \subseteq \mathcal{PC}$ we denote by $\hat{B}_k$, $k = 0, 1, \dots, m$, the set $\hat{B}_k = \{\hat{x}_k : x \in B\}$. The notation $B_r(x, H)$ stands for the closed ball with center at x and radius $r > 0$ in H.

Lemma 2.7. *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\hat{B}_k$ is relatively compact in $C([t_k, t_{k+1}]; L_2(\Omega, H))$, for every $k = 0, 1, \dots, m$.*

Proof. Let $B \subseteq \mathcal{PC}$ be a subset and $\{x^{(i)}(\cdot)\}$ be any sequence of B. Since $\hat{B}_0$ is a relatively compact subset of $C([0, t_1]; L_2(\Omega, H))$. Then, there exists a subsequence of $x^{(i)}$, labeled $\{x_1^{(i)}\} \subset B$, and $x_1 \in C([0, t_1]; L_2(\Omega, H))$, such that

$$x_1^{(i)} \to x_1 \quad \text{in } C([0, t_1]; L_2(\Omega, H)) \quad \text{as } i \to \infty.$$

Similarly, $\hat{B}_k$ is a relatively compact subset of $C([t_k, t_{k+1}]; L_2(\Omega, H))$, for $k = 1, 2, \dots, m$. Then, there exists a subsequence of $x^{(i)}$, labeled $\{x_k^{(i)}\} \subset B$, such that $x_k \in C([t_k, t_{k+1}]; L_2(\Omega, H))$, and

$$x_k^{(i)} \to x^k \quad \text{in } C([t_k, t_{k+1}]; L_2(\Omega, H)) \text{ as } i \to \infty.$$

Setting

$$x(t) = \begin{cases} x_1(t), & t \in [0, t_1], \\ x_2(t), & t \in (t_1, t_2], \\ \dots \\ x_m(t), & t \in (t_m, b], \end{cases}$$

then

$$x_m^{(i)} \to x \quad \text{in } \mathcal{PC} \text{ as } i \to \infty.$$

Thus, the set B is relatively compact.

If set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ and $\{x^{(i)}(\cdot)\}$ be any sequence of B. Then, for each $t \in [0, t_1]$, there exists a subsequence of $x^{(i)}$, labeled $\{x_1^{(i)}\} \subset B$, and $x_1 \in \mathcal{PC}$, such that $x_1^{(i)} \to x_1$ in $\mathcal{PC}$ as $i \to \infty$. From the definition of the set $\hat{B}_0$, we can get

$$\hat{x}_1^{(i)} \to \hat{x}_1 \quad \text{in } C([0, t_1]; L_2(\Omega, H)) \text{ as } i \to \infty.$$

Similarly, for each $t \in [t_k, t_{k+1}] (k = 1, 2, \dots, m)$, there exists a subsequence of $x^{(i)}$, labeled $\{x_k^{(i)}\} \subset B$ and $x_k \in \mathcal{PC}$, such that $x_k^{(i)} \to x_k$ in $\mathcal{PC}$ as $i \to \infty$. From the definition of the set $\hat{B}_k$, we can get

$$\hat{x}_k^{(i)} \to \hat{x}_k \quad \text{in } C([t_k, t_{k+1}]; L_2(\Omega, H)) \text{ as } i \to \infty.$$

Thus, the set $\hat{B}_k$ is relatively compact in $C([t_k, t_{k+1}]; L_2(\Omega, H))$, for every $k = 0, 1, \dots, m$. The proof is complete. $\square$

In this article, we assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into H, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [18]).

- (A) If $x : (-\infty, \sigma + b] \to H$, $b > 0$, is such that $x|_{[\sigma, \sigma+b]} \in C([\sigma, \sigma + b], H)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:
  - (i) x_t is in $\mathcal{B}$;
  - (ii) $\|x(t)\| \le \tilde{H} \|x_t\|_{\mathcal{B}}$;
  - (iii) $\|x_t\|_{\mathcal{B}} \le K(t - \sigma) \sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$, where $\tilde{H} \ge 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous and M is locally bounded, and $\tilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma+b]$ into $\mathcal{B}$.
(C) The space $\mathcal{B}$ is complete.

The next result is a consequence of the phase space axioms.

Lemma 2.8. *Let $x : (-\infty, b] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$ and $x|_J \in \mathcal{PC}(J, H)$, then*

$$\|x_s\|_{\mathcal{B}} \le M_b E\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\le s\le b} E\|x(s)\|,$$

where $K_b = \sup\{K(t) : 0 \le t \le b\}$, $M_b = \sup\{M(t) : 0 \le t \le b\}$.

Proof. For each fixed $x \in H$, we consider the function $\xi(t)$ defined by $\xi(t) = \sup\{\|x_s\|_{\mathcal{B}} : 0 \le s \le t\}$, $0 \le t \le b$. Obviously, ξ is increasing. This combined with the phase space axioms, we have

$$\begin{aligned} \xi(t) &\le M(t)\|\varphi\|_{\mathcal{B}} + K(t) \sup_{0\le s\le t} \|x(s)\| \\ &\le M_b\|\varphi\|_{\mathcal{B}} + K_b\|x(t)\|. \end{aligned}$$

Since $E\|\varphi\|_{\mathcal{B}} < \infty, E\|x(t)\| < \infty$, the previous inequality holds. Consequently

$$\begin{aligned} E(\xi(t)) &\le E(M_b\|\varphi\|_{\mathcal{B}} + K_b\|x(t)\|) \\ &\le M_b E\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\le s\le b} E\|x(s)\| \end{aligned}$$

for each $t \in J$. By the definition of ξ, we have

$$\xi(b) = E(\xi(b)) \le M_b E\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\le s\le b} E\|x(s)\|,$$

and $\|x_s\|_{\mathcal{B}} \le \xi(b)$ for each $s \in J$; therefore,

$$\|x_s\|_{\mathcal{B}} \le M_b E\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\le s\le b} E\|x(s)\|.$$

The proof is complete. □

Let $\mathcal{P}(H)$ denote all the nonempty subsets of H. Let $\mathcal{P}_{bd,cl}(H)$, $\mathcal{P}_{cp,cv}(H)$, $\mathcal{P}_{bd,cl,cv}(H)$, and $\mathcal{P}_{cd}(H)$ denote respectively the family of all nonempty bounded-closed, compact-convex, bounded-closed-convex and compact-acyclic subsets of H (see [17]). For $x \in H$ and $Y, Z \in \mathcal{P}_{bd,cl}(H)$, we denote by $D(x, Y) = \inf\{\|x - y\| : y \in Y\}$ and $\tilde\rho(Y, Z) = \sup_{a\in Y} D(a, Z)$, and the Hausdorff metric $H_d : \mathcal{P}_{bd,cl}(H) \times \mathcal{P}_{bd,cl}(H) \to \mathbb{R}^+$ by $H_d(A, B) = \max\{\tilde\rho(A, B), \tilde\rho(B, A)\}$.

A multi-valued map G is called upper semicontinuous (u.s.c.) on H if, for each $x_0 \in H$, the set $G(x_0)$ is a nonempty, closed subset of H and if, for each open set S of H containing $G(x_0)$, there exists an open neighborhood S of x_0 such that $G(S) \subseteq V$. F is said to be completely continuous if $G(V)$ is relatively compact, for every bounded subset $V \subseteq H$.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if F has a closed graph, i.e. $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

A multi-valued map $G : J \to \mathcal{P}_{bd,cl,cv}(H)$ is measurable if for each $x \in H$, the function $t \mapsto D(x, G(t))$ is a measurable function on J.

Definition 2.9 ([17]). Let $G : H \to \mathcal{P}_{bd,cl}(H)$ be a multi-valued map. Then G is called a multi-valued contraction if there exists a constant $\kappa \in (0, 1)$ such that for each $x, y \in H$ we have

$$H_d(G(x) - G(y)) \leq \kappa \|x - y\|.$$

The constant κ is called a contraction constant of G.

A closed and linear operator A is said to be sectorial of type ω if there exist $0 < \theta < \pi/2$, $M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector $\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C} | \arg(-\lambda) < \theta\}$ and $\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \lambda \notin \omega + S_\theta$. To give an operator theoretical approach we recall the following definition.

Definition 2.10 ([10]). Let A be a closed and linear operator with domain $D(A)$ defined on a Hilbert space H. We call A the generator of a solution operator if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(H)$ such that $\{\lambda^\alpha : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)dt, \operatorname{Re}(\lambda) > \omega, x \in H$. In this case, $S_\alpha(\cdot)$ is called the solution operator generated by A.

We note that, if A is sectorial of type ω with $0 < \theta < \pi(1 - \frac{\alpha}{2})$ then A is the generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Sigma e^{-\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda,$$

where Σ is a suitable path lying outside the sector $\omega + S_\alpha$.

Cuesta [10] proved that, if A is a sectorial operator of type $\omega < 0$, for some $M > 0$ and $0 < \theta < \pi(1 - \frac{\alpha}{2})$, there is $C > 0$ such that

$$\|S_\alpha(t)\| \leq \frac{CM}{1 + |\omega| t^\alpha}, \ t \geq 0. \tag{2.1}$$

Moreover, we have the following results.

Lemma 2.11 ([10]). *Let $S_\alpha(t)$ be a solution operator on H with generator A. Then, we have*

(a) *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A), t \geq 0$;*
(b) *Let $x \in D(A)$ and $t \geq 0$. Then $S_\alpha(t)x = x + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AS_\alpha(s)xds$;*
(c) *Let $x \in H$ and $t > 0$. Then $\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} S_\alpha(s)x\, ds \in D(A)$ and*

$$S_\alpha(t)x = x + A \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} S_\alpha(s)xds.$$

Note that the Laplace transform of the abstract function $f \in L^2(\mathbb{R}^+, L(K, H))$ is defined by

$$\tilde{f}(\varsigma) = \int_0^\infty e^{-\varsigma t} f(t) dw(t).$$

Now we consider the problem

$$dx(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ax(s)\, ds\, dt + f(t)dw(t), \quad t > 0, 1 < \alpha < 2, \tag{2.2}$$

$$x_0 = \varphi \in H. \tag{2.3}$$

Formally applying the Laplace transform, we obtain

$$\lambda \tilde{x}(\varsigma) - \varphi = \lambda^{1-\alpha} A \tilde{x}(\varsigma) + \tilde{f}(\lambda) dw(\lambda),$$

which establishes the result

$$\lambda\tilde{x}(\varsigma) = \lambda^{\alpha-1}R(\lambda^\alpha, A)\varphi + \lambda^{\alpha-1}R(\lambda^\alpha, A)\tilde{f}(\lambda)dw(\lambda).$$

This implies that

$$x(t) = S_\alpha(t)\varphi + \int_0^t S_\alpha(t-s)f(s)dw(s).$$

Let $x : (-\infty, b] \to H$ be a function such that $x, x' \in \mathcal{PC}$. If x is a solution of (1.1)-(1.3), from the partial neutral integro-differential inclusions theory, we obtain

$$x(t) \in S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) + \int_0^t S_\alpha(t-s)F(s, x_{\rho(s,x_s)})dw(s), \quad t \in [0, t_1].$$

By using that $x(t_1^+) = x(t_1^-) + I_k(x_{t_1})$, for $t \in (t_1, t_2]$ we have

$$\begin{aligned} x(t) &\in S_\alpha(t-t_1)[x(t_1^+) - G(t_1, x_{t_1^+})] + G(t, x_t) + \int_{t_1}^t S_\alpha(t-s)F(s, x_{\rho(s,x_s)})dw(s) \\ &= S_\alpha(t-t_1)[x(t_1^-) + I_1(x_{t_1}) - G(t_1, x_{t_1^+})] + G(t, x_t) \\ &\quad + \int_{t_1}^t S_\alpha(t-s)F(s, x_{\rho(s,x_s)})dw(s). \end{aligned}$$

By repeating the same procedure, we can easily deduce that

$$\begin{aligned} x(t) \in\ & S_\alpha(t-t_k)[x(t_k^-) + I_k(x_{t_k}) - G(t_1, x_{t_k^+})] + G(t, x_t) \\ & + \int_{t_k}^t S_\alpha(t-s)F(s, x_{\rho(s,x_s)})dw(s) \end{aligned}$$

holds for any $t \in (t_k, t_{k+1}], k = 2, \dots, m$. This expression motivates the following definition.

Definition 2.12. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b] \to H$ is called a mild solution of the system (1.1)-(1.3) if $x_0 = \varphi, x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in J$ and $\Delta x(t_k) = I_k(x_{t_k}), k = 1, \dots, m$, the restriction of $x(\cdot)$ to the interval $(t_k, t_{k+1}](k = 0, 1, \dots, m)$ is continuous, and

$$x(t) \in \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) & \\ + \int_0^t S_\alpha(t-s)F(s, x_{\rho(s,x_s)})dw(s), & t \in [0, t_1], \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x_{t_1}) - G(t_1, x_{t_1^+})] + G(t, x_t) & \\ + \int_{t_1}^t S_\alpha(t-s)F(s, x_{\rho(s,x_s)})dw(s), & t \in (t_1, t_2], \\ \dots & \\ S_\alpha(t-t_m)[x(t_m^-) + I_m(x_{t_m}) - G(t_m, x_{t_m^+})] + G(t, x_t) & \\ + \int_{t_m}^t S_\alpha(t-s)F(s, x_{\rho(s,x_s)})dw(s), & t \in (t_m, b]. \end{cases}$$

Now we have a nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan.

Lemma 2.13 ([29]). *Let H be a Hilbert space with V an open,convex subset of H and $y \in H$. Suppose*

(a) $\Phi : \overline{V} \to \mathcal{P}_{cd}(H)$ *has closed graph, and*
(b) $\Phi : \overline{V} \to \mathcal{P}_{cd}(H)$ *is a condensing map with $\Phi(\overline{V})$ a subset of a bounded set in H hold.*

Then either

(i) Φ *has a fixed point in* $\overline{V}$; *or*

(ii) *There exist* $y \in \partial V$ *and* $\lambda \in (0,1)$ *with* $y \in \lambda\Phi(y) + (1-\lambda)\{y_0\}$.

3. Main results

In this section we shall present and prove our main result. Assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. In addition, we make the following hypotheses:

(H1) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\psi) \le 0, (s,\psi) \in J \times \mathcal{B}\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \le J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for each $t \in \mathcal{R}(\rho^-)$.

(H2) The multi-valued map $F : J \times \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(L(K,H))$; for each $t \in J$, the function $F(t,\cdot) : \mathcal{B} \to \mathcal{P}_{bd,cl,cv}(L(K,H))$ is u.s.c. and for each $\psi \in \mathcal{B}$, the function $F(\cdot,\psi)$ is measurable; for each fixed $\psi \in \mathcal{B}$, the set

$$S_{F,\psi} = \{f \in L^2(J, L(K,H)) : f(t) \in F(t,\psi) \quad \text{for a.e } t \in J\}$$

is nonempty.

(H3) There exists a positive function $l : J \to \mathbb{R}^+$ such that the function $s \mapsto (\frac{1}{1+|\omega|(t-s)^\alpha})^2 l(s)$ belongs to $L^1([0,t], \mathbb{R}^+), t \in J$, and

$$\limsup_{\|\psi\|_{\mathcal{B}}^2 \to \infty} \frac{\|F(t,\psi)\|^2}{l(t)\|\psi\|_{\mathcal{B}}^2} = \gamma$$

uniformly in $t \in J$ for a nonnegative constant γ, where

$$\|F(t,\psi)\|^2 = \sup\{E\|f\|^2 : f \in F(t,\psi)\}.$$

(H4) The function $G : J \times \mathcal{B} \to H$ is continuous and there exist $L, L_1 > 0$ such that

$$E\|G(t,\psi_1) - G(t,\psi_2)\|^2 \le L\|\psi_1 - \psi_2\|_{\mathcal{B}}^2, \quad t \in J, \psi_1, \psi_2 \in \mathcal{B},$$
$$E\|G(t,\psi)\|^2 \le L_1(\|\psi\|_{\mathcal{B}}^2 + 1), \quad t \in J, \psi \in \mathcal{B},$$

with $4[(CM)^2 + 1]LK_b^2 < 1$.

(H5) The functions $I_k : \mathcal{B} \to H$ are completely continuous and there exist constants c_k such that

$$\limsup_{\|\psi\|_{\mathcal{B}}^2 \to \infty} \frac{E\|I_k(\psi)\|^2}{\|\psi\|_{\mathcal{B}}^2} = c_k$$

for every $\psi \in \mathcal{B}$, $k = 1, \dots, m$.

Remark 3.1. Let $\varphi \in \mathcal{B}$ and $t \le 0$. The notation φ_t represents the function defined by $\varphi_t(\tau) = \varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom (A) is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well-defined for $t < 0$ since the domain of φ is $(-\infty, 0]$. We also note that, in general, $\varphi_t \notin \mathcal{B}$; consider, for instance, a discontinuous function in $C_r \times L^p(h,H)$ for $r > 0$ (see [21]).

Remark 3.2. The condition (H1) is frequently verified by continuous and bounded functions. In fact, if $\mathcal{B}$ verifies axiom (C_2) in the nomenclature of [21], then there exists $\tilde{L} > 0$ such that $\|\varphi\|_{\mathcal{B}} \le \tilde{L}\sup_{\tau \le 0}\|\varphi(\tau)\|$ for every $\varphi \in \mathcal{B}$ continuous and bounded, see [21, Proposition 7.1.1] for details. Consequently,

$$\|\varphi_t\|_{\mathcal{B}} \le \tilde{L}\frac{\sup_{\tau \le 0}\varphi(\tau)}{\|\varphi\|_{\mathcal{B}}},$$

for every continuous and bounded function $\varphi \in \mathcal{B} \setminus \{0\}$ and every $t \leq 0$. We also observe that the space $C_r \times L^p(h, H)$ verifies axiom (C$_2$) see [21, p. 10] for details.

Lemma 3.3. *Let $x : (-\infty, b] \to H$ such that $x_0 = \varphi$ and $x|_{[0,b]} \in \mathcal{PC}(J, H)$. If (H1) be hold, then*

$$\|x_s\|_{\mathcal{B}} \leq (M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{\|x(\theta)\|; \theta \in [0, \max\{0, s\}]\}, s \in \mathcal{R}(\rho^-) \cup J,$$

where $J_0^{\varphi} = \sup_{t \in \mathcal{R}(\rho^-)} J^{\varphi}(t)$.

Proof. For any $s \in \mathcal{R}(\rho^-)$, by (H1), we have

$$\|x_s\|_{\mathcal{B}} \leq \|\varphi_s\|_{\mathcal{B}} \leq J^{\varphi}(s)\|\varphi\|_{\mathcal{B}} \leq J_0^{\varphi}\|\varphi\|_{\mathcal{B}}.$$

For any $s \in [0, b]$, $x \in \mathcal{PC}(J, H)$. Using the phase spaces axioms, we have

$$\begin{aligned} \|x_s\|_{\mathcal{B}} &\leq M(s)\|\varphi\|_{\mathcal{B}} + K(s) \sup\{\|x(s)\| : 0 \leq s \leq t\} \\ &\leq M_b\|\varphi\|_{\mathcal{B}} + K_b \sup\{\|x(s)\| : 0 \leq s \leq t\}. \end{aligned}$$

Then, for $s \in (-\infty, b]$, we have

$$\|x_s\|_{\mathcal{B}} \leq (M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{\|x(\theta)\|; \theta \in [0, \max\{0, s\}]\}, s \in \mathcal{R}(\rho^-) \cup J.$$

The proof is complete. □

Lemma 3.4 ([26])**.** *Let J be a compact interval and H be a Hilbert space. Let F be a multivalued map satisfying* (H2) *and Γ be a linear continuous operator from $L^2(J, H)$ to $C(J, H)$. Then the operator $\Gamma \circ S_F : C(J, H) \to \mathcal{P}_{cp,cv}(C(J, H))$ is a closed graph in $C(J, H) \times C(J, H)$.*

Theorem 3.5. *Let* (H1)–(H5) *be satisfied and $x_0 \in L_2^0(\Omega, H)$, with $\rho(t, \psi) \leq t$ for every $(t, \psi) \in J \times \mathcal{B}$. Then problem* (1.1)-(1.3) *has at least one mild solution on J, provided that*

$$\max_{1 \leq k \leq m}\{9(CM)^2[1 + 2K_b^2 c_k + 2K_b^2 L_1] + 6K_b^2 L_1\} < 1. \tag{3.1}$$

Proof. Consider the space $\mathcal{BPC} = \{x : (-\infty, b] \to H; x_0 = 0, x|_J \in \mathcal{PC}\}$ endowed with the uniform convergence topology and define the multi-valued map $\Phi : \mathcal{BPC} \to \mathcal{P}(\mathcal{BPC})$ by Φx the set of $h \in \mathcal{BPC}$ such that

$$h(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ S_\alpha(t)[\varphi(0) - G(0, \varphi)] + G(t, \bar{x}_t) + \int_0^t S_\alpha(t-s)f(s)dw(s), & t \in [0, t_1], \\ S_\alpha(t-t_1)[\bar{x}(t_1^-) + I_1(\bar{x}_{t_1}) - G(t_1, \bar{x}_{t_1^+})] + G(t, \bar{x}_t) & \\ + \int_{t_1}^t S_\alpha(t-s)f(s)dw(s), & t \in (t_1, t_2], \\ \dots & \\ S_\alpha(t-t_m)[\bar{x}(t_m^-) + I_m(\bar{x}_{t_m}) - G(t_m, \bar{x}_{t_m^+})] + G(t, \bar{x}_t) & \\ + \int_{t_m}^t S_\alpha(t-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

where $f \in S_{F,\bar{x}_\rho} = \{f \in L^2(L(K, H)) : f(t) \in F(t, \bar{x}_{\rho(s,\bar{x}_t)})$ a.e. $t \in J\}$ and $\bar{x} : (-\infty, 0] \to H$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on J. In what follows, we aim to show that the operator Φ has a fixed point, which is a solution of the problem (1.1)-(1.3).

Let $\bar{\varphi} : (-\infty, 0) \to H$ be the extension of $(-\infty, 0]$ such that $\bar{\varphi}(\theta) = \varphi(0) = 0$ on J and $J_0^{\varphi} = \sup\{J^{\varphi}(s) : s \in \mathcal{R}(\rho^-)\}$. We now show that Φ satisfies all the conditions of Lemma 2.13. The proof will be given in several steps.

Step 1. We shall show there exists an open set $V \subseteq \mathcal{BPC}$ with $x \in \lambda\Phi x$ for $\lambda \in (0,1)$ and $x \notin \partial V$. Let $\lambda \in (0,1)$ and let $x \in \lambda\Phi x$, then there exists an $f \in S_{F,\bar{x}_\rho}$ such that we have

$$x(t) = \begin{cases} \lambda S_\alpha(t)[\varphi(0) - G(0,\varphi)] + \lambda G(t,\bar{x}_t) + \lambda \int_0^t S_\alpha(t-s)f(s)dw(s), & t \in [0,t_1], \\ \lambda S_\alpha(t-t_1)[\bar{x}(t_1^-) + I_1(\bar{x}_{t_1}) - G(t_1,\bar{x}_{t_1^+})] + \lambda G(t,\bar{x}_t) \\ +\lambda \int_{t_1}^t S_\alpha(t-s)f(s)dw(s), & t \in (t_1,t_2], \\ \dots \\ \lambda S_\alpha(t-t_m)[\bar{x}(t_m^-) + I_m(\bar{x}_{t_m}) - G(t_m,\bar{x}_{t_m^+})] + \lambda G(t,\bar{x}_t) \\ +\lambda \int_{t_m}^t S_\alpha(t-s)f(s)dw(s), & t \in (t_m,b], \end{cases}$$

for some $\lambda \in (0,1)$. It follows from assumption (H3) that there exist two nonnegative real numbers a_1 and a_2 such that for any $\psi \in \mathcal{B}$ and $t \in J$,

$$\|F(t,\psi)\|^2 \le a_1 l(t) + a_2 l(t)\|\psi\|_{\mathcal{B}}^2. \tag{3.2}$$

On the other hand, from condition (H5), we conclude that there exist positive constants $\epsilon_k (k = 1,\dots,m), \gamma_1$ such that, for all $\|\psi\|_{\mathcal{B}}^2 > \gamma_1$,

$$\begin{gathered} E\|I_k(\psi)\|^2 \le (c_k + \epsilon_k)\|\psi\|_{\mathcal{B}}^2, \\ \max_{1\le k\le m}\{9(CM)^2[1 + 2K_b^2(c_k+\epsilon_k) + 2K_b^2 L_1] + 6K_b^2 L_1\} < 1. \end{gathered} \tag{3.3}$$

Let

$$\begin{gathered} F_1 = \{\psi : \|\psi\|_{\mathcal{B}}^2 \le \gamma_1\}, \quad F_2 = \{\psi : \|\psi\|_{\mathcal{B}}^2 > \gamma_1\}, \\ C_1 = \max\{E\|I_k(\psi)\|^2, x \in F_1\}. \end{gathered}$$

Therefore,

$$E\|I_k(\psi)\|^2 \le C_1 + (c_k+\epsilon_k)\|\psi\|_{\mathcal{B}}^2. \tag{3.4}$$

Then, by (H4), (3.2) and (3.4), from the above equation, for $t \in [0,t_1]$, we have

$$\begin{aligned} E\|x(t)\|^2 &\le 3E\|S_\alpha(t)[\varphi(0) - G(0,\varphi)]\|^2 + 3E\|G(t,\bar{x}_t)\|^2 \\ &\quad + 3E\Big\|\int_0^t S_\alpha(t-s)f(s)dw(s)\Big\|^2 \\ &\le 6(CM)^2[E\|\varphi(0)\|^2 + L_1(\|\varphi\|_{\mathcal{B}}^2 + 1)] + 3L_1(\|\bar{x}_t\|_{\mathcal{B}}^2 + 1) \\ &\quad + 3(CM)^2 \operatorname{Tr}(Q) \int_0^t \Big(\frac{1}{1+|\omega|(t-s)^\alpha}\Big)^2 [a_1 l(s) + a_2 l(s)\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}}^2]ds \\ &\le 6(CM)^2[\tilde{H}^2 E\|\varphi\|_{\mathcal{B}}^2 + L_1(\|\varphi\|_{\mathcal{B}}^2 + 1)] + 3L_1(\|\bar{x}_t\|_{\mathcal{B}}^2 + 1) \\ &\quad + 3(CM)^2 \operatorname{Tr}(Q) a_1 \int_0^{t_1} \Big(\frac{1}{1+|\omega|(t_1-s)^\alpha}\Big)^2 l(s)ds \\ &\quad + 3(CM)^2 \operatorname{Tr}(Q) a_2 \int_0^t \Big(\frac{1}{1+|\omega|(t-s)^\alpha}\Big)^2 l(s)\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}}^2 ds. \end{aligned}$$

Similarly, for any $t \in (t_k,t_{k+1}]$, $k = 1,\dots,m$, we have

$$\begin{aligned} &E\|x(t)\|^2 \\ &\le 3E\|S_\alpha(t-t_k)[\bar{x}(t_k^-) + I_k(\bar{x}_{t_k}) - G(t_k,\bar{x}_{t_k^+})]\|^2 + 3E\|G(t,\bar{x}_t)\|^2 \\ &\quad + 3E\Big\|\int_{t_k}^t S_\alpha(t-s)f(s)dw(s)\Big\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le 9(CM)^2[E\|\bar{x}(t_k^-)\|^2 + C_1 + (c_k+\epsilon_k)\|\bar{x}_{t_k}\|_{\mathcal{B}}^2 + L_1(\|\bar{x}_{t_k^+}\|_{\mathcal{B}}^2+1)] \\
&\quad + 3L_1(\|\bar{x}_t\|_{\mathcal{B}}^2+1) + 3(CM)^2 a_1 \operatorname{Tr}(Q)\int_{t_k}^{t_{k+1}}\Big(\frac{1}{1+|\omega|(t_{k+1}-s)^\alpha}\Big)^2 l(s)ds \\
&\quad + 3(CM)^2 a_2 \operatorname{Tr}(Q)\int_{t_k}^{t}\Big(\frac{1}{1+|\omega|(t-s)^\alpha}\Big)^2 l(s)\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}}^2 ds.
\end{aligned}
$$

Then, for all $t\in[0,b]$, we have

$$
\begin{aligned}
&E\|x(t)\|^2 \\
&\le \widetilde{M} + 9(CM)^2[E\|\bar{x}(t_k^-)\|^2 + (c_k+\epsilon_k)\|\bar{x}_{t_k}\|_{\mathcal{B}}^2 + L_1\|\bar{x}_{t_k^+}\|_{\mathcal{B}}^2] \\
&\quad + 3L_1\|\bar{x}_t\|_{\mathcal{B}}^2 + 3(CM)^2 a_2 \operatorname{Tr}(Q)\int_0^t\Big(\frac{1}{1+|\omega|(t-s)^\alpha}\Big)^2 l(s)\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}}^2 ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{M} = \max\Big\{&6(CM)^2[\tilde{H}^2E\|\varphi\|_{\mathcal{B}}^2 + L_1(\|\varphi\|_{\mathcal{B}}^2+1)] + 3L_1 \\
&+ 3(CM)^2\operatorname{Tr}(Q)a_1\int_0^{t_1}\Big(\frac{1}{1+|\omega|(b-s)^\alpha}\Big)^2 l(s)ds,\ 9(CM)^2(C_1+L_1) \\
&+ 3L_1 + 3(CM)^2 a_1\operatorname{Tr}(Q)\int_{t_k}^{t_{k+1}}\Big(\frac{1}{1+|\omega|(t_{k+1}-s)^\alpha}\Big)^2 l(s)ds\Big\}.
\end{aligned}
$$

By Lemmas 2.8 and 3.3, it follows that $\rho(s,\overline{x}_s)\le s, s\in[0,t], t\in[0,b]$ and

$$
\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}^2 \le 2[(M_b+J_0^\varphi)E\|\varphi\|_{\mathcal{B}}]^2 + 2K_b^2\sup_{0\le s\le b}E\|x(s)\|^2. \tag{3.5}
$$

For each $t\in[0,b]$, we have

$$
\begin{aligned}
E\|x(t)\|^2 &\le M_* + \{9(CM)^2[1+2K_b^2(c_k+\epsilon_k)+2K_b^2L_1]+6K_b^2L_1\}\sup_{t\in[0,b]}E\|x(t)\|^2 \\
&\quad + 6(CM)^2a_2K_b^2\operatorname{Tr}(Q)\int_0^t\Big(\frac{1}{1+|\omega|(t-s)^\alpha}\Big)^2 l(s)\sup_{\tau\in[0,s]}E\|x(\tau)\|^2 ds,
\end{aligned}
$$

where

$$
\begin{gathered}
M_* = \widetilde{M} + 9(CM)^2[C_1 + (c_k+\epsilon_k)C^* + L_1(C^*+1)] + 3L_1(C^*+1) \\
+ 3(CM)^2\operatorname{Tr}(Q)a_2C^*\int_0^b\Big(\frac{1}{1+|\omega|(b-s)^\alpha}\Big)^2 l(s)ds, \\
C^* = 2[(M_b+J_0^\varphi)\|\varphi\|_{\mathcal{B}}]^2.
\end{gathered}
$$

Since $L_* = \max_{1\le k\le m}\{9(CM)^2[1+2K_b^2(c_k+\epsilon_k)+2K_b^2L_1]+6K_b^2L_1\} < 1$, we have

$$
\begin{aligned}
&\sup_{t\in[0,b]}E\|x(t)\|^2 \\
&\le \frac{M_*}{1-L_*} + \frac{6(CM)^2a_2K_b^2\operatorname{Tr}(Q)}{1-L_*}\int_0^b\Big(\frac{1}{1+|\omega|(b-s)^\alpha}\Big)^2 l(s)\sup_{\tau\in[0,s]}E\|x(\tau)\|^2 ds.
\end{aligned}
$$

Applying Gronwall's inequality in the above expression, we obtain

$$
\sup_{t\in[0,b]}E\|x(s)\|^2 \le \frac{M_*}{1-L_*}\exp\Big\{\frac{6(CM)^2a_2K_b^2\operatorname{Tr}(Q)}{1-L_*}\int_0^b\Big(\frac{1}{1+|\omega|(b-s)^\alpha}\Big)^2 l(s)ds\Big\}
$$

and, therefore,

$$\|x\|^2_{\mathcal{PC}} \le \frac{M_*}{1-L_*} \exp\Big\{\frac{6(CM)^2 a_2 K_b^2 \operatorname{Tr}(Q)}{1-L_*}\int_0^b \Big(\frac{1}{1+|\omega|(b-s)^\alpha}\Big)^2 l(s)ds\Big\} < \infty.$$

Then, there exists r^* such that $\|x\|^2_{\mathcal{PC}} \neq r^*$. Set

$$V = \{x \in \mathcal{BPC} : \|x\|^2_{\mathcal{PC}} < r^*\}.$$

From the choice of V, there is no $x \in \partial V$ such that $x \in \lambda \Phi x$ for $\lambda \in (0,1)$.

Step 2. Φ has a closed graph. Let $x^{(n)} \to x^*, h_n \in \Phi x^{(n)}, x^{(n)} \in \overline{V} = B_{r^*}(0, \mathcal{BPC})$ and $h_n \to h_*$. From Axiom (A), it is easy to see that $(\overline{x^{(n)}})_s \to \overline{x^*}_s$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. We prove that $h_* \in \Phi\overline{x^*}$. Now $h_n \in \Phi\overline{x^{(n)}}$ means that there exists $f_n \in S_{F,\overline{x^{(n)}}_\rho}$ such that, for each $t \in [0,t_1]$,

$$h_n(t) = S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t,(\overline{x^{(n)}})_t) + \int_0^t S_\alpha(t-s)f_n(s)dw(s), \quad t \in [0,t_1].$$

We must prove that there exists $f_* \in S_{F,\overline{x^*}_\rho}$ such that, for each $t \in [0,t_1]$,

$$h_*(t) = S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t,(\overline{x^*})_t) + \int_0^t S_\alpha(t-s)f_*(s)dw(s), \quad t \in [0,t_1].$$

Now, for every $t \in [0,t_1]$, we have

$$\begin{aligned}
&\Big\|\Big(h_n(t) - S_\alpha(t)[\varphi(0) - G(0,\varphi)] - G(t,(\overline{x^{(n)}})_t) - \int_0^t S_\alpha(t-s)f_n(s)dw(s)\Big) \\
&- \Big(h_*(t) - S_\alpha(t)[\varphi(0) - G(0,\varphi)] - G(t,(\overline{x^*})_t) \\
&- \int_0^t S_\alpha(t-s)f_*(s)dw(s)\Big)\Big\|^2_{\mathcal{PC}} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Consider the linear continuous operator $\Psi : L([0,t_1],H) \to C([0,t_1],H)$,

$$\Psi(f)(t) = \int_0^t S_\alpha(t-s)f(s)dw(s).$$

From Lemma 3.4, it follows that $\Psi \circ S_F$ is a closed graph operator. Also, from the definition of Ψ, we have that, for every $t \in [0,t_1]$,

$$h_n(t) - S_\alpha(t)[\varphi(0) - G(0,\varphi)] - G(t,(\overline{x^{(n)}})_t) - \int_0^t S_\alpha(t-s)f_n(s)dw(s)\Big) \in \Gamma(S_{F,\overline{x^{(n)}}}).$$

Since $\overline{x^{(n)}} \to \overline{x^*}$, for some $f_* \in S_{F,\overline{x^*}_\rho}$ it follows that, for every $t \in [0,t_1]$,

$$h_*(t) - S_\alpha(t)[\varphi(0) - G(0,\varphi)] - G(t,(\overline{x^*})_t) = \int_0^t \mathcal{S}_\alpha(t-s)f_* dw(s).$$

Similarly, for any $t \in (t_k, t_{k+1}], k = 1,\dots,m$, we have

$$\begin{aligned}
h_n(t) &= S_\alpha(t-t_k)[\overline{x^{(n)}}(t_k^-) + I_k(\overline{x^{(n)}}_{t_k}) - G(t_k,(\overline{x^{(n)}})_{t_k^+})] + G(t,(\overline{x^{(n)}})_t) \\
&\quad + \int_{t_k}^t S_\alpha(t-s)f_n(s)dw(s), \quad t \in (t_k,t_{k+1}].
\end{aligned}$$

We must prove that there exists $f_* \in S_{F,\overline{x^*}_\rho}$ such that, for each $t \in (t_k,t_{k+1}]$,

$$h_*(t) = S_\alpha(t-t_k)[\overline{x^*}(t_k^-) + I_k(\overline{x^*}_{t_k}) - G(t_k,(\overline{x^*})_{t_k^+})] + G(t,(\overline{x^*})_t)$$

$$+\int_{t_k}^{t} S_\alpha(t-s)f_*(s)dw(s), \quad t\in(t_k,t_{k+1}].$$

Now, for every $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$, we have

$$\Big\|\Big(h_n(t)-S_\alpha(t-t_k)[\overline{x^{(n)}}(t_k^-)+I_k(\overline{x^{(n)}}_{t_k})-G(t_k,(\overline{x^{(n)}})_{t_k^+})]-G(t,(\overline{x^{(n)}})_t)$$
$$-\int_{t_k}^{t} S_\alpha(t-s)f_n(s)dw(s)\Big)-\Big(h_*(t)-S_\alpha(t-t_k)\big[\overline{x^*}(t_k^-)+I_k(\overline{x^*}_{t_k})$$
$$-G(t_k,(\overline{x^*})_{t_k^+})\big]-G(t,(\overline{x^*})_t)-\int_{t_k}^{t} S_\alpha(t-s)f_*(s)dw(s)\Big)\Big\|_{\mathcal{PC}}^2\to 0 \quad \text{as } n\to\infty.$$

Consider the linear continuous operator $\Psi: L^2((t_k,t_{k+1}],H)\to C((t_k,t_{k+1}],H)$, $k=1,\dots,m$,

$$\Psi(f)(t)=\int_{t_k}^{t} S_\alpha(t-s)f(s)dw(s).$$

From Lemma 3.4, it follows that $\Psi\circ S_F$ is a closed graph operator. Also, from the definition of Ψ, we have that, for every $t\in(t_k,t_{k+1}],k=1,\dots,m$,

$$h_n(t)-S_\alpha(t-t_k)[\overline{x^{(n)}}(t_k^-)+I_k(\overline{x^{(n)}}_{t_k})-G(t_k,(\overline{x^{(n)}})_{t_k^+})]-G(t,(\overline{x^{(n)}})_t)\in\Gamma(S_{F,\overline{x^{(n)}}_\rho}).$$

Since $\overline{x^{(n)}}\to\overline{x^*}$, for some $f_*\in S_{F,\overline{x^*}_\rho}$ it follows that, for every $t\in(t_k,t_{k+1}]$, we have

$$\begin{aligned}&h_*(t)-S_\alpha(t-t_k)[\overline{x^*}(t_k^-)+I_k(\overline{x^*}_{t_k})-G(t_k,(\overline{x^*})_{t_k^+})]-G(t,(\overline{x^*})_t)\\&=\int_{t_k}^{t} S_\alpha(t-s)f_*(s)dw(s).\end{aligned}$$

Therefore, Φ has a closed graph.

Step 3. We show that the operator Φ condensing. For this purpose, we decompose Φ as $\Phi_1+\Phi_2$, where the map $\Phi_1:\overline{V}\to\mathcal{P}(\mathcal{BPC})$ be defined by $\Phi_1 x$, the set $h_1\in\mathcal{BPC}$ such that

$$h_1(t)=\begin{cases}0, & t\in(-\infty,0],\\ -S_\alpha(t)G(0,\varphi)+G(t,\bar{x}_t), & t\in[0,t_1],\\ -S_\alpha(t-t_1)G(t_1,\bar{x}_{t_1^+})+G(t,\bar{x}_t), & t\in(t_1,t_2],\\ \dots\\ -S_\alpha(t-t_m)G(t_m,\bar{x}_{t_m^+})+G(t,\bar{x}_t), & t\in(t_m,b],\end{cases}$$

and the map $\Phi_2:\overline{V}\to\mathcal{P}(\mathcal{BPC})$ be defined by $\Phi_2 x$, the set $h_2\in\mathcal{BPC}$ such that

$$h_2(t)=\begin{cases}0, & t\in(-\infty,0],\\ S_\alpha(t)\varphi(0)+\int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t\in[0,t_1],\\ S_\alpha(t-t_1)[\bar{x}(t_1^-)+I_1(\bar{x}_{t_1})]+\int_{t_1}^t S_\alpha(t-s)f(s)dw(s), & t\in(t_1,t_2],\\ \dots\\ S_\alpha(t-t_m)[\bar{x}(t_m^-)+I_m(\bar{x}_{t_m})]+\int_{t_m}^t S_\alpha(t-s)f(s)dw(s), & t\in(t_m,b].\end{cases}$$

We first show that Φ_1 is a contraction while Φ_2 is a completely continuous operator.

Claim 1. Φ_1 is a contraction on $\mathcal{BPC}$. Let $t\in[0,t_1]$ and $v^*,v^{**}\in\mathcal{BPC}$. From (H4), Lemmas 2.8 and 3.3, we have

$$E\|(\Phi_1 v^*)(t)-(\Phi_1 v^{**})(t)\|^2\le E\|G(t,\overline{v^*}_t)-G(t,\overline{v^{**}}_t)\|^2$$

$$
\begin{aligned}
&\le L\|\overline{v^*}_t - \overline{v^{**}}_t\|_{\mathcal{B}}^2 \\
&\le 2LK_b^2 \sup\{\|\overline{v^*}(\tau) - \overline{v^{**}}(\tau)\|^2, 0 \le \tau \le t\} \\
&\le 2LK_b^2 \sup_{s\in[0,b]} \|\overline{v^*}(s) - \overline{v^{**}}(s)\|^2 \\
&= 2LK_b^2 \sup_{s\in[0,b]} \|v^*(s) - v^{**}(s)\|^2 \quad (\text{since } \bar{v} = v \text{ on } J) \\
&= 2LK_b^2 \|v^* - v^{**}\|_{\mathcal{PC}}^2.
\end{aligned}
$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, we have

$$
\begin{aligned}
&E\|(\Phi_1 v^*)(t) - (\Phi_1 v^{**})(t)\|^2 \\
&\le 2E\|S_\alpha(t-t_k)[-G(t_k, \overline{v^*}_{t_k^+}) + G(t_k, \overline{v^{**}}_{t_k^+})]\|^2 + 2E\|G(t, \overline{v^*}_t) - G(t, \overline{v^{**}}_t)\|^2 \\
&\le 2(CM)^2 L\|\overline{v^*}_{t_k^+} - \overline{v^{**}}_{t_k^+}\|_{\mathcal{B}}^2 + 2L\|\overline{v^*}_t - \overline{v^{**}}_t\|_{\mathcal{B}}^2 \\
&\le 4((CM)^2+1)LK_b^2 \sup_{s\in[0,b]} \|\overline{v^*}(s) - \overline{v^{**}}(s)\|^2 \\
&= 4((CM)^2+1)LK_b^2 \sup_{s\in[0,b]} \|v^*(s) - v^{**}(s)\|^2 \quad (\text{since } \bar{v} = v \text{ on } J) \\
&= 4[(CM)^2+1)]LK_b^2 \|v^* - v^{**}\|_{\mathcal{PC}}^2,
\end{aligned}
$$

Thus, for all $t \in [0, b]$, we have

$$
E\|(\Phi_1 v^*)(t) - (\Phi_1 v^{**})(t)\|^2 \le L_0 \|v^* - v^{**}\|_{\mathcal{PC}}^2.
$$

Taking supremum over t,

$$
\|\Phi_1 v^* - \Phi_1 v^{**}\|_{\mathcal{PC}}^2 \le L_0 \|v^* - v^{**}\|_{\mathcal{PC}}^2,
$$

where $L_0 = 4[(CM)^2 + 1]LK_b^2 < 1$. Hence, Φ_1 is a contraction on $\mathcal{BPC}$.

Claim 2. Φ_2 is convex for each $x \in \overline{V}$. In fact, if h_2^1, h_2^2 belong to $\Phi_2 x$, then there exist $f_1, f_2 \in S_{F,\bar{x}_\rho}$ such that

$$
h_2^i(t) = S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(t-s) f_i(s) dw(s), \quad t \in [0, t_1],\ i = 1, 2.
$$

Let $0 \le \lambda \le 1$. For each $t \in [0, t_1]$ we have

$$
(\lambda h_2^1 + (1-\lambda) h_2^2)(t) = S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(t-s)[\lambda f_1(s) + (1-\lambda) f_2(s)] dw(s).
$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, we have

$$
h_2^i(t) = S_\alpha(t-t_k)[\bar{x}(t_k^-) + I_k(\bar{x}_{t_k})] + \int_{t_k}^t S_\alpha(t-s) f_i(s) dw(s), \quad i = 1, 2.
$$

Let $0 \le \lambda \le 1$. For each $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, we have

$$
\begin{aligned}
(\lambda h_2^1 + (1-\lambda) h_2^2)(t) &= S_\alpha(t-t_k)[\bar{x}(t_k^-) + I_k(\bar{x}_{t_k})] \\
&\quad + \int_{t_k}^t S_\alpha(t-s)[\lambda f_1(s) + (1-\lambda) f_2(s)] dw(s).
\end{aligned}
$$

Since $S_{F,\bar{x}_\rho}$ is convex (because F has convex values) we have $(\lambda h_2^1 + (1-\lambda) h_2^2) \in \Phi_2 x$.

Claim 3. $\Phi_2(\overline{V})$ is completely continuous. We begin by showing $\Phi_2(\overline{V})$ is equicontinuous. If $x \in \overline{V}$, from Lemmas 2.8 and 3.3, it follows that

$$
\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}}^2 \le 2[(M_b + J_0^\varphi)\|\varphi\|_{\mathcal{B}}]^2 + 2K_b^2 r^* := r'.
$$

Let $0 < \tau_1 < \tau_2 \le t_1$. For each $x \in \overline{V}$, $h_2 \in \Phi_2 x$, there exists $f \in S_{F,\bar{x}_\rho}$, such that

$$h_2(t) = S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(t-s)f(s)dw(s). \tag{3.6}$$

Then

$$\begin{aligned}
&E\|h_2(\tau_2) - h_2(\tau_1)\|^2 \\
&\le 4E\|[S_\alpha(\tau_2) - S_\alpha(\tau_1)]\varphi(0)\|^2 + 4E\Big\| \int_0^{\tau_1-\varepsilon} [S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)]f(s)dw(s)\Big\|^2 \\
&\quad + 4E\Big\| \int_{\tau_1-\varepsilon}^{\tau_1} [S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)]f(s)dw(s)\Big\|^2 \\
&\quad + 4E\Big\| \int_{\tau_1}^{\tau_2} S_\alpha(\tau_2 - s)f(s)dw(s)\Big\|^2 \\
&\le 4E\|[S_\alpha(\tau_2) - S_\alpha(\tau_1)]\varphi(0)\|^2 + 4(CM)^2(a_1 + a_2 r')[1 + |\omega| b^\alpha]^2 \operatorname{Tr}(Q) \\
&\quad \times \int_0^{\tau_1-\varepsilon} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|^2 \Big(\frac{1}{1 + |\omega|(\tau_1 - \varepsilon - s)^\alpha}\Big)^2 l(s)ds \\
&\quad + 4(CM)^2(a_1 + a_2 r') \operatorname{Tr}(Q) \int_{\tau_1-\varepsilon}^{\tau_1} \Big(\frac{1}{1 + |\omega|(\tau_2 - s)^\alpha}\Big)^2 l(s)ds \\
&\quad + 4(CM)^2(a_1 + a_2 r') \operatorname{Tr}(Q) \int_{\tau_1-\varepsilon}^{\tau_1} \Big(\frac{1}{1 + |\omega|(\tau_1 - s)^\alpha}\Big)^2 l(s)ds \\
&\quad + 4(CM)^2(a_1 + a_2 r') \operatorname{Tr}(Q) \int_{\tau_1}^{\tau_2} \Big(\frac{1}{1 + |\omega|(\tau_2 - s)^\alpha}\Big)^2 l(s)ds.
\end{aligned}$$

Similarly, for any $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, $\tau_1 < \tau_2$, $k = 1, \dots, m$, we have

$$h_2(t) = S_\alpha(t - t_k)[\bar{x}(t_k^-) + I_k(\bar{x}_{t_k})] + \int_{t_k}^t S_\alpha(t-s)f(s)dw(s). \tag{3.7}$$

Then

$$\begin{aligned}
&E\|h_2(\tau_2) - h_2(\tau_1)\|^2 \\
&\le 4E\|[S_\alpha(\tau_2) - S_\alpha(\tau_1)]\varphi(0)\|^2 + 4E\Big\| \int_{t_k}^{\tau_1-\varepsilon} [S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)]f(s)dw(s)\Big\|^2 \\
&\quad + 4E\Big\| \int_{\tau_1-\varepsilon}^{\tau_1} [S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)]f(s)dw(s)\Big\|^2 \\
&\quad + 4E\Big\| \int_{\tau_1}^{\tau_2} S_\alpha(\tau_2 - s)f(s)dw(s)\Big\|^2 \\
&\le 4E\|[S_\alpha(\tau_2) - S_\alpha(\tau_1)]\varphi(0)\|^2 + 4(CM)^2(a_1 + a_2 r')[1 + |\omega| b^\alpha]^2 \operatorname{Tr}(Q) \\
&\quad \times \int_{t_k}^{\tau_1-\varepsilon} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|^2 \Big(\frac{1}{1 + |\omega|(\tau_1 - \varepsilon - s)^\alpha}\Big)^2 l(s)ds \\
&\quad + 4(CM)^2(a_1 + a_2 r') \operatorname{Tr}(Q) \int_{\tau_1-\varepsilon}^{\tau_1} \Big(\frac{1}{1 + |\omega|(\tau_2 - s)^\alpha}\Big)^2 l(s)ds \\
&\quad + 4(CM)^2(a_1 + a_2 r') \operatorname{Tr}(Q) \int_{\tau_1-\varepsilon}^{\tau_1} \Big(\frac{1}{1 + |\omega|(\tau_1 - s)^\alpha}\Big)^2 l(s)ds \\
&\quad + 4(CM)^2(a_1 + a_2 r') \operatorname{Tr}(Q) \int_{\tau_1}^{\tau_2} \Big(\frac{1}{1 + |\omega|(\tau_2 - s)^\alpha}\Big)^2 l(s)ds.
\end{aligned}$$

From the above inequalities, we see that the right-hand side of $E\|h_2(\tau_2)-h_2(\tau_1)\|^2$ tends to zero independent of $x \in \overline{V}$ as $\tau_2-\tau_1 \to 0$ with ε sufficiently small, since $I_k, k=1,2,\dots,m$, are completely continuous in H and the compactness of $S_\alpha(t)$ for $t>0$ imply the continuity in the uniform operator topology. Indeed, the fact of $S_\alpha(\cdot)$ is compact in H since it is generated by the dense operator A. Thus the set $\{\Phi_2 x : x \in \overline{V}\}$ is equicontinuous. The equicontinuities for the other cases $\tau_1<\tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2 \le b$ are very simple.

Next, we prove that $\Phi_2(\overline{V})(t)=\{h_2(t) : h_2(t) \in \Phi_2(\overline{V})\}$ is relatively compact for every $t \in [0,b]$. To this end, we decompose Φ_2 by $\Phi_2(\overline{V})=\Gamma_1(\overline{V})+\Gamma_2(\overline{V})$, where the map Γ_1 is defined by $\Gamma_1 x, x \in \overline{V}$ the set $\tilde{h}_1$ such that

$$\tilde{h}_1(t)=\begin{cases} \int_0^t S_\alpha(t-s)f(s)dw(s), & t \in [0,t_1],\\ \int_{t_k}^t S_\alpha(t-s)f(s)dw(s), & t \in (t_1,t_2],\\ \dots \\ \int_{t_m}^t S_\alpha(t-s)f(s)dw(s), & t \in (t_m,b], \end{cases}$$

and the map Γ_2 is defined by $\Gamma_2 x, x \in \overline{V}$ the set $\tilde{h}_2$ such that

$$\tilde{h}_2(t)=\begin{cases} S_\alpha(t)\varphi(0), & t \in [0,t_1],\\ S_\alpha(t-t_1)[\bar{x}(t_1^-)+I_1(\bar{x}_{t_1})], & t \in (t_1,t_2],\\ \dots \\ S_\alpha(t-t_m)[\bar{x}(t_m^-)+I_m(\bar{x}_{t_m})], & t \in (t_m,b]. \end{cases}$$

We now prove that $\Gamma_1(\overline{V}))(t)=\{\tilde{h}_1(t) : \tilde{h}_1(t) \in \Gamma_1(\overline{V}))\}$ is relatively compact for every $t \in [0,b]$. Let $0<t \le s \le t_1$ be fixed and let ε be a real number satisfying $0<\varepsilon<t$. For $x \in \overline{V}$, we define

$$\tilde{h}_{1,\varepsilon}(t)=\int_0^{t-\varepsilon} S_\alpha(t-s)f(s)dw(s),$$

where $f \in S_{F,\bar{x}_\rho}$. Using the compactness of $\mathcal{S}_\alpha(t)$ for $t>0$, we deduce that the set $U_\varepsilon(t)=\{\tilde{h}_{1,\varepsilon}(t) : x \in \overline{V}\}$ is relatively compact in H for every $\varepsilon, 0<\varepsilon<t$. Moreover, for every $x \in \overline{V}$ we have

$$\begin{aligned} E\|\tilde{h}_1(t)-\tilde{h}_{1,\varepsilon}(t)\|^2 &\le \Big\| \int_{t-\varepsilon}^t \mathcal{S}_\alpha(t-s)f(s)dw(s)\Big\| \\ &\le (CM)^2(a_1+a_2r')\operatorname{Tr}(Q)\int_{t-\varepsilon}^t \Big(\frac{1}{1+|\omega|(t-s)^\alpha}\Big)^2 l(s)ds. \end{aligned}$$

Similarly, for any $t \in (t_k,t_{k+1}]$ with $k=1,\dots,m$. Let $t_k<t \le s \le t_{k+1}$ be fixed and let ε be a real number satisfying $0<\varepsilon<t$. For $x \in \overline{V}$, we define

$$\tilde{h}_{1,\varepsilon}(t)=\int_{t_k}^{t-\varepsilon} S_\alpha(t-s)f(s)dw(s),$$

where $f \in S_{F,\bar{x}_\rho}$. Using the compactness of $\mathcal{S}_\alpha(t)$ for $t>0$, we deduce that the set $U_\varepsilon(t)=\{\tilde{h}_{1,\varepsilon}(t) : x \in \overline{V}\}$ is relatively compact in H for every $\varepsilon, 0<\varepsilon<t$. Moreover, for every $x \in \overline{V}$ we have

$$E\|\tilde{h}_1(t)-\tilde{h}_{1,\varepsilon}(t)\|^2 \le \Big\| \int_{t-\varepsilon}^t \mathcal{S}_\alpha(t-s)f(s)dw(s)\Big\|$$

$$\leq (CM)^2(a_1 + a_2 r')\operatorname{Tr}(Q)\int_{t-\varepsilon}^{t}\Big(\frac{1}{1+|\omega|(t-s)^\alpha}\Big)^2 l(s)ds.$$

The right hand side of the above inequality tends to zero as $\varepsilon \to 0$. Since there are relatively compact sets arbitrarily close to the set $U(t) = \{\tilde{h}_1(t) : x \in \overline{V}\}$. Hence the set $U(t)$ is relatively compact in H. By Arzelá-Ascoli theorem, we conclude that $\Gamma_1(\overline{V})$ is completely continuous.

Next, we show that $\Gamma_2(\overline{V})(t) = \{\tilde{h}_2(t) : \tilde{h}_2(t) \in \Gamma_2(\overline{V})\}$ is relatively compact for every $t \in [0, b]$. For all $t \in [0, t_1]$, since $\tilde{h}_2(t) = S_\alpha(t)\varphi(0)$, by the $S_\alpha(\cdot)$ is compact operator, it follows that $\{\tilde{h}_2(t) : t \in [0, t_1], x \in \overline{V}\}$ is a compact subset of H. On the other hand, for $t \in (t_k, t_{k+1}], k = 1, \dots, m$, and $x \in \overline{V}$, there exists $r' > 0$ such that

$$\widehat{[\tilde{h}_2]}_k(t) \in \begin{cases} S_\alpha(t-t_k)[\bar{x}(t_k^-) + I_k(\bar{x}_{t_k})], & t \in (t_k, t_{k+1}),\ x \in \overline{V}_{r''},\\ S_\alpha(t_{k+1}-t_k)[\bar{x}(t_k^-) + I_k(\bar{x}_{t_k})], & t = t_{k+1},\ x \in \overline{V}_{r''},\\ \bar{x}(t_k^-) + I_k(\bar{x}_{t_k}), & t = t_k,\ x \in \overline{V}_{r''}, \end{cases}$$

where $\overline{V}_{r''}$ is an open ball of radius r''. From (H5), it follows that $\widehat{[\tilde{h}_2]}_k(t)$ is relatively compact in H, for all $t \in [t_k, t_{k+1}]$, $k = 1, \dots, m$. By Lemma 2.7, we infer that $\Gamma_2(\overline{V})$ is relatively compact. Moreover, using the compactness of $\{I_k\}(k = 1, \dots, m)$ and the continuity of the operator $S_\alpha(t)$, for all $t \in [0, b]$, $\Gamma_2(\overline{V})$ is completely continuous, and hence $\Phi_2(\overline{V})$ is completely continuous.

As a consequence of the above steps 1-3, we conclude that $\Phi = \Phi_1 + \Phi_2$ is a condensing map. All of the conditions of Lemma 2.13 are satisfied, we deduce that Φ has a fixed point $x \in \mathcal{BPC}$, which is in turn a mild solution of the problem (1.1)-(1.3). The proof is complete. □

Remark 3.6. Note that by the condition $\rho(s, \overline{x}_s) \leq s$, $s \in [0, t]$, $t \in [0, b]$ and using Lemma 3.3, we have

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}} \leq (M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{\|\bar{x}(s)\| : 0 \leq s \leq t\}.$$

By lemma 2.8 this implies that

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}} \leq (M_b + J_0^{\varphi})E\|\varphi\|_{\mathcal{B}} + K_b \sup_{0\leq s\leq b} E\|x(s)\|,$$

and so (3.5) holds.

4. Application

Consider the following impulsive fractional partial neutral stochastic functional integro-differential inclusions of the form

$$\begin{aligned} dD(t, z_t)(x) \in{}& J_t^{\alpha-1}\Big(\frac{\partial^2}{\partial x^2} - \nu\Big)D(t, z_t)(x)dt\\ &+ \int_{-\infty}^{t} \mu_2(t, s-t, x, z(s - \rho_1(t)\rho_2(\|z(t)\|), x))dw(s),\\ &0 \leq t \leq b,\ 0 \leq x \leq \pi, \end{aligned} \tag{4.1}$$

$$z(t, 0) = z(t, \pi) = 0, \quad 0 \leq t \leq b, z(\tau, x) = \varphi(\tau, x), \quad \tau \leq 0,\ 0 \leq x \leq \pi, \tag{4.2}$$

$$\triangle z(t_k, x) = \int_{-\infty}^{t_k} \eta_k(s - t_k)z(s, x)ds, \quad k = 1, 2, \dots, m, \tag{4.3}$$

where $1 < \alpha < 2$, $\nu > 0$ and φ is continuous and $w(t)$ denotes a standard cylindrical Wiener process in H defined on a stochastic space $(\Omega, \mathcal{F}, P)$. In this system,

$$D(t, z_t)(x) = z(t, x) - \int_{-\infty}^{t} \mu_1(s-t)z(s, x)ds.$$

Let $H = L^2([0, \pi])$ with the norm $\|\cdot\|$ and define the operator $A : D(A) \subset H \to H$ is the operator given by $A\omega = \omega'' - \nu\omega$ with the domain

$$D(A) := \{\omega \in H : \omega'' \in H, \omega(0) = \omega(\pi) = 0\}.$$

It is well known that $\Delta x = x''$ is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ on H. Hence, A is sectorial of type $\mu = -\nu < 0$.

Let $r \geq 0, 1 \leq p < 1$ and let $h : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (h-5), (h-6) in the terminology of Hino et al [21]. Briefly, this means that h is locally integrable and there is a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $h(\xi + \tau) \leq \gamma(\xi)h(\tau)$ for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set whose Lebesgue measure zero. We denote by $\mathcal{PC}_r \times L^2(h, H)$ the set consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi_{|_{[-r,0]}} \in \mathcal{PC}([-r, 0], H)$, $\varphi(\cdot)$ is Lebesgue measurable on $(-\infty, -r)$, and $h\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm is given by

$$\|\varphi\|_{\mathcal{B}} = \sup_{-r\leq\tau\leq 0} \|\varphi(\tau)\| + \Big(\int_{-\infty}^{-r} h(\tau)\|\varphi\|^p d\tau\Big)^{1/p}.$$

The space $\mathcal{B} = \mathcal{PC}_r \times L^p(h, H)$ satisfies axioms (A)–(C). Moreover, when $r = 0$ and $p = 2$, we can take $\tilde{H} = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + (\int_{-t}^{0} h(\tau)d\tau)^{1/2}$, for $t \geq 0$ (see [21, Theorem 1.3.8] for details).

Additionally, we will assume that

(i) The functions $\rho_i : [0, \infty) \to [0, \infty), i = 1, 2$, are continuous.

(ii) The functions $\mu_1 : \mathbb{R} \to \mathbb{R}$, are continuous, and $l_1 = (\int_{-\infty}^{0} \frac{(\mu_1(s))^2}{h(s)} ds)^{1/2} < \infty$.

(iii) The function $\mu_2 : \mathbb{R}^4 \to \mathbb{R}$ is continuous and there exist continuous functions $b_1, b_2 : \mathbb{R} \to \mathbb{R}$ such that

$$|\mu_2(t, s, x, y)| \leq b_1(t)b_2(s)|y|, \quad (t, s, x, y) \in \mathbb{R}^4$$

with $l_2 = (\int_{-\infty}^{0} \frac{(b_2(s))^2}{h(s)} ds)^{1/2} < \infty$.

(iv) The functions $\eta_k : \mathbb{R} \to \mathbb{R}, k = 1, 2, \ldots, m$, are continuous, and $L_k = (\int_{-\infty}^{0} \frac{(\eta_k(s))^2}{h(s)} ds)^{1/2} < \infty$ for every $k = 1, 2, \ldots, m$.

In the sequel, $\mathcal{B}$ will be the phase space $\mathcal{PC}_0 \times L^2(h, H)$. Set $\varphi(\theta)(x) = \varphi(\theta, x) \in \mathcal{B}$, defining the maps $G : [0, b] \times \mathcal{B} \to H$, $F : [0, b] \times \mathcal{B} \to \mathcal{P}(H)$ by

$$G(t, \varphi)(x) = \int_{-\infty}^{0} \mu_1(\theta)\varphi(\theta)(x)d\theta,$$

$$D(t, \varphi)(x) = \varphi(0)x + G(t, \varphi)(x), \quad J_t^{\alpha-1}G(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} G(s)ds,$$

$$F(t, \varphi)(x) = \int_{-\infty}^{0} \mu_2(t, \theta, x, \varphi(\theta))(x)d\theta, \quad \rho(t, \varphi) = \rho_1(t)\rho_2(\|\varphi(0)\|).$$

From these definitions, it follows that G, F are bounded linear operators on $\mathcal{B}$ with $\|G\| \le L_G$ and $\|F\| \le L_F, \|I_k\| \le L_k$, $k = 1, 2, \dots, m$, where $L_G = l_1, L_F = \|b_1\|_\infty l_2$. Then the problem (4.1)-(4.3) can be written as system (1.1)-(1.3). Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 3.5, we can conclude that system (4.1)-(4.3) has at least one mild solution on $[0, b]$.

Conclusion. We have studied the existence of mild solutions for a class of impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay and solution operator, which is new and allow us to develop the existence of various partial fractional integro-differential inclusions and partial stochastic integro-differential inclusions. An application is provided to illustrate the applicability of the new result. The results presented in this paper extend and improve the corresponding ones announced by Chauhan et al [8], Shu et al [31], Hu and Ren [22], Lin et al [27], and others.

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