Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 82, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

POSITIVE SOLUTIONS AND GLOBAL BIFURCATION OF STRONGLY COUPLED ELLIPTIC SYSTEMS

JAGMOHAN TYAGI

Abstract. In this article, we study the existence of positive solutions for the coupled elliptic system

$$\begin{gathered}
-\Delta u = \lambda(f(u,v) + h_1(x)) \quad \text{in } \Omega,\\
-\Delta v = \lambda(g(u,v) + h_2(x)) \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

under certain conditions on f, g and allowing h_1, h_2 to be singular. We also consider the system

$$\begin{gathered}
-\Delta u = \lambda(a(x)u + b(x)v + f_1(v) + f_2(u)) \quad \text{in } \Omega,\\
-\Delta v = \lambda(b(x)u + c(x)v + g_1(u) + g_2(v)) \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

and prove a Rabinowitz global bifurcation type theorem to this system.

1. Introduction

The investigation on the existence questions of positive solutions to semilinear elliptic equations and systems has been of great interest to many researchers. Many problems in mathematical physics, for example, wave phenomena [22], nonlinear field equations [4], combustion theory [3, 13], fluid dynamics [2] etc. lead to nonlinear eigenvalue problem of the type

$$-\Delta u = \lambda f(u),$$

where a positive solution is meaningful, see for example [4, 5]. In the recent years, a good amount of research is established for reaction–diffusion systems. Reaction–diffusion systems model many phenomena in biology, ecology, combustion theory, chemical reactions, population dynamics etc. A typical example of these models is

$$\begin{gathered}
-\Delta u = f(v) \quad \text{in } \Omega,\\
-\Delta v = g(u) \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$.

2000 *Mathematics Subject Classification.* 35J57, 35B32, 35B09.
Key words and phrases. Elliptic system; bifurcation; positive solutions.

Submitted April 18, 2012. Published March 31, 2013.

Using Schauder's fixed point theorem and degree theoretic arguments, Dalmasso [10] obtain the existence and uniqueness of positive solution to (1.1). de Figueiredo et al [11] obtain the existence of positive solution to (1.1) by an Orlicz space setting for $N \geq 3$. Hulshof and Van der Vorst [17] establish the existence of positive solution to (1.1). For the existence and non–existence of positive solutions to (1.1) in a ball, we refer the reader to [14] for $N \geq 4$. By the method of sub and supersolutions and Schauder's fixed point theorem, Hai and Shivaji [16] establish the existence of a positive solution to the system

$$\begin{aligned} -\Delta u &= \lambda f(v) \quad \text{in } \Omega, \\ -\Delta v &= \lambda g(u) \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

for λ large. Using the monotonicity of f and g and degree theory and L^∞ priori estimates, Clément et al [9] obtain the existence of at least one positive solution to (1.2) in bounded, convex domains. The existence of a nonnegative solutions to (1.2) with indefinite weights can be seen in [23, 24]. Hai and Shivaji [16] point out that, using the similar arguments as in [16], the existence of a positive solution can be obtained to the following coupled system

$$\begin{aligned} -\Delta u &= \lambda f(u,v) \quad \text{in } \Omega. \\ -\Delta v &= \lambda g(u,v) \quad \text{in } \Omega. \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

for λ sufficiently large. So it is natural to ask that under what conditions on the nonlinearities, we have the existence of positive solutions to (1.3) for λ sufficiently small. Recently, Chern et al [8] establish the existence of positive solutions to (1.3) by the method of monotone iteration. In this paper, we show the existence of positive solutions to the nonhomogeneous elliptic system

$$\begin{aligned} -\Delta u &= \lambda(f(u,v) + h_1(x)) \quad \text{in } \Omega, \\ -\Delta v &= \lambda(g(u,v) + h_2(x)) \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega, \lambda$ is a positive parameter and $h_1, h_2 \in L^\infty(\Omega)$. We allow the sign changing nature of h_1 and h_2.

We will also consider the following coupled system for the existence of a positive solution

$$\begin{aligned} -\Delta u &= \lambda(a(x)u + b(x)v + f_1(v) + f_2(u) + h_1(x)) \quad \text{in } \Omega, \\ -\Delta v &= \lambda(b(x)u + c(x)v + g_1(u) + g_2(v) + h_2(x)) \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

where the conditions on $a, b, c, f_1, f_2, g_1, g_2, h_1, h_2$ will be specified later. By an application of implicit function theorem in a functional framework, Anoop and the present author [1] obtain the existence of a positive solution of scalar equation in $\mathbb{R}^N$. Chern et al [8] also obtain the existence and uniqueness of a solution to (1.5), where $a = b = c = h_1 = h_2 = 0$, by implicit function theorem. Mitidieri and Sweers [20] study the $n \times n$ weakly coupled system of type (1.5), where $f_i, g_i, h_i = 0$, for $i = 1, 2$. They show the preservance of the positive cone under the weakly coupled system. We also use the similar arguments as in [1, 8] to obtain the existence of a unique positive solution to (1.5).

We make the following hypotheses on the nonlinearity and weights:

(H1) Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ are continuous and there exist $L > 0$ and $k > 0$ such that $f(x, y) \geq L$, $g(x, y) \geq L$, for all $x \geq k$ and for all $y \geq k$.

(H2) The boundary value problems

$$\begin{aligned} -\Delta z &= h_1 \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta z &= h_2 \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

have positive solutions z_{h_1} and z_{h_2}, respectively.

(H3) Let $f_1, f_2, g_1, g_2 \in C^1(\mathbb{R}, \mathbb{R})$ be such that $f_1(0)+f_2(0) > 0$, $g_1(0)+g_2(0) > 0$.

(H4) Let $a, b, c \in C(\overline{\Omega}, \mathbb{R})$.

(H5) Let $|f_1(s) + f_2(s')| \leq |f_1(s + s') + f_2(s + s')|$, $|g_1(s) + g_2(s')| \leq |g_1(s + s') + g_2(s+s')|$ for s and s' near 0, and $|f_1(S)+f_2(S')| \leq |f_1(S+S')+f_2(S+S')|$, $|g_1(S) + g_2(S')| \leq |g_1(S + S') + g_2(S + S')|$ for S and S' near ∞.

(H6) $\lim_{s\to 0} \frac{f_1(s)}{s} = \lim_{s\to 0} \frac{f_2(s)}{s} = \lim_{s\to 0} \frac{g_1(s)}{s} = \lim_{s\to 0} \frac{g_2(s)}{s} = 0$.

(H7) There is a $1 < q < 2^*$, such that $\lim_{|s|\to\infty} \frac{f_1(s)}{|s|^{q-1}} = \lim_{|s|\to\infty} \frac{f_2(s)}{|s|^{q-1}} = \lim_{|s|\to\infty} \frac{g_1(s)}{|s|^{q-1}} = \lim_{|s|\to\infty} \frac{g_2(s)}{|s|^{q-1}} = 0$.

We organize this paper as follows: Section 2 deals with the proof of Theorems 1.1, 1.2. While Section 3 shows the bifurcation results. We state now the main results.

Theorem 1.1. *Let* (H1) *and* (H2) *hold. Then there exists a positive solution* (u, v) *to* (1.4) *for* λ *sufficiently small.*

Theorem 1.2. *Let* $a, b, c \in L^\infty(\Omega)$. *Let* (H2) *and* (H3) *hold. Then there exists a unique positive solution* (u, v) *to* (1.5) *for* λ *sufficiently small.*

2. Proof of main results

Proof of Theorem 1.1. Let $f(x) = f(0)$ and $g(x) = g(0)$ for $x < 0$. Let ξ_0 be the solution of

$$\begin{aligned} -\Delta \xi_0 &= 1 \quad \text{in } \Omega. \\ \xi_0 &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Now using strong maximum principle and boundary point lemma [18, p.34], we have $\xi_0(x) \geq cd(x, \partial\Omega)$ for some $c > 0$. Let $X = C(\overline{\Omega})$ and $A : X \times X \to X \times X$ be defined by

$$A(u, v)(x) = \Big(\lambda \int_\Omega G(x, y)(f(v(y)) + h_1(y))dy, \lambda \int_\Omega G(x, y)(g(u(y)) + h_2(y))dy\Big),$$

where $G(x, y)$ is the Green's function of $-\Delta$ associated to Dirichlet boundary condition. It is easy to see that A is a completely continuous operator and fixed points of A are solutions to the problem (1.4). Let $\psi = (\xi_\lambda, \xi_\lambda)$, where $\xi_\lambda = \frac{\lambda L \xi_0}{2}$ and let there exist $w_0 > 0, w_1 > 0$ and let $\phi = (w_0, w_1)$. We note that $\psi \leq \phi$ for $\lambda > 0$ sufficiently small. Let $\mathcal{K}$ be a cone in $X \times X$ defined as:

$$\mathcal{K} = \{(u, v) \in X \times X : \psi \leq (u, v) \leq \phi\}.$$

Now we claim that $\mathcal{K}$ is invariant under A.

To prove the claim, we will verify the following:

(i) If $(u,v) \geq \psi$, then $A(u,v) \geq \psi$.
(ii) If $(u,v) \leq \phi$, then $A(u,v) \leq \phi$.

Claim (i). Let $(u,v) \geq \psi$. We show that

$$\int_\Omega G(x,y)(f(u(y),v(y)) + h_1(y))\, dy \geq \frac{1}{2}L \int_\Omega G(x,y)\, dy. \tag{2.2}$$

Let C be a positive upper bound of $L - f(x,y)$ and let D be a subregion of Ω such that $\overline{D} \subset \Omega$ and

$$\frac{1}{2}L\xi_0 - C\int_{\Omega/D} G(x,y)dy \geq 0 \quad \text{on } \overline{\Omega}.$$

To see such a choice of D exists, let

$$z(x) = \int_{\Omega/D} G(x,y)dy.$$

Then z satisfies $-\Delta z = \chi_{\Omega/D}$ in Ω, $z = 0$ on $\partial\Omega$. By Sobolev's embedding theorem, there exists a positive constant C_1 such that

$$\|z\|_{W^{2,p}} \leq C_1\Big(\int_{\Omega/D} dx\Big)^{1/p} \quad \text{for } p > N, \tag{2.3}$$

and hence $z(x) \leq \epsilon d(x,\partial\Omega)$, where $\epsilon \to 0$ as $d(D,\partial\Omega) \to 0$. But $\xi_0 \geq cd(x,\partial\Omega)$ for some positive constant c. Hence

$$\frac{1}{2}L\xi_0 - C\int_{\Omega/D} G(x,y)\, dy \geq \frac{1}{2}Lcd(x,\partial\Omega) - C\epsilon d(x,\partial\Omega) \geq 0 \tag{2.4}$$

for ϵ small enough. Now since $u,v \geq \xi_\lambda$, we have $u,v \geq 0$ in D and hence $f(v) \geq L$ in D. Consequently,

$$\begin{aligned}
&\int_\Omega G(x,y)(f(u(y),v(y)) + h_1(y))dy - \frac{1}{2}L\int_\Omega G(x,y)dy \\
&= \int_\Omega G(x,y)h_1(y)dy + \int_\Omega G(x,y)f(u(y),v(y))dy - \frac{1}{2}L\int_\Omega G(x,y)dy \\
&= z_{h_1}(x) + \int_D G(x,y)(f(u(y),v(y)) - \frac{1}{2}L)dy \\
&\quad + \int_{\Omega/D} G(x,y)(f(u(y),v(y)) - \frac{1}{2}L)dy \\
&\geq \frac{1}{2}L\int_D G(x,y)dy + \int_{\Omega/D} G(x,y)(f(u(y),v(y)) - \frac{1}{2}L)dy \\
&\geq \frac{1}{2}L\xi_0 - \int_{\Omega/D} G(x,y)(L - f(u(y),v(y)))dy \\
&\geq \frac{1}{2}L\xi_0 - C\int_{\Omega/D} G(x,y)dy \geq 0.
\end{aligned} \tag{2.5}$$

Using the same arguments, we obtain

$$\int_\Omega G(x,y)(g(u(y),v(y)) + h_2(y))dy \geq \frac{1}{2}L\int_\Omega G(x,y)dy.$$

Claim (ii). Let us define

$$\tilde{f}(x_1,x_2)=\sup_{0\le y_1\le x_1}\sup_{0\le y_2\le x_2} f(y_1,y_2),\quad \tilde{g}(x_1,x_2)=\sup_{0\le y_1\le x_1}\sup_{0\le y_2\le x_2} g(y_1,y_2).$$

It is easy to see that $\tilde{f}$ and $\tilde{g}$ are nondecreasing functions. Let $(u,v)\le\phi=(w_0,w_1)$. We show that $A(u,v)\le\phi$; i.e.,

$$\begin{gathered}
\lambda\int_\Omega G(x,y)(f(u(y),v(y))+h_1(y))dy\le w_0,\\
\lambda\int_\Omega G(x,y)(g(u(y),v(y))+h_2(y))dy\le w_1.
\end{gathered}$$

For this,

$$\begin{aligned}
&\lambda\int_\Omega G(x,y)(f(u(y),v(y))+h_1(y))dy\\
&\le\lambda\int_\Omega G(x,y)h_1(y)dy+\lambda\int_\Omega G(x,y)\tilde{f}(u(y),v(y))dy,\ \text{(by definition of }\tilde{f})\\
&\le\lambda z_{h_1}(x)+\lambda\int_\Omega G(x,y)\tilde{f}(w_0,w_1)dy \qquad (2.6)\\
&=\lambda\Big[z_{h_1}(x)+\tilde{f}(w_0,w_1)\xi_0\Big]\\
&\le\lambda[\|z_{h_1}\|_{L^\infty(\Omega)}+M\|\xi_0\|_{L^\infty(\Omega)}]\quad(\text{where }\tilde{f}(w_0,w_1)\le M\text{ for some }M>0)\\
&\le w_0,\quad\text{for }\lambda>0\text{ sufficiently small,}
\end{aligned}$$

which proves the claim. Next, again using the same arguments, we obtain

$$\lambda\int_\Omega G(x,y)(g(u(y))+h_2(y))dy\le w_1, \tag{2.7}$$

which proves the second claim. This completes the proof of this theorem. □

Proof of Theorem 1.2. We extend f_i and g_i to be defined on $\mathbb{R}$ for $u,v<0$ in the following manner. Let $f_i(x)=f(0)$ and $g_i(x)=g(0)$ for $x<0$, $i=1,2$. Let $F:\mathbb{R}\times[H_0^1(\Omega)]^2\to H^{-1}(\Omega)]^2$ defined by

$$F(\lambda,u,v)=\begin{pmatrix}\Delta u+\lambda(a(x)u+b(x)v+f_1(v)+f_2(u)+h_1(x))\\ \Delta v+\lambda(b(x)u+c(x)v+g_1(u)+g_2(u)+h_1(x))\end{pmatrix}.$$

It is clear that $(\lambda,u,v)=(0,0,0)$ is a solution of (1.5). To obtain the solution of (1.5) in a small neighborhood of $\lambda=0$, we apply implicit function theorem at $(\lambda,u,v)=(0,0,0)$. The Fréchet derivative of F is given by

$$F_{(u,v)}(\lambda,u,v)(\phi,\psi)^T=\begin{pmatrix}\Delta\phi+\lambda(a(x)\phi+b(x)\psi+f_1'(v)\psi+f_2'(u)\phi)\\ \Delta\psi+\lambda(b(x)\phi+c(x)\psi+g_1'(u)\phi+g_2'(v)\psi)\end{pmatrix}.$$

Thus $F_{(u,v)}(0,0,0)(\phi,\psi)^T=(\Delta\phi,\Delta\psi)^T$, which is an isomorphism from $[H_0^1(\Omega)]^2$ to $[H^{-1}(\Omega)]^2$. Also, one can see that F is a C^1 map. By an application of implicit function theorem, see [25, Theorem 4B, pp. 150],

$$F(\lambda,u,v)=0 \tag{2.8}$$

has a unique solution $(\lambda,u(\lambda),v(\lambda))$ for $\lambda\in(0,\lambda_0)$ for some small $\lambda_0>0$ and $u(0)=v(0)=0$. Furthermore, u and v are C^1 maps and satisfy (2.8) in the weak sense. Now we differentiate (2.8) with respect to λ and evaluate at $(0,0,0)$, which yields that $(u'(0), v'(0))$ is the unique solution of

$$\begin{gathered}
-\Delta u'(0) = f_1(0) + f_2(0) + h_1(x) \quad \text{in } \Omega, \\
-\Delta v'(0) = g_1(0) + g_2(0) + h_2(x) \quad \text{in } \Omega, \\
u'(0) = v'(0) = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.9}$$

Now by (H3), using $f_1(0)+f_2(0) > 0$, $g_1(0)+g_2(0) > 0$ and an application of (H2) implies that $u'(0) > 0, v'(0) > 0$ in Ω and using the fact that $u(0) = v(0) = 0$, we get $u(\lambda) > 0, v(\lambda) > 0$ for $\lambda \in (0,\delta)$, $\delta \le \lambda_0$. Since $h_1, h_2 \in L^\infty(\Omega)$, by the classical regularity theory, $u'(0), v'(0) \in C^1(\Omega)$ and one can see that $u(\lambda)(x), v(\lambda)(x) > 0$, $\forall\, x \in \Omega$, which completes the proof. $\square$

3. Bifurcation

Let us consider the coupled system

$$\begin{gathered}
-\Delta u = \lambda(a(x)u + b(x)v + f_1(v) + f_2(u)) \quad \text{in } \Omega, \\
-\Delta v = \lambda(b(x)u + c(x)v + g_1(u) + g_2(v)) \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.1}$$

where $a, b, c, f_1, f_2, g_1, g_2$ are defined earlier. Let $\overline{F} : \mathbb{R} \times X = [H_0^1(\Omega)]^2 \to Y = [H^{-1}(\Omega)]^2$ defined by

$$\overline{F}(\lambda, u, v) = \begin{pmatrix} \Delta u + \lambda(a(x)u + b(x)v + f_1(v) + f_2(u)) \\ \Delta v + \lambda(b(x)u + c(x)v + g_1(u) + g_2(u)) \end{pmatrix}.$$

Let $f_1, f_2, g_1, g_2 \in C^1(\mathbb{R}, \mathbb{R})$ such that $f_1(0) = 0 = f_2(0) = g_1(0) = g_2(0)$. Then we observe that $\overline{F}(\lambda, u, v) = 0$, for all $\lambda \in \mathbb{R}$; i.e., $(\lambda, 0, 0)$ is a solution of (3.1) for every $\lambda \in \mathbb{R}$. These kind of pairs are called trivial solutions. The set

$$\Sigma = \{(\lambda, u, v) \in \mathbb{R} \times X : \overline{F}(\lambda, u, v) = 0,\ u \neq 0, v \neq 0\}$$

is called a set of nontrivial solutions of (3.1). We say that $(\overline{\lambda}, 0, 0)$ is a bifurcation point of (3.1) if in any neighborhood of $(\overline{\lambda}, 0, 0)$ in $\mathbb{R} \times X$, there exists a nontrivial solution of (3.1). We recall some qualitative results from [6, 7, 12, 15]. Let us denote $S_2(\Omega)$ be the set of all symmetric matrices of the form

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix},$$

where $a, b, c \in C(\overline{\Omega}, \mathbb{R})$ satisfy

(i) A is cooperative, i.e., $b(x) \ge 0$, for all $x \in \overline{\Omega}$.
(ii) $\max_{x\in\Omega} \max\{a(x), c(x)\} > 0$.

Given $A \in S_2(\Omega)$, let us consider the weighted eigenvalue problem

$$\begin{gathered}
-\Delta u = \lambda(a(x)u + b(x)v) \quad \text{in } \Omega, \\
-\Delta v = \lambda(b(x)u + c(x)v) \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.2}$$

In view of (i) and (ii) above, we can use spectral theory for compact operators [12] to obtain a sequence of eigenvalues

$$0 < \lambda_1(A) < \lambda_2(A) \le \lambda_3(A) \le \dots \le \lambda_k(A) \dots$$

such that $\lambda_k(A) \to \infty$ as $k \to \infty$. From [6, 7, 12, 15], we know that $\lambda_1 = \lambda_1(A)$ is positive, simple and isolated.

The next proposition deals with a connection between the eigenvalue of (3.2) and the bifurcation point of (3.1) and is adapted from [19] to the coupled system.

Proposition 3.1. *Let* (H4)–(H7) *be satisfied. If* $(\overline{\lambda}, 0, 0)$ *is a bifurcation point of* (3.1)*, then* $\overline{\lambda}$ *is an eigenvalue of* (3.2)*.*

Proof. Let $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$. It is well-known that the auxiliary problem

$$-\Delta U = H \text{ in } \Omega; \quad u = v = 0 \text{ on } \partial\Omega \tag{3.3}$$

has a unique solution U for each $H \in [H^{-1}(\Omega)]^2$; i.e., $U \in [H_0^1(\Omega)]^2$ such that

$$\begin{gathered} \int_\Omega \nabla u.\nabla v_1 = \langle h_1, v_1\rangle, \quad \forall v_1 \in H^{-1}(\Omega), \\ \int_\Omega \nabla v.\nabla v_2 = \langle h_2, v_2\rangle, \quad \forall v_2 \in H^{-1}(\Omega), \end{gathered} \tag{3.4}$$

and (u, v) is unique. In the above equations $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Let us denote by $(-\Delta)^{-1}(H)$ the unique weak solution of (3.3). Then

$$(-\Delta)^{-1} : [H^{-1}(\Omega)]^2 \to [H_0^1(\Omega)]^2$$

is a continuous operator. Also, since $[H_0^1(\Omega)]^2$ embeds compactly into $[L^r]^2$ for each $r \in (1, 2^*)$ so it follows that the restriction of $(-\Delta)^{-1}$ to $[L^{r'}]^2$ is a completely continuous operator. It is easy to observe that (λ, u, v) is a solution of (3.1) if and only if (λ, u, v) satisfies

$$\begin{pmatrix} u \\ v \end{pmatrix} = (-\Delta)^{-1}\left(\lambda A(x)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(v) + F_2(u) \\ G_1(u) + G_2(v) \end{pmatrix}\right), \tag{3.5}$$

where F_1, F_2, G_1, G_2 denote the usual Nemitsky operator associated with f_1, f_2, g_1, g_2, respectively. From (H8), the right-hand side of (3.5) defines a completely continuous operator from $[H_0^1(\Omega)]^2$ to itself. Let us assume that $(\overline{\lambda}, 0, 0)$ is a bifurcation point of (3.1). We show that $\overline{\lambda}$ is an eigenvalue of (3.2). Since $(\overline{\lambda}, 0, 0)$ is a bifurcation point so there exists a sequence $\{\lambda_n, u_n, v_n\}_{n=1}^{\infty}$ of nontrivial solutions of (3.1) such that $\lambda_n \to \overline{\lambda}$ in $\mathbb{R}$ and $u_n \to 0$, $v_n \to 0$ in $H_0^1(\Omega)$. Since (λ_n, u_n, v_n) satisfies (3.5), we have

$$\begin{pmatrix} \hat{u}_n \\ \hat{v}_n \end{pmatrix} = (-\Delta)^{-1}\left(\lambda A(x)\begin{pmatrix} \hat{u}_n \\ \hat{v}_n \end{pmatrix} + \begin{pmatrix} \frac{F_1(v_n)+F_2(u_n)}{\|u_n\|} \\ \frac{G_1(u_n)+G_2(v_n)}{\|v_n\|} \end{pmatrix}\right), \tag{3.6}$$

where $\hat{u}_n = \frac{u_n}{\|u_n\|}$, $\hat{v}_n = \frac{v_n}{\|v_n\|}$, $\|u_n\| = \|u_n\|_{1,2} = (\int_\Omega |\nabla u_n|^2 dx)^{1/2}$. Since $\|\hat{u}_n\| = 1$ and $\|\hat{v}_n\| = 1$, for all $n \in \mathcal{N}$, so we can assume that

$$\hat{u}_n \to \hat{u},\ \hat{v}_n \to \hat{v} \quad \text{in } H_0^1(\Omega),$$

up to a subsequence as $n \to \infty$. We claim that

$$\frac{F_1(v_n) + F_2(u_n)}{\|u_n\|} \to 0 \text{ in } L^{q'}, \quad \frac{G_1(u_n) + G_2(v_n)}{\|v_n\|} \to 0 \text{ in } L^{q'}, \tag{3.7}$$

where q is chosen in (H8) and without any loss of generality $2 < q$. We note that

$$\begin{aligned}\frac{F_1(v_n)+F_2(u_n)}{\|u_n\|} &= \frac{(F_1(v_n)+F_2(u_n))}{(v_n+u_n)}\frac{(u_n+v_n)}{\|u_n\|},\\ \frac{G_1(u_n)+G_2(v_n)}{\|v_n\|} &= \frac{(G_1(u_n)+G_2(v_n))}{(u_n+v_n)}\frac{(u_n+v_n)}{\|v_n\|}.\end{aligned} \tag{3.8}$$

Thus from (3.8) and Hölder inequality, to prove the claim it is sufficient to find a real number $r > 1$ and a constant $C > 0$ so that

$$\Big|\frac{F_1(v_n)+F_2(u_n)}{(v_n+u_n)}\Big|^{q'} \to 0, \quad \Big|\frac{G_1(u_n)+G_2(v_n)}{(u_n+v_n)}\Big|^{q'} \to 0, \quad \text{in } L^r \tag{3.9}$$

and

$$\||\frac{(u_n+v_n)}{\|u_n\|}|^{q'}\|_{L^{r'}} \le C, \quad \||\frac{(u_n+v_n)}{\|v_n\|}|^{q'}\|_{L^{r'}} \le C, \quad \forall\, n \in \mathcal{N}. \tag{3.10}$$

In view of (H7) and (H8), let us fix $\epsilon > 0$ and choose positive numbers $\delta = \delta(\epsilon), \overline{\delta}$ and $M = M(\delta, \overline{\delta})$ such that for every $x \in \Omega$ and $n \in \mathcal{N}$, the following inequalities hold:

$$|f_1(s)| \le \epsilon|s|, \quad |f_2(s)| \le \epsilon|s|, \quad |g_1(s)| \le \epsilon|s|, \quad |g_2(s)| \le \epsilon|s| \quad \text{for } |s| \le \delta \tag{3.11}$$

and

$$\begin{gathered}|f_1(s)| \le M|s|^{q-1}, \quad |f_2(s)| \le M|s|^{q-1},\\ |g_1(s)| \le M|s|^{q-1}, \quad |g_2(s)| \le M|s|^{q-1} \quad \text{for } |s| > \delta.\end{gathered} \tag{3.12}$$

Let r be a real number greater than 1. Then from (3.11), we get

$$\begin{aligned}&\||\frac{(F_1(v_n)+F_2(u_n))}{(v_n+u_n)}|^{q'}\|_r^r\\ &= \int_\Omega |\frac{f_1(v_n)+f_2(u_n)}{v_n+u_n}|^{q'r}dx\\ &= \int_{\{x\in\Omega||u_n+v_n|<\delta\}} |\frac{f_1(v_n)+f_2(u_n)}{v_n+u_n}|^{q'r}dx\\ &\quad + \int_{\{x\in\Omega|\delta\le|u_n+v_n|\le\overline{\delta}\}} |\frac{f_1(v_n)+f_2(u_n)}{v_n+u_n}|^{q'r}dx\\ &\quad + \int_{\{x\in\Omega||u_n+v_n|>\overline{\delta}\}} |\frac{f_1(v_n)+f_2(u_n)}{v_n+u_n}|^{q'r}dx\\ &\le \int_{\{x\in\Omega||u_n+v_n|<\delta\}} |\frac{f_1(u_n+v_n)+f_2(u_n+v_n)}{v_n+u_n}|^{q'r}dx\\ &\quad + \int_{\{x\in\Omega|\delta\le|u_n+v_n|\le\overline{\delta}\}} |\frac{f_1(v_n)+f_2(u_n)}{v_n+u_n}|^{q'r}dx\\ &\quad + \int_{\{x\in\Omega||u_n+v_n|>\overline{\delta}\}} |\frac{f_1(u_n+v_n)+f_2(u_n+v_n)}{v_n+u_n}|^{q'r}dx\\ &\le \epsilon|\Omega| + M^{q'r}\int_\Omega |u_n+v_n|^{q'r(q-2)}dx \quad (\text{by (3.11), (3.12)})\\ &\le \epsilon|\Omega| + 2^{(q-2)q'r-1}M^{q'r}\int_\Omega (|u_n|^{q'r(q-2)} + |v_n|^{q'r(q-2)})dx.\end{aligned}$$

From the above inequality and using the fact that $u_n \to 0$ and $v_n \to 0$ in $H_0^1(\Omega)$, we find that

$$\frac{F_1(v_n) + F_2(u_n)}{\|u_n\|} \to 0 \text{ in } L^{q'}, \quad \text{if } q'r(q-2) < 2^*. \tag{3.13}$$

Using the same arguments as above, one can also see that

$$\frac{G_1(u_n) + G_2(v_n)}{\|v_n\|} \to 0 \text{ in } L^{q'}, \quad \text{if } q'r(q-2) < 2^* \tag{3.14}$$

and therefore, (3.9) is satisfied if

$$q'r(q-2) < 2^*. \tag{3.15}$$

Also, using the boundedness of $\hat{u}_n$ in L^{2^*}, we see that (3.10) is satisfied if

$$q'r' < 2^*. \tag{3.16}$$

To obtain an r satisfying (3.15) and (3.16) is equivalent to obtain an r such that

$$\frac{q'(q-2)}{2^*} < \frac{1}{r} < \frac{2^* - q'}{2^*} \tag{3.17}$$

and the above inequality always holds because of $q < 2^*$ and this choice of r satisfying (3.17), proves (3.9) and (3.10).

Now from (3.6), (3.7), and the compactness of $(-\Delta)^{-1}$, we can assume that (passing a subsequence if necessary) $\hat{u}_n \to \hat{u}$, $\hat{v}_n \to \hat{v}$ in $H_0^1(\Omega)$. Now we pass the limit in (3.6) and find that

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = (-\Delta)^{-1} \left(\lambda A(x) \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \right). \tag{3.18}$$

Since $\|\hat{u}_n\| = 1$, so $\hat{u} \neq 0$, which implies that $\overline{\lambda}$ is an eigenvalue of (3.2), which proves the claim. $\square$

From [6, 7, 12, 15], we know that $\lambda_1(A)$ is an isolated eigenvalue of (3.2) so if we let

$$\lambda_2(A) = \inf\{\lambda > \lambda_1(A) | \lambda \text{ is an eigenvalue of (3.2)}\}, \tag{3.19}$$

then $\lambda_1(A) < \lambda_2(A)$. By definition, there is no eigenvalue of (3.2) less than $\lambda_1(A)$, therefore for $\lambda < \lambda_1(A)$ or $\lambda_1(A) < \lambda < \lambda_2(A)$, the system

$$\begin{pmatrix} u \\ v \end{pmatrix} = (-\Delta)^{-1} \left(\lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} \right) \tag{3.20}$$

admits only the trivial solution $u \equiv v \equiv 0$. Let us define the completely continuous operator $S_\lambda : [H_0^1(\Omega)]^2 \to [H_0^1(\Omega)]^2$ by

$$S_\lambda \begin{pmatrix} u \\ v \end{pmatrix} = (-\Delta)^{-1} \left(\lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} \right). \tag{3.21}$$

It is clear that when $\lambda < \lambda_1(A)$ or $\lambda_1(A) < \lambda < \lambda_2(A)$, the Leray–Schauder degree

$$\deg_{[H_0^1(\Omega)]^2}(I - S_\lambda, B(0, r), \mathbf{0})$$

is well defined for any $r > 0$. The next lemma is well-known for a scalar equation, see [21] and the same proof works for a system also and it is given in [19] for any $p > 1$. We omit the proof of this lemma.

Proposition 3.2. *Let $r > 0$ and $\lambda \in \mathbb{R}$. Then*

$$\deg_{[H_0^1(\Omega)]^2}(I - S_\lambda, B(0,r), \mathbf{0}) = \begin{cases} 1 & \text{if } \lambda < \lambda_1(A) \\ -1 & \text{if } \lambda_1(A) < \lambda < \lambda_2(A). \end{cases}$$

Our main result on bifurcation is the following theorem.

Theorem 3.3. *Let* (H4)–(H7) *and* (i)–(ii) be satisfied. Then $(\lambda_1, 0, 0)$ is a bifurcation point of (3.1). Moreover, there is a component of the set of nontrivial solutions of (3.1) in $\mathbb{R} \times [H_0^1(\Omega)]^2$ whose closure contains $(\lambda_1, 0, 0)$ and is either unbounded or contains a pair $(\overline{\lambda}, 0, 0)$ for some eigenvalue $\overline{\lambda}$ of (3.2) with $\overline{\lambda} \neq \lambda_1$.

Proof. Let us set

$$T_\lambda \begin{pmatrix} u \\ v \end{pmatrix} = (-\Delta)^{-1} \left(\lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(v) + F_2(u) \\ G_1(u) + G_2(v) \end{pmatrix} \right). \tag{3.22}$$

Suppose that $(\lambda_1, 0, 0)$ is not a bifurcation point of (3.1). Then there exist $\epsilon > 0, \delta_0 > 0$ such that there is no nontrivial solution of the system

$$\begin{pmatrix} u \\ v \end{pmatrix} - T_\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.23}$$

for $|\lambda_1| < \epsilon$ and $\delta < \delta_0$ with $\|u\| = \delta = \|v\|$. Since degree is invariant under compact homotopy so we obtain that

$$\deg_{[H_0^1(\Omega)]^2}(I - S_\lambda, B(0,r), \mathbf{0}) = \text{constant, for } \lambda \in [\lambda_{\mathbf{1}} - \epsilon, \lambda_{\mathbf{1}} + \epsilon]. \tag{3.24}$$

With the choice of ϵ small enough, there is no eigenvalue of (3.2) in $(\lambda_1, \lambda_1 + \epsilon]$. We fix $\lambda \in (\lambda_1, \lambda_1 + \epsilon]$. It is easy to see that if we choose δ sufficiently small then the system

$$\begin{pmatrix} u \\ v \end{pmatrix} - (-\Delta)^{-1} \left(\lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} + s \begin{pmatrix} F_1(v) + F_2(u) \\ G_1(u) + G_2(v) \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.25}$$

has no solution (u, v) with $\|u\| = \delta = \|v\|$ for every $s \in [0,1]$. In fact, assuming the contrary and by the similar lines of the proof as Proposition 3.1, we find that λ is an eigenvalue of (3.2). From the invariance of the degree under homotopies and Proposition 3.2, we then obtain

$$\deg_{[H_0^1(\Omega)]^2}(I - T_\lambda, B(0,r), \mathbf{0}) = \deg_{[H_0^1(\Omega)]^2}(I - S_\lambda, B(0,r), \mathbf{0}) = -1. \tag{3.26}$$

Similarly, for $\lambda \in [\lambda_1 - \epsilon, \lambda_1)$, we find that

$$\deg_{[H_0^1(\Omega)]^2}(I - T_\lambda, B(0,r), \mathbf{0}) = 1. \tag{3.27}$$

(3.26) and (3.27) lead a contradiction to (3.24) and hence $(\lambda_1, 0, 0)$ is a bifurcation point of (3.1). The rest of the proof of this theorem is completely similar to the classical Rabinowitz global bifurcation theorem, see [21]. For the sake of brevity we omit the details. $\square$

References

[1] T. V. Anoop, J. Tyagi; *A positive solution branch for nonlinear eigenvalue problems in $\mathbb{R}^N$*, Nonlinear Anal. **74** (2011), pp. 2191–2200.

[2] D. Aronson, M. G. Crandall, L. A. Peletier; *Stabilization of solutions of a degenerate nonlinear diffusion problem, Nonlinear Anal.*, **6** (1982), pp. 1001–1022.

[3] J. W. Bebernes, D. R. Kassoy; *A mathematical analysis of blow up for thermal reactions-the spatially inhomogenous case*, SIAM J. Appl. Math., **40** (1) (1981), pp. 476–484.

[4] H. Berestycki, P.-L. Lions; *Existence of a ground state in nonlinear equations of the type Klein-Gordon*, in Variational Inequalities, (Cottle, Gianessi and J. L. Lions, editors), J. Wiley, New York, 1980.
[5] H. Berestycki, P.-L. Lions, L. A. Peletier; *An ODE approach to the existence of positive solutions for semilinear problems in* $\mathbb{R}^N$, Indiana Uni. Math. Journal, **30** (1) (1981), pp. 141–157.
[6] K. C. Chang; *An extension of the Hess-Kato theorem to elliptic systems and its applications to multiple solutions problems*, Acta Math. Sinica **15** (1999), 439–454.
[7] K. C. Chang; *Principal eigenvalue for weight matrix in elliptic systems*, Nonlinear Anal. **46** (2001), 419–433.
[8] J. L. Chern, Y. L. Tang, C. S. Lin, J. Shi; *Existence, uniqueness and stability of positive solutions to sublinear elliptic systems*, Proc. Roy. Soc. Edin., 141 A, **46** (2011), 45–64.
[9] Ph. Clément, D. G. de Figueiredo, E. Mitidieri; *Positive solutions of semilinear elliptic systems*, Commu. Partial Diff. Equ.,**17** (1992), pp. 923–940.
[10] R. Dalmasso; *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal. **39** (2000), pp. 559–568.
[11] D. G. de Figueirdo, J. Marcos do Ó, B. Ruf; *An Orlicz–space approach to superlinear elliptic systems*, J. Func. Anal., **224** (2005), pp. 471–496.
[12] D. G. de Figueiredo; *Positive solutions of semilinear elliptic problems*, in: Differential Equations, São Paulo, 1981, in: Lecture Notes in Math., Vol. 957, Springer, Berlin, 1982, pp. 34–87.
[13] A. M. Fink; *The radial Laplacian Gelfand problem*, in "Delay and Differential Equations", pp. 93–98, World
[14] J. García–Melián, Julio D. Rossi; *Boundary blow–up solutions to elliptic systems of competitive type*, J. Diff. Eqns., **206** (2004), pp. 156–181.
[15] M. F. Furtado, F. O. V. de Paiva; *Multiplicity of solutions for resonant elliptic systems*, J. Math. Anal. Appl. **319** (2006), 435–449.
[16] D. D. Hai, R. Shivaji; *An existence result on positive solutions for a class of semilinear elliptic systems*, Proc. Roy. Soc. Edinburgh, **134 A** (2004), pp. 137–141.
[17] D. Hulshof, R. Van der Vorst; *Differential systems with strongly indefinite variational structure*, J. Func. Anal., **114** (1993), pp. 32–58.
[18] D. Gilberg, Neil S. Trudinger; *Elliptic partial differential equations of second order*, Second Edition, Springer Verlag, 1983.
[19] Manuel A. del Pino, R. F. Manasevich; *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Diff. Eqns., **92** (1991), pp. 226–251.
[20] E. Mitidieri, G. Sweers; *Weakly coupled elliptic systems and positivity*, Math. Nach., **173** (1995), pp. 259–286.
[21] P. H. Rabinowitz; *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math. **3** (1973), pp. 161–202.
[22] W. A. Strauss; *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., **55** (1977), pp. 149–162.
[23] J. Tyagi; *Existence of nonnegative solutions for a class of semilinear elliptic systems with indefinite weight*, Nonlinear Anal. **73** (2010), pp. 2882–2889.
[24] J. Tyagi; *Existence of a non-negative solution for prey-predator elliptic systems with sign-changing nonlinearity*, Electronic J. Diff. Equations, Vol. 2011, no. 153 (2011), pp. 1–9.
[25] E. Zeidler; *Nonlinear Functional Analysis and its Applications I, Fixed Point Theorems*, Springer–Verlag, 1984.

Jagmohan Tyagi
Indian Institute of Technology Gandhinagar, Vishwakarma Government Engineering College Complex, Chandkheda, Visat-Gandhinagar Highway, Ahmedabad, Gujarat, India, 382424
E-mail address: jtyagi1@gmail.com, jtyagi@iitgn.ac.in