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CHARACTERIZATION OF POSITIVE SOLUTION TO STOCHASTIC COMPETITOR-COMPETITOR-COOPERATIVE MODEL

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Abstract. In this article we study a randomized three-dimensional Lotka-Volterra model with competitor-competitor-mutualist interaction. We show the existence, uniqueness, moment boundedness, stochastic boundedness and global asymptotic stability of positive global solutions for this stochastic model. Analytical results are validated by numerical examples.

1. Introduction

Gyllenberg et al [10] investigated the following Lotka-Volterra system describing a competitor-competitor-mutualist interaction:

$$\frac{dx_1(t)}{dt} = x_1(t)(r_1 - a_{11}x_1(t) - a_{12}x_2(t) + a_{13}x_3(t)), \tag{1.1}$$

$$\frac{dx_2(t)}{dt} = x_2(t)(r_2 - a_{21}x_1(t) - a_{22}x_2(t) + a_{23}x_3(t)), \tag{1.2}$$

$$\frac{dx_3(t)}{dt} = x_3(t)(r_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)), \tag{1.3}$$

subjected to the biologically feasible initial condition $x_1(0) \equiv x_{10} > 0$, $x_2(0) \equiv x_{20} > 0$, $x_3(0) \equiv x_{30} > 0$. $x_1(t)$, $x_2(t)$ are the population densities of two competitive species and $x_3(t)$ denotes the population density of the cooperative species. Intrinsic growth rates of the three species are denoted by r_i, $(i = 1, 2, 3)$ and intra-specific competition coefficients for the limited resources are denoted by a_{ii}, $(i = 1, 2, 3)$. The strengths of inter-specific interactions are denoted by a_{ij}'s $(i, j = 1, 2, 3, ;\ i \neq j)$. All the parameters involved with the model are positive. Competitive and cooperative interactions are characterized by the negative and positive signs before the inter-specific interaction terms. Gyllenberg et al gave the detailed mathematical analysis of the above system. However, to the best of my knowledge, the deterministic model (1.1)-(1.3) is not studied so far by anyone, in presence of environmental fluctuation. In this paper, We will study the stochastic

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model corresponding to the model (1.1)-(1.3). The stochastic model takes into account the environmental fluctuation. Therefore the main contribution of this paper is clear.

The rest of this article is organized as follows: In section 2, we formulate the stochastic model perturbing the growth rate terms of (1.1)-(1.3) by white noise terms, which governs a system of stochastic differential equations(SDEs). Then section 2 is divided into three subsections. In subsection (2.1), we show the existence of unique positive global solution to the given SDE system. In subsection (2.2), we establish the stochastic boundedness of the solution to the formulated SDE system. Global asymptotic stability results are derived in subsection (2.3). In section 3, we validated the analytical findings with the help of numerical example. Finally, we closed this paper with a detail discussion in section 4. The key method used in this paper is the analysis of Lyapunov functions.

Throughout this paper, we will use the following notation:

$$\mathbb{R}^3_+ = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \ge 0, x_2 \ge 0, x_3 \ge 0\big\},$$
$$\mathrm{Int}(\mathbb{R}^3_+) = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\big\}.$$

2. Stochastic model

In this section, we study the effect of environmental driving forces on the dynamics of the model system (1.1)-(1.3) after introducing the multiplicative white noise terms into the growth equations of each species. To formulate the stochastic model, we introduce randomness into the deterministic model system (1.1)-(1.3) by perturbing r_1, r_2, r_3 by $\sigma_1\xi_1(t)$, $\sigma_2\xi_2(t)$ and $\sigma_3\xi_3(t)$ respectively. Therefore, we obtain the following modified version:

$$\frac{dx_1(t)}{dt} = x_1(t)(r_1 - a_{11}x_1(t) - a_{12}x_2(t) + a_{13}x_3(t)) + \sigma_1 x_1(t)\xi_1(t), \tag{2.1}$$
$$\frac{dx_2(t)}{dt} = x_2(t)(r_2 - a_{21}x_1(t) - a_{22}x_2(t) + a_{23}x_3(t)) + \sigma_2 x_2(t)\xi_2(t), \tag{2.2}$$
$$\frac{dx_3(t)}{dt} = x_3(t)(r_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)) + \sigma_3 x_3(t)\xi_3(t), \tag{2.3}$$

subjected to the initial conditions $x_1(0), x_2(0), x_3(0) > 0$. $\xi_1(t)$, $\xi_2(t)$ and $\xi_3(t)$ are three mutually independent white noise terms [11] characterized by $\langle\xi_1(t)\rangle = \langle\xi_2(t)\rangle = \langle\xi_3(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(t_1)\rangle = \delta_{ij}\delta(t - t_1)$ where δ_{ij} is Kronecker delta and $\delta(.)$ is the 'Dirac-δ' function [13, 14]. Here parameters σ_1, σ_2 and σ_3 denote the intensities of white noise. Now we can write the stochastic model system (2.1)-(2.3) into the following system of SDEs:

$$dx_1(t) = x_1(t)(r_1 - a_{11}x_1(t) - a_{12}x_2(t) + a_{13}x_3(t)) + \sigma_1 x_1(t)dB_1(t), \tag{2.4}$$
$$dx_2(t) = x_2(t)(r_2 - a_{21}x_1(t) - a_{22}x_2(t) + a_{23}x_3(t)) + \sigma_2 x_2(t)dB_2(t), \tag{2.5}$$
$$dx_3(t) = x_3(t)(r_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)) + \sigma_3 x_3(t)dB_3(t), \tag{2.6}$$

where $B_1(t)$, $B_2(t)$ and $B_3(t)$ are three standard one-dimensional independent Wiener processes defined over the complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets [16]). The relations between the white noise terms and Wiener processes are defined by $dB_r = \xi_r(t)dt$, $r = 1, 2, 3$ [15]. Since the main objective of this paper is to study the effect of environmental noise on the dynamics of the system, we assume that the noise intensities are positive. As each $x_i(t)$ denote the population density, it should be nonnegative. Now we show that the system (2.4)-(2.6) has a unique positive global solution

2.1. **Existence, uniqueness and stochastic boundedness solutions.**

Theorem 2.1. *For any initial value* $x_0 \equiv (x_{10}, x_{20}, x_{30}) \in \mathrm{Int}(\mathbb{R}^3_+)$, *there is a unique solution* $x(t) \equiv (x_1(t), x_2(t), x_3(t))$ *of system*(2.4)-(2.6) *for* $t \geq 0$ *and the solution will remain in* $\mathrm{Int}(\mathbb{R}^3_+)$ *with probability 1, namely* $x(t) \in \mathrm{Int}(\mathbb{R}^3_+)$ *for all* $t \geq 0$ *a.s.*

The proof of the above theorem is similar to the one of [9, Theorem 2.1]; Hence it is omitted here.

Stochastic boundedness is one of the most important properties for stochastically perturbed population system, because boundedness of a system guarantees its ecological validity. Now we are interested to discuss the stochastic boundedness of the solution to the system (2.4)-(2.6). We use the following definition of stochastic boundedness from [18].

Definition 2.2. The solution of (2.4)-(2.6) is said to be stochastically bounded if for any $\epsilon_1 > 0$, there is a constant $Z \equiv Z(\epsilon_1)$ such that for any $x_0 \in \mathrm{Int}(\mathbb{R}^3_+)$, we have

$$\limsup_{t\to\infty} P\big\{|x(t)| := \sqrt{(x_1^2(t) + x_2^2(t) + x_3^2(t))} \leq Z\big\} \geq 1 - \epsilon_1. \tag{2.7}$$

As a deterministic system, we can not find the uniform bound of the solution to the stochastic system. Instead, we can find the uniform bound of the higher order moments of the solution for the stochastic system. The nice property about the solution of system (2.4)-(2.6), discussed in the previous theorem provides with a great opportunity to discuss about the boundedness of p th order moment of the solution, which will be required to prove the stochastic boundedness of the solution. Before going to prove the next theorem, we will state a lemma which will be required to prove the theorem.

Lemma 2.3. *If* $a > 0$, $b > 0$ *and* $\frac{dx}{dt} \geq (\leq) x\,(b - ax^{\alpha})$, *where* α *is a positive constant, when* $t \geq 0$ *and* $x(0) > 0$, *we have*

$$x(t) \geq (\leq) \big(\frac{b}{a}\big)^{1/\alpha} \big[1 + \big(\frac{bx^{-\alpha}(0)}{a} - 1\big) e^{-b\alpha t}\big]^{-1/\alpha}$$

Remark 2.4. From Lemma 2.3, we have

(a) $\liminf_{t\to+\infty} x(t) \geq (\frac{b}{a})^{1/\alpha} \big(\limsup_{t\to+\infty} x(t) \leq (\frac{b}{a})^{1/\alpha}\big)$,

(b) $x(t) \geq \min\big\{x(0), (\frac{b}{a})^{1/\alpha}\big\} \big(x(t) \leq \max\{x(0), (\frac{b}{a})^{1/\alpha}\}\big)$, for all $t \geq 0$.

For a detailed proof of the above lemma and of the remark, one can see [21].

Theorem 2.5. *Assume that* $a_{11} - a_{13} > 0$, $a_{22} - a_{23} > 0$, $a_{33} - (a_{31} + a_{32}) > 0$. *Then there exists a positive constant* $K^*(p)$ *such that for any initial value* $x_0 \in \mathrm{Int}(\mathbb{R}^3_+)$, *the solution* $x(t)$ *of system* (2.4)-(2.6) *has the following property:*

$$E\big[\sum_{i=1}^{3} x_i^p(t)\big] \leq K^*(p) < \infty, \quad \forall t \in [0, \infty),\ p \geq 1.$$

Proof. Calculating the differential of the function $\frac{1}{p}\sum_{i=1}^{3} c_i x_i^p$, $p \geq 1$, (where c_j, $j = 1, 2, 3$ are three positive constants to be defined later) along the solution trajectories of the system (2.4)-(2.6), with the help of Itô's formula, we obtain

$$
\begin{aligned}
& d\Big(\frac{1}{p}\sum_{i=1}^{3} c_i x_i^p\Big) \\
& = \Big[\sum_{i=1}^{3} c_i\Big(r_i + \frac{p-1}{2}\sigma_i^2\Big)x_i^p - c_1\Big(a_{11}x_1^{p+1} + a_{12}x_2x_1^p - a_{13}x_3x_1^p\Big) \\
& \quad - c_2\Big(a_{21}x_1x_2^p - a_{22}x_2^{p+1} - a_{23}x_3x_2^p\Big) + c_3\Big(a_{31}x_1x_3^p + a_{32}x_2x_3^p - a_{33}x_3^{p+1}\Big)\Big]dt \\
& \quad + \sum_{i=1}^{3} c_i\sigma_i x_i^p dB_i(t).
\end{aligned}
$$

Now integrating above result from 0 to t and then taking expectation of both sides, we find

$$
\begin{aligned}
& \frac{1}{p}E\Big[\sum_{i=1}^{3} c_i x_i^p\Big] \\
& = E\int_0^T \Big[\sum_{i=1}^{3}\Big(r_i + \frac{p-1}{2}\sigma_i^2\Big)c_i x_i^p - c_1\Big(a_{11}x_1^{p+1} + a_{12}x_2x_1^p - a_{13}x_3x_1^p\Big) \\
& \quad - c_2\Big(a_{21}x_1x_2^p + a_{22}x_2^{p+1} - a_{23}x_3x_2^p\Big) + c_3\Big(a_{31}x_1x_3^p + a_{32}x_2x_3^p - a_{33}x_3^{p+1}\Big)\Big]ds,
\end{aligned}
$$

with the help of mean zero property of Itô's integral [12]. Next applying Fubini's theorem [19, 20] and differentiating both sides with respect to t, we obtain,

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}E\Big[\sum_{i=1}^{3} c_i x_i^p(t)\Big] &= E\Big[\sum_{i=1}^{3}\Big(r_i + \frac{p-1}{2}\sigma_i^2\Big)c_i x_i^p(t)\Big] \\
& \quad - c_1E\Big[a_{11}x_1^{p+1}(t) + a_{12}x_2(t)x_1^p(t) - a_{13}x_3(t)x_1^p(t)\Big] \\
& \quad - c_2E\Big[a_{21}x_1(t)x_2^p(t) + a_{22}x_2^{p+1}(t) - a_{23}x_3(t)x_2^p(t)\Big] \\
& \quad + c_3E\Big[a_{31}x_1(t)x_3^p(t) + a_{32}x_2(t)x_3^p(t) - a_{33}x_3^{p+1}(t)\Big].
\end{aligned}
$$

Applying Young's inequality on the terms $x_i(t)x_j^p(t)$, $1 \leq i, j \leq 3$, $i \neq j$ and with the help of the positivity of solution, we obtain

$$
\begin{aligned}
& \frac{d}{dt}E\Big[\sum_{i=1}^{3} c_i x_i^p(t)\Big] \\
& \leq E\Big[p\sum_{i=1}^{3}\Big(r_i + \frac{p-1}{2}\sigma_i^2\Big)c_i x_i^p(t)\Big] \\
& \quad - \Big(c_1a_{11}p - c_1a_{13}\frac{p^2}{p+1} - c_3a_{31}\frac{p}{p+1}\Big)E\big(x_1^{p+1}(t)\big) \\
& \quad - \Big(c_2a_{22}p - c_2a_{23}\frac{p^2}{p+1} - c_3a_{32}\frac{p}{p+1}\Big)E\big(x_2^{p+1}(t)\big)
\end{aligned}
$$

$$-\Big(c_3a_{33}p - c_3(a_{31}+a_{32})\frac{p^2}{p+1} - (c_1a_{13}+c_2a_{23})\frac{p}{p+1}\Big)E\big(x_3^{p+1}(t)\big). \tag{2.8}$$

Now we establish the existence of three positive constants c_i's, $i = 1, 2, 3$, such that

$$\begin{gathered}
c_1a_{11}p - c_1a_{13}\frac{p^2}{p+1} - c_3a_{31}\frac{p}{p+1} > 0,\\
c_2a_{22}p - c_2a_{23}\frac{p^2}{p+1} - c_3a_{32}\frac{p}{p+1} > 0,\\
c_3a_{33}p - c_3(a_{31}+a_{32})\frac{p^2}{p+1} - (c_1a_{13}+c_2a_{23})\frac{p}{p+1} > 0.
\end{gathered}$$

One can easily verify that the restrictions on the parameters, defined by $a_{11} > a_{13}$, $a_{22} > a_{23}$ and $a_{33} > a_{31} + a_{32}$ ensure that we can find three positive c_i's which satisfy above mentioned three inequalities. Let us define six quantities as follows:

$$\begin{gathered}
A_i = p\big(r_i + \frac{p-1}{2}\sigma_i^2\big), \quad i = 1, 2, 3,\\
\beta_1 = c_1^{\frac{-(p+1)}{p}}\Big(c_1a_{11}p - c_1a_{13}\frac{p^2}{p+1} - c_3a_{31}\frac{p}{p+1}\Big),\\
\beta_2 = c_2^{\frac{-(p+1)}{p}}\Big(c_2a_{22}p - c_2a_{23}\frac{p^2}{p+1} - c_3a_{32}\frac{p}{p+1}\Big),\\
\beta_3 = c_3^{\frac{-(p+1)}{p}}\Big(c_3a_{33}p - c_3(a_{31}+a_{32})\frac{p^2}{p+1} - (c_1a_{13}+c_2a_{23})\frac{p}{p+1}\Big),
\end{gathered}$$

where $A_i > 0$ as $r_i, \sigma_i > 0$ and $p \geq 1$ and positivity of β_i's depend upon the satisfaction of the inequalities and the choices of c_i's. Hence from (2.8), we can write

$$\frac{d}{dt}E\Big[\sum_{i=1}^{3} c_i x_i^p(t)\Big] \leq E\Big[\sum_{i=1}^{3} A_i c_i x_i^p(t)\Big] - \sum_{i=1}^{3} c_i^{\frac{(p+1)}{p}}\beta_i E\big(x_i^{p+1}(t)\big).$$

Defining $M_1 = \max\{A_1, A_2, A_3\}$ and $M_2 = \min\{\beta_1, \beta_2, \beta_3\}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}E\Big[\sum_{i=1}^{3} c_i x_i^p(t)\Big]\\
&\leq M_1E(c_1x_1^p + c_2x_2^p + c_3x_3^p) - M_2E\Big(c_1^{\frac{(p+1)}{p}}x_1^{p+1} + c_2^{\frac{(p+1)}{p}}x_2^{p+1} + c_3^{\frac{(p+1)}{p}}x_3^{p+1}\Big)\\
&\leq M_1E\big(c_1x_1^p + c_2x_2^p + c_3x_3^p\big) - \frac{M_2}{3^p}E\big(c_1^{1/p}x_1 + c_2^{1/p}x_2 + c_3^{1/p}x_3\big)^{p+1}.
\end{aligned}$$

By Hölder's inequality,

$$E(c_1^{1/p}x_1 + c_2^{1/p}x_2 + c_3^{1/p}x_3)^{p+1} \geq \Big[E(c_1^{1/p}x_1 + c_2^{1/p}x_2 + c_3^{1/p}x_3)^p\Big]^{\frac{p+1}{p}}.$$

Therefore using above inequality ,

$$\begin{aligned}
&\frac{d}{dt}E\Big[\sum_{i=1}^{3} c_i x_i^p(t)\Big]\\
&\leq M_1E\left(c_1x_1^p + c_2x_2^p + c_3x_3^p\right) - \frac{M_2}{3^p}\Big[E(c_1^{1/p}x_1 + c_1^{1/p}x_2 + c_1^{1/p}x_3)^p\Big]^{\frac{p+1}{p}} \qquad (2.9)\\
&\leq M_1E\left(c_1x_1^p + c_2x_2^p + c_3x_3^p\right) - \frac{M_2}{3^p}\left[E(c_1x_1^p + c_2x_2^p + c_3x_3^p)\right]^{\frac{p+1}{p}}.
\end{aligned}$$

Using the result (b) of Remark 2.4, we obtain

$$E[c_1x_1^p + c_2x_2^p + c_3x_3^p] \le \max\left\{c_1x_1^p(0) + c_2x_2^p(0) + c_3x_3^p(0), \left(\frac{M_1 3^p}{M_2}\right)^p\right\}$$

Hence

$$E[x_1^p + x_2^p + x_3^p] \le \Big[\frac{M_1 3^p}{M_2^*}\Big]^p := K^*(p) < \infty, \quad \text{for all } t \ge 0 \tag{2.10}$$

where $M_2^* = \min(\beta_1^*, \beta_2^*, \beta_3^*)$ and β_1^*, β_2^*, β_3^* are obtained from expressions of β_1, β_2, β_3 by putting $c_1 = c_2 = c_3 = 1$. Hence the result follows. □

Theorem 2.6. *Assume conditions of Theorem 2.5. Then the solution of* (2.4)-(2.6) *is stochastically bounded starting from* $x_0 \in \mathrm{Int}(\mathbb{R}_+^3)$.

Proof. Observe that $|x(t)|^p \le 3^{p/2}\sum_{i=1}^3 x_i(t)^p$. Thus it follows from Theorem 2.5 that for any $t \ge 0$,

$$E[|x(t)|^p] \le 3^{p/2}E\Big[\sum_{i=1}^3 x_i^p(t)\Big] \le K_1(p) < \infty,$$

where $K_1(p) = 3^{p/2}K^*(p)$. Applying Tchebychev's inequality[16], for $Z > 0$, we have

$$P\{|x(t)| > Z\} \le \frac{E\left[|x(t)|^2\right]}{Z^2} \le \frac{K_1(2)}{Z^2}.$$

Therefore, by choosing Z sufficiently large, (2.7) follows. □

2.2. Global asymptotic stability and global asymptotic stability in mean.

In this section, we discuss the global asymptotic stability of the solution to the system (2.4)-(2.6). We will use the following definitions from [6, 22].

Definition 2.7. Let $(x_{11}(t), x_{21}(t), x_{31}(t))$ denote the positive solution of (2.4)-(2.6) with initial value $(x_{11}(0), x_{21}(0), x_{31}(0)) \in \mathrm{Int}(\mathbb{R}_+^3)$. This solution is said to be globally asymptotically stable if for any other positive solution $(x_{12}(t), x_{22}(t), x_{32}(t))$ with initial value $(x_{12}(0), x_{22}(0), x_{32}(0)) \in \mathrm{Int}(\mathbb{R}_+^3)$, we have

$$\lim_{t\to\infty}|x_{11}(t) - x_{12}(t)| = \lim_{t\to\infty}|x_{21}(t) - x_{22}(t)| = \lim_{t\to\infty}|x_{31}(t) - x_{32}(t)| = 0, \quad \text{a.s.} \tag{2.11}$$

System (2.4)-(2.6) is said to be globally asymptotically stable if (2.11) holds for any two positive solutions.

Definition 2.8. Let $(x_{11}(t), x_{21}(t), x_{31}(t))$ denote the positive solution of (2.4)-(2.6) with initial value $(x_{11}(0), x_{21}(0), x_{31}(0)) \in \mathrm{Int}(\mathbb{R}_+^3)$. This solution is said to be globally asymptotically stable in mean if for any other positive solution $(x_{12}(t), x_{22}(t), x_{32}(t))$ with initial value $(x_{12}(0), x_{22}(0), x_{32}(0)) \in \mathrm{Int}(\mathbb{R}_+^3)$, we have

$$P\Big\{\lim_{t\to+\infty} E\left(|\left(x_{11}(t), x_{21}(t), x_{31}(t)\right) - \left(x_{12}(t), x_{22}(t), x_{32}(t)\right)|\right) = 0\Big\} = 1.$$

Now we state a lemma which will be useful for proving the main theorem of this subsection.

Lemma 2.9. *Suppose that an* n*-dimensional stochastic process* $X(t)$ *on* $t \ge 0$ *satisfies the condition*

$$E|X(t) - X(s)|^{\nu_1} \le m|t-s|^{1+\nu_2}, \quad 0 \le s, t < \infty,$$

for some positive constants ν_1, ν_2 and m. Then there exists a continuous modification $\bar{X}(t)$ of $X(t)$ which has the property that for every $\gamma \in \left(0, \frac{\nu_2}{\nu_1}\right)$ there is a positive random variable $h(\omega)$ such that

$$P\Big\{\omega : \sup_{0<|t-s|<h(\omega), 0\le s,t<\infty} \frac{|\bar{X}(t,\omega) - \bar{X}(t,\omega)|}{|t-s|^\gamma} \le \frac{2}{1-2^{-\gamma}}\Big\} = 1.$$

In other words, almost every sample path of $\bar{X}(t)$ is locally but uniformly Hölder continuous with exponent γ.

The proof of the above lemma can be found in [3].

Lemma 2.10. *Under the assumptions of Theorem 2.5, let $(x_1(t), x_2(t), x_3(t))$ be a solution of (2.4)-(2.6) with initial condition $(x_1(0), x_2(0), x_3(0)) \in \operatorname{Int}(\mathbb{R}^3_+)$ for $t \ge 0$. Then almost every sample path of $(x_1(t), x_2(t), x_3(t))$ is uniformly continuous on $t \ge 0$.*

Proof. Consider the following stochastic integral equation which is equivalent to (2.4),

$$x_1(t) = x_1(0) + \int_0^T f\left(s, x_1(s), x_2(s), x_3(s)\right) ds + \int_0^T g\left(s, x_1(s), x_2(s), x_3(s)\right) dB_1(s),$$

where

$$\begin{gathered} f\left(s, x_1(s), x_2(s), x_3(s)\right) = x_1(s)\big(r_1 - a_{11}x_1(s) - a_{12}x_2(s) + a_{13}x_3(s)\big), \\ g\left(s, x_1(s), x_2(s), x_3(s)\right) = \sigma_1 x_1(s). \end{gathered}$$

Then

$$\begin{aligned}
& E\big(\big|f\left(s, x_1(s), x_2(s), x_3(s)\right)\big|^p\big) \\
&= E\Big(x_1^p(s)\big|r_1 - a_{11}x_1(s) - a_{12}x_2(s) + a_{13}x_3(s)\big|^p\Big) \\
&\le \frac{1}{2}E\Big(x_1^{2p}(s)\Big) + \frac{1}{2}E\Big(\big|r_1 - a_{11}x_1(s) - a_{12}x_2(s) + a_{13}x_3(s)\big|^{2p}\Big) \\
&\le \frac{1}{2}E\big(x_1^{2p}(s)\big) + \frac{2^{2p-1}}{2}E\Big(|r_1 + a_{13}x_3|^{2p}\Big) + \frac{2^{2p-1}}{2}E\Big(|a_{11}x_1 + a_{12}x_2|^{2p}\Big) \\
&\le \frac{2^{4p-2}}{2}(r_1)^{2p} + \Big(\frac{1}{2} + a_{11}^{2p}\frac{2^{4p-2}}{2}\Big)E\big(x_1^{2p}\big) \\
&\quad + (a_{12})^{2p}\frac{2^{4p-2}}{2}E\big(x_2^{2p}\big) + (a_{13})^{2p}\frac{2^{4p-2}}{2}E\big(x_3^{2p}\big) \\
&\le \frac{2^{4p-2}}{2}(r_1)^{2p} + U_b E\left(x_1^{2p} + x_2^{2p} + x_3^{2p}\right) \\
&\le \frac{2^{4p-2}}{2}(r_1)^{2p} + U_b K^*(2p) \equiv F(p),
\end{aligned} \tag{2.12}$$

where $U_b = \max\left\{\frac{1}{2} + (a_{11})^{2p}\frac{2^{4p-2}}{2}, (a_{12})^{2p}\frac{2^{4p-2}}{2}, (a_{13})^{2p}\frac{2^{4p-2}}{2}\right\}$ and

$$\begin{aligned}
E\left(|g\left(s, x_1(s), x_2(s), x_3(s)\right)|^p\right) &= E\left(|\sigma_1|^p |x_1(s)|^p\right) \\
&= \sigma_1^p E\left(|x_1(s)|^p\right) \\
&< \sigma_1^p K^*(p) \equiv G(p).
\end{aligned} \tag{2.13}$$

Assume that $p > 2$. Now applying moment inequality [2] for stochastic integral, for $0 \le t_1 < t_2 < \infty$ and $p > 2$, we have

$$\begin{aligned} &E\Big|\int_{t_1}^{t_2} g\left(s, x_1(s), x_2(s), x_3(s)\right) dB_1(s)\Big|^p \\ &\le \big[\frac{p(p-1)}{2}\big]^{p/2}(t_2-t_1)^{\frac{p-2}{2}}\int_{t_1}^{t_2} E|g\left(s, x_1(s), x_2(s), x_3(s)\right)|^p ds. \end{aligned} \tag{2.14}$$

Let $0 < t_1 < t_2 < \infty$, $t_2 - t_1 \le 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then from (2.12)-(2.14), by applying Hölder's inequality, we obtain

$$\begin{aligned} &E|x_1(t_2) - x_1(t_1)|^p \\ &\le 2^{p-1}E\Big(\int_{t_1}^{t_2} |f\left(s, x_1(s), x_2(s), x_3(s)\right)|ds\Big)^p \\ &\quad + 2^{p-1}E\Big|\int_{t_1}^{t_2} g\left(s, x_1(s), x_2(s), x_3(s)\right) dB_1(s)\Big|^p \\ &\le 2^{p-1}E\Big(\int_{t_1}^{t_2} |f\left(s, x_1(s), x_2(s), x_3(s)\right)|^p ds\Big)\Big(\int_{t_1}^{t_2} 1^q ds\Big)^{p/q} \\ &\quad + 2^{p-1}\big[\frac{p(p-1)}{2}\big]^{p/2}(t_2-t_1)^{\frac{p-2}{2}}\int_{t_1}^{t_2} E|g\left(s, x_1(s), x_2(s), x_3(s)\right)|^p ds \\ &\le 2^{p-1}(t_2-t_1)^p F(p) + 2^{p-1}\big[\frac{p(p-1)}{2}\big]^{p/2}(t_2-t_1)^{p/2}G(p) \\ &\le 2^{p-1}(t_2-t_1)^{p/2}\big\{(t_2-t_1)^{p/2} + \big[\frac{p(p-1)}{2}\big]^{p/2}\big\}M(p) \\ &\le 2^{p-1}(t_2-t_1)^{p/2}\big\{1 + \big[\frac{p(p-1)}{2}\big]^{p/2}\big\}M(p), \end{aligned}$$

where $M(p) = \max\{F(p), G(p)\}$.

Therefore, it follows from Lemma 2.9 that almost every sample path $x_1(t)$ is locally but uniformly Hölder continuous with exponent γ for every $\gamma \in \Big(0, \frac{p-2}{2p}\Big)$ and hence almost every sample path of $x_1(t)$ is uniformly continuous on $t \ge 0$. In a similar manner, one can easily show that almost every sample path of $x_2(t)$ and $x_3(t)$ are also uniformly continuous. A property of an uniformly continuous function defined over $\mathbb{R}_+$ is recalled in the following lemma, its proof can be found in [1]. □

Lemma 2.11. *Let $h(t)$ be a non-negative function defined on $\mathbb{R}_+$ such that $h(t)$ is integrable and uniformly continuous on $\mathbb{R}_+$, then $\lim_{t\to+\infty} h(t) = 0$.*

Theorem 2.12. *Assume that the conditions of Theorem 2.5 hold. Further if,*

$$a_{11} < a_{21} + a_{31},\ a_{22} < a_{12} + a_{32},\ a_{33} < a_{13} + a_{23} \tag{2.15}$$

then system (2.4)-(2.6) *is globally asymptotically stable.*

Proof. Consider the Lyapunov function $V(t)$ defined by

$$\begin{aligned} V(t) = &|\log(x_{11}(t)) - \log(x_{12}(t))| + |\log(x_{21}(t)) - \log(x_{22}(t))| \\ &+ |\log(x_{31}(t)) - \log(x_{32}(t))|, \end{aligned} \tag{2.16}$$

for $t \geq 0$. Calculating the right differential $d^+V(t)$ of $V(t)$ along the solutions of (2.4)-(2.6) and using Itô's formula, we obtain

$$\begin{aligned}
d^+V(t) &= \operatorname{sign}(x_{11}(t)-x_{12}(t))\big\{-a_{11}(x_{11}(t)-x_{12}(t)) - a_{12}\,(x_{21}(t)-x_{22}(t)) \\
&\quad + a_{13}(x_{31}(t)-x_{32}(t))\big\}dt \\
&\quad + \operatorname{sign}(x_{21}(t)-x_{22}(t))\big\{-a_{21}\,(x_{11}(t)-x_{12}(t)) - a_{22}\,(x_{21}(t)-x_{22}(t)) \\
&\quad + a_{23}(x_{31}(t)-x_{32}(t))\big\}dt \\
&\quad + \operatorname{sign}(x_{31}(t)-x_{32}(t))\big\{a_{31}\,(x_{11}(t)-x_{12}(t)) + a_{32}\,(x_{21}(t)-x_{22}(t)) \\
&\quad - a_{33}(x_{31}(t)-x_{32}(t))\big\}dt \\
&\leq \big\{-(a_{11}-a_{21}-a_{31})|x_{11}(t)-x_{12}(t)| - (a_{22}-a_{12}-a_{32})|x_{21}(t)-x_{22}(t)| \\
&\quad - (a_{33}-a_{13}-a_{23})|x_{31}(t)-x_{32}(t)|\big\}dt
\end{aligned}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}
V(t) \leq V(0) &- \int_0^T (a_{11}-a_{21}-a_{31})|x_{11}(s)-x_{12}(s)|ds \\
&- \int_0^T (a_{22}-a_{12}-a_{32})|x_{21}(s)-x_{22}(s)|ds \\
&- \int_0^T (a_{33}-a_{13}-a_{23})|x_{31}(s)-x_{32}(s)|ds
\end{aligned}$$

Consequently,

$$\begin{aligned}
&V(t) + \int_0^T (a_{11}-a_{21}-a_{31})|x_{11}(s)-x_{12}(s)|ds \\
&+ \int_0^T (a_{22}-a_{12}-a_{32})|x_{21}(s)-x_{22}(s)|ds \\
&+ \int_0^T (a_{33}-a_{13}-a_{23})|x_{31}(s)-x_{32}(s)|ds \\
&\leq V(0) < \infty.
\end{aligned} \tag{2.17}$$

Since $V(t) \geq 0$, $a_{11} > a_{21} + a_{31}$, $a_{22} > a_{12} + a_{32}$ and $a_{33} > a_{13} + a_{23}$, we have $|x_{11}(t) - x_{12}(t)| \in L^1[0,\infty)$, $|x_{21}(t) - x_{22}(t)| \in L^1[0,\infty)$, $|x_{31}(t) - x_{32}(t)| \in L^1[0,\infty)$. Now using Lemmas 2.10 and 2.11, we obtain $\lim_{t\to\infty} |x_{11}(t)-x_{12}(t)| = 0$, $\lim_{t\to\infty} |x_{21}(t) - x_{22}(t)| = 0$ and $\lim_{t\to\infty} |x_{31}(t) - x_{32}(t)| = 0$, a.s. This completes the proof of the theorem. □

Corollary 2.13. *System* (2.4)-(2.6) *is globally asymptotically stable in mean under the same parametric restrictions as Theorem 2.12.*

Proof. Taking expectation of both sides of (2.17) and applying Fubini's theorem,

$$\begin{aligned}
&E\,(V(t)) + \int_0^T (a_{11}-a_{21}-a_{31})E|x_{11}(s)-x_{12}(s)|ds \\
&+ \int_0^T (a_{22}-a_{12}-a_{32})E|x_{21}(s)-x_{22}(s)|ds \\
&+ \int_0^T (a_{33}-a_{13}-a_{23})E|x_{31}(s)-x_{32}(s)|ds
\end{aligned}$$

$$\leq V(0) < \infty.$$

Using the conditions of Theorem 2.12 and $V(t) \geq 0$, it follows that $E|x_{11}(t) - x_{12}(t)| \in L^1[0,\infty)$, $E|x_{21}(t) - x_{22}(t)| \in L^1[0,\infty)$, $E|x_{31}(t) - x_{32}(t)| \in L^1[0,\infty)$. Also

$$\begin{aligned}
&E\big(|(x_{11}(t), x_{21}(t), x_{31}(t)) - (x_{12}(t), x_{22}(t), x_{32}(t))|\big) \\
&= E\big\{[|x_{11}(t) - x_{12}(t)|^2 + |x_{21}(t) - x_{22}(t)|^2 + |x_{31}(t) - x_{32}(t)|^2]^{\frac{1}{2}}\big\} \\
&\leq E\big(|x_{11}(t) - x_{12}(t)|\big) + E\big(|x_{21}(t) - x_{22}(t)|\big) + E\big(|x_{31}(t) - x_{32}(t)|\big) \in L^1[0,\infty).
\end{aligned}$$

Therefore, using Lemmas 2.10 and 2.11, one obtains

$$\lim_{t\to+\infty} E\left(|\,(x_{11}(t), x_{21}(t), x_{31}(t)) - (x_{12}(t), x_{22}(t), x_{32}(t))\,|\right) = 0, \quad \text{a.s.}$$

Hence under the same parametric conditions like Theorem 2.12, system (2.4)-(2.6) is globally asymptotically stable in mean. □

3. Numerical simulation

To confirm the analytical findings, we simulate the solution of system (2.4)-(2.6) using Milstein's method having strong order of convergence $\gamma = 1$ [15]. The discretized scheme for the stochastic system (2.4)-(2.6), following the Milstein's method is

$$\begin{aligned}
x_{1,j+1} &= x_{1,j} + x_{1,j}\Big[\left(r_1 - a_{11}x_{1,j} - a_{12}x_{2,j} + a_{13}x_{3,j}\right)\Delta t + \sigma_1\epsilon_{1j}\sqrt{\Delta t} \\
&\quad + \frac{1}{2}\sigma_1^2\Delta t(\epsilon_{1j}^2 - 1)\Big], \\
x_{2,j+1} &= x_{2,j} + x_{2,j}\Big[\left(r_2 - a_{21}x_{1,j} - a_{22}x_{2,j} + a_{23}x_{3,j}\right)\Delta t + \sigma_2\epsilon_{2j}\sqrt{\Delta t} \\
&\quad + \frac{1}{2}\sigma_2^2\Delta t(\epsilon_{2j}^2 - 1)\Big], \\
x_{3,j+1} &= x_{3,j} + x_{3,j}\Big[\left(r_3 + a_{31}x_{1,j} + a_{32}x_{2,j} - a_{33}x_{3,j}\right)\Delta t + \sigma_3\epsilon_{3j}\sqrt{\Delta t} \\
&\quad + \frac{1}{2}\sigma_3^2\Delta t(\epsilon_{3j}^2 - 1)\Big],
\end{aligned}$$

where ϵ_{1j}, ϵ_{2j} and ϵ_{3j} are three independent Gaussian random variables $N(0,1)$ for $j = 1, 2, \dots, n$ and Δt is the time step.

Now consider the numerical example

$$dx_1(t) = x_1(t)(0.9 - 3x_1(t) - x_2(t) + 1.2x_3(t)) + 0.01x_1(t)dB_1(t), \tag{3.1}$$

$$dx_2(t) = x_2(t)(2257/850 - 2257/850x_1(t) - 2.2x_2(t) + x_3(t)) + 0.01x_2(t)dB_2(t), \tag{3.2}$$

$$dx_3(t) = x_3(t)(0.1 + 0.1x_1(t) + x_2(t) - 2.4x_3(t)) + 0.01x_3(t)dB_3(t), \tag{3.3}$$

Some of the parameter values of the above example are chosen from [10] and others are chosen hypothetically. For simulation purpose, we choose the time step $\Delta t = 0.01$. Comparing the above example with (2.4)-(2.6), we obtain $r_1 = 0.9$, $r_2 = 2257/850$, $r_3 = 0.1$, $a_{11} = 3$, $a_{12} = 1.4$, $a_{13} = 1.2$, $a_{21} = 2257/850$, $a_{22} = 2.2$, $a_{23} = 1$, $a_{31} = 0.5$, $a_{32} = 1$, $a_{33} = 2$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0.01$. Therefore, $a_{11} - a_{13} = 1.8 > 0$, $a_{22} - a_{23} = 1.2 > 0$, $a_{33} - a_{31} - a_{32} = 0.5 > 0$, $a_{21} + a_{31} - a_{11} = 0.1553 > 0$, $a_{12} + a_{32} - a_{22} = 0.2 > 0$ and $a_{13} + a_{23} - a_{33} = 0.2 > 0$. Therefore, all the conditions of the Theorem 2.12 are satisfied. Hence system (3.1)-(3.3) is globally asymptotically stable (see Figure 1) as well as globally asymptotically stable in mean (see Figure 2).

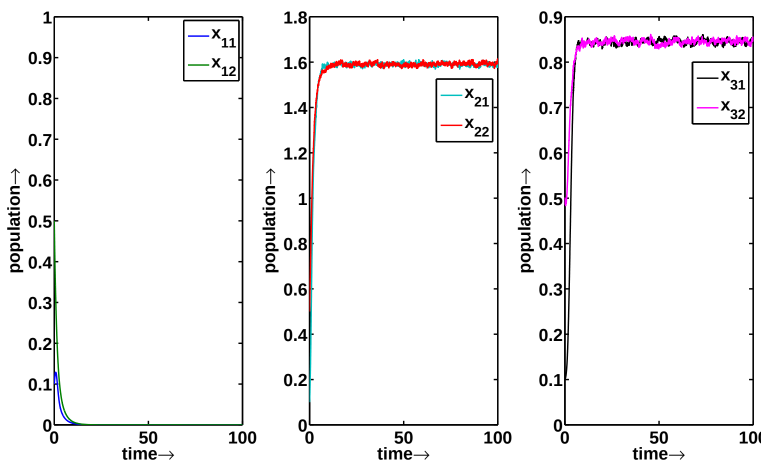

FIGURE 1. Trajectories for the SDE system (3.1)-(3.3) with two sets of initial conditions $(x_{11}(0), x_{21}(0), x_{31}(0)) = (0.1, 0.1, 0.1)$ and $(x_{12}(0), x_{22}(0), x_{32}(0)) = (0.5, 0.5, 0.5)$

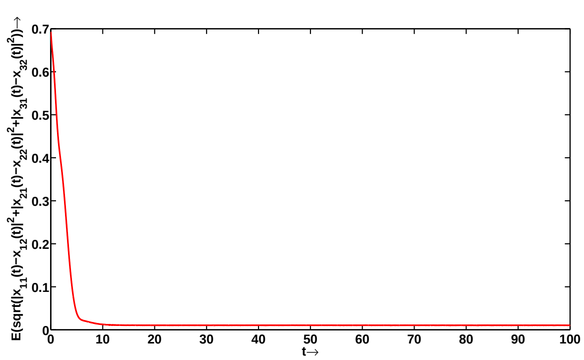

FIGURE 2. Global asymptotic stability in mean of the SDE system (3.1)-(3.3)

Discussion. In this work, we have studied a competitor-competitor-mutualist Lotka-Volterra model in presence of environmental noise. First, we have proved that the given SDE system has a unique positive global solution which ensures that the solution will not go to explosion at a finite time. Second, we have shown that the sum of higher order moments of each component to the solution of the concerned SDE system is uniformly bounded under some parametric conditions though it is impossible to find out the uniform bound of the moments of each component separately.

Uniform bound of the moments ensures that the solution of the given SDE system is stochastically bounded. It is interesting to observe that due to the presence of both the competitive and cooperative terms in the working model, moment boundedness depends on the parametric conditions as this is not the case always (see [6, 22]). Finally, I have shown that the given SDE system is globally asymptotically stable and as well as it globally asymptotically stable in mean under the same parametric restrictions. There are several literatures dealing with global asymptotic stability results [4, 5, 6, 8]. In[4] Jiang et al derived the sufficient conditions for global asymptotic stability of unique positive solution for one dimensional non-autonomous model of population growth with logistic growth law. This analysis is extended for a single species stochastic logistic model with impulsive perturbation term by Liu and Wang in[8]. Sufficient conditions for global attractivity (global asymptotic stability) for a non-autonomous Lotka-Volterra type competitive model with multiplicative noise terms are derived by Li and Mao in[5]. In [22], Ji et al proved that the positive solution of a stochastic logistic model is globally asymptotically stable in mean. In[6], Liu and Wang also showed that the positive solution of a stochastic non-autonomous predator-prey model is globally asymptotically stable. Recently, Liu and Wang [7] obtained the sufficient condition for global asymptotic stability of the classical n-species mutualism system and hence it would be interesting to study the global asymptotic stability for a n-dimensional model having m-competitors and p-mutualists such that $m + p = n$.

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