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COUNTEREXAMPLES TO MEAN SQUARE ALMOST PERIODICITY OF THE SOLUTIONS OF SOME SDES WITH ALMOST PERIODIC COEFFICIENTS

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Abstract. We show that, contrarily to what is claimed in some papers, the nontrivial solutions of some stochastic differential equations with almost periodic coefficients are never mean square almost periodic (but they can be almost periodic in distribution).

1. Introduction

Almost periodicity for stochastic processes and in particular for solutions of stochastic differential equations is investigated in an increasing number of papers since the works of Tudor and his collaborators [1, 8, 11, 12], who proved almost periodicity in distribution of solutions of some SDEs with almost periodic coefficients. More recently, Bezandry and Diagana [3, 4, 5] claimed that some SDEs with almost periodic coefficients have solutions which satisfy the stronger property of mean square almost periodicity. These claims are repeated in some subsequent papers and a book by different authors.

The aim of this short note is to give counterexamples to the results of [3, 4, 5].

Notation. We denote by $\operatorname{law} Y$ the distribution of a random variable Y. If $\mathbb{X}$ is a metrizable topological space, we denote by $\mathcal{M}^{1,+}(\mathbb{X})$ the set of Borel probability measures on $\mathbb{X}$, endowed with the topology of narrow (or weak) convergence; i.e., the coarsest topology such that the mappings $\mu \mapsto \mu(\varphi)$, $\mathcal{M}^{1,+}(\mathbb{X}) \to \mathbb{R}$ are continuous for all bounded continuous $\varphi : \mathbb{X} \to \mathbb{R}$.

Let $(\mathbb{X}, d)$ be a metric space. A continuous mapping $f : \mathbb{R} \to \mathbb{X}$ is said to be *almost periodic* (in Bohr's sense) if, for every $\varepsilon > 0$, there exists a number $l(\varepsilon) > 0$ such that every interval I of length greater than $l(\varepsilon)$ contains an ε-*almost period*; that is, a number $\tau \in I$ such that $d(f(t+\tau), f(t)) \le \varepsilon$ for all $t \in \mathbb{R}$. Equivalently, by a criterion of Bochner, f is almost periodic if and only if the set $\{x(t+.),\, t \in \mathbb{R}\}$ is totally bounded in the space $C(\mathbb{R}, \mathbb{X})$ endowed with the topology of uniform convergence. Thanks to another criterion of Bochner [6], almost periodicity of f does not depend on the metric d nor on the uniform structure of $(\mathbb{X}, d)$, but only

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on f and the topology generated by d (see [2] for details). We refer to e.g. [7, 13] for beautiful expositions of almost periodic functions and their many properties.

Let $X = (X_t)_{t\in\mathbb{R}}$ be a continuous stochastic process with values in a separable Banach space $\mathbb{E}$:

- We say that X is *mean square almost periodic* if X_t is square integrable for each t and the mapping $t \mapsto X_t$, $\mathbb{R} \to L^2(\mathbb{E})$ is almost periodic.
- We say that X is *almost periodic in distribution* (in Bohr's sense) if the mapping $t \mapsto \operatorname{law} X_{t+.}$, $\mathbb{R} \to \mathcal{M}^{1,+}(C(\mathbb{R},\mathbb{E}))$ is almost periodic, where $C(\mathbb{R},\mathbb{E})$ is endowed with the topology of uniform convergence on compact subsets.

It is shown in [2] that, if X is mean square almost periodic, then X is almost periodic in distribution. The counterexamples of this paper also show that the converse implication is false (actually, it is proved in [2] that the converse implication is true under a tightness condition).

2. Two explicit counterexamples

The following very simple counterexample, inspired by [2, Counterexample 2.16], was suggested to us by Adam Jakubowski. It contradicts [3, Theorem 3.2], [4, Theorem 3.3], and [5, Theorem 4.2].

Example 2.1 (stationary Ornstein-Uhlenbeck process). Let $W = (W_t)_{t\in\mathbb{R}}$ be a standard Brownian motion on the real line. Let $\alpha, \sigma > 0$, and let X be the stationary Ornstein-Uhlenbeck process (see [10]) defined by

$$X_t = \sqrt{2\alpha\sigma} \int_{-\infty}^{t} e^{-\alpha(t-s)} dW_s. \tag{2.1}$$

Then X is the only L^2-bounded solution of the following SDE, which is a particular case of Equation (3.1) in [3]:

$$dX_t = -\alpha X_t \, dt + \sqrt{2\alpha}\sigma \, dW_t.$$

The process X is Gaussian with mean 0, and we have, for all $t \in \mathbb{R}$ and $\tau \geq 0$,

$$\operatorname{Cov}(X_t, X_{t+\tau}) = \sigma^2 e^{-\alpha\tau}.$$

Assume that X is mean square almost periodic, and let (t_n) be any increasing sequence of real numbers which converges to ∞. By Bochner's characterization, we can extract a sequence (still denoted by (t_n) for simplicity) such that (X_{t_n}) converges in L^2 to a random variable Y. Necessarily Y is Gaussian with law $\mathcal{N}(0, 2\alpha\sigma^2)$, and Y is $\mathcal{G}$-measurable, where $\mathcal{G} = \sigma(X_{t_n} : n \geq 0)$. Moreover (X_{t_n}, Y) is Gaussian for every n, and we have, for any integer n,

$$\operatorname{Cov}(X_{t_n}, Y) = \lim_{m\to\infty} \operatorname{Cov}(X_{t_n}, X_{t_{n+m}}) = 0$$

because $(X_t^2)_{t\in\mathbb{R}}$ is uniformly integrable. This proves that Y is independent of X_{t_n} for every n, thus Y is independent of $\mathcal{G}$. Thus Y is constant, a contradiction.

Thus (2.1) has no mean square almost periodic solution.

A similar reasoning applies to the next counterexample, which also contradicts [3, Theorem 3.2], [4, Theorem 3.3], and [5, Theorem 4.2]:

Example 2.2. Again, $W = (W_t)_{t\in\mathbb{R}}$ is a standard Brownian motion on the real line. Let X be defined by

$$X_t = e^{-t+\sin(t)} \int_{-\infty}^{t} e^{s-\sin(s)} \sqrt{1-\cos(s)}\, dW_s.$$

Then X satisfies the SDE with periodic coefficients

$$dX_t = (-1+\cos(t))X_t\, dt + \sqrt{1-\cos(t)}\, dW_t.$$

The process X is Gaussian, with $EX_t = 0$ and

$$\begin{aligned}\operatorname{Cov}(X_t, X_{t+\tau}) &= e^{-t-\tau+\sin(t+\tau)} e^{-t+\sin(t))} \int_{-\infty}^{t} e^{2(s-\sin(s))}(1-\cos(s))\, ds \\ &= \frac{1}{2}\, e^{-\tau+\sin(t+\tau)-\sin(t)} \to 0 \quad \text{as } \tau \to +\infty\end{aligned}$$

in particular $EX_t^2 = \frac{1}{2}\, e^{2\sin(t)} \geq \frac{1}{2}\, e^{-2}$ thus the same reasoning as in Example 2.1 shows that X is not mean square almost periodic, because if X_{t_n} converges in L^2 to Y, with $t_n \to \infty$, then $Y = 0$ and $EY^2 \geq e^{-2}/2$.

By [8, Theorem 4.1], the process X is periodic in distribution.

The argument in the previous counterexamples can be slightly generalized for non necessarily Gaussian processes as follows:

Lemma 2.3. *Let X be a continuous square integrable stochastic process with values in a Banach space $\mathbb{E}$. Assume that $(\|x_t\|^2)_{t\in\mathbb{R}}$ is uniformly integrable and that there exists a sequence (t_n) of real numbers, $t_n \to \infty$, such that for any $x^* \in \mathbb{E}^*$ and any integer $n \geq 0$,*

$$\lim_{m\to\infty} \operatorname{Cov}\big(\langle x^*, X_{t_n}\rangle, \langle x^*, X_{t_m}\rangle\big) = 0, \tag{2.2}$$

$$\lim_{m\to\infty} \operatorname{Var}(\|X_{t_m}\|) > 0. \tag{2.3}$$

Then X is not mean square almost periodic.

Proof. Assume that X is mean square almost periodic. Then, for some subsequence (t'_n) of (t_n), $X_{t'_n}$ converges in L^2 to some random vector Y. By (2.3) and the uniform integrability hypothesis, Y is not constant. On the other hand, by (2.2) and the uniform integrability hypothesis, we have

$$\operatorname{Cov}\big(\langle x^*, X_{t'_n}\rangle, \langle x^*, Y\rangle\big) = 0$$

for every $x^* \in \mathbb{E}^*$ and every integer n. Then

$$\operatorname{Var}\langle x^*, Y\rangle = \lim_n \operatorname{Cov}\big(\langle x^*, X_{t'_n}\rangle, \langle x^*, Y\rangle\big) = 0,$$

thus Y is constant, a contradiction. □

3. Generalization

We present a generalization of Counterexamples 2.1 and 2.2 in a Hilbert space setting. Other generalizations in the same setting are possible.

For the rest of this article, $\mathbb{H}$ and $\mathbb{U}$ are separable Hilbert spaces, Q is a symmetric nonnegative operator on $\mathbb{U}$ with finite trace, and $(W_t)_{t\in\mathbb{R}}$ is a Q-Brownian motion with values in $\mathbb{U}$. We denote $\mathbb{U}_0 = Q^{1/2}\mathbb{U}$ and $L_2^0 = L_2(\mathbb{U}_0, \mathbb{H})$ the space of Hilbert-Schmidt operators from $\mathbb{U}_0$ to $\mathbb{H}$, endowed with the Hilbertian norm

$$\|\Psi\|_{L_2^0}^2 = \|\Psi Q^{1/2}\|_{L_2}^2 = \operatorname{Tr}(\Psi Q \Psi^*).$$

It is well known that, if Φ is a predictable stochastic process with values in L_2^0 such that $\int_0^t \|\Phi_s\|_{L_2^0}^2\,ds < +\infty$, then we have the Ito isometry

$$E\Big(\big\|\int_0^t \Phi_s\,dW_s\big\|^2\Big) = \int_0^t \|\Phi_s\|_{L_2^0}^2\,ds.$$

Recall (see e.g. [9, Definitions 1.4.1 and 1.4.2]) that a linear operator $A(t)$ on $\mathbb{H}$ with domain $D(A(t))$ generates an *evolution semigroup* $(U(t,s))_{t\geq s}$ on $\mathbb{H}$, if $(U(t,s))_{t\geq s}$ is a family of bounded linear operators on $\mathbb{H}$ such that

(i) $U(t,r)\,U(r,s) = U(t,s)$ for all $t,r,s\in\mathbb{R}$ such that $s\leq r\leq t$, and, for every $t\in\mathbb{R}$, $U(t,t)=I$ the identity operator on $\mathbb{H}$,
(ii) for every $x\in\mathbb{H}$, the mapping $(t,s)\mapsto U(t,s)$ from $\{(t,s): t\geq s\}$ to $\mathbb{H}$ is continuous,
(iii) for every $T>0$, there exists $K_T<\infty$ such that $\|U(t,s)\|\leq K_T$ for $0\leq s\leq t\leq T$,
(iv) for all $t,s\in\mathbb{R}$ such that $s\leq t$, the domain $D(A(t))$ is dense in $\mathbb{H}$, $U(t,s)\,D(A(s))\subset D(A(t))$, and
$$\frac{\partial}{\partial t}U(t,s)\,x = A(t)\,U(t,s)\,x \text{ for } t>s \text{ and } x\in D(A(s)).$$

The following theorem contains Counterexamples 2.1 and 2.2. Counterexample 2.2 can be seen as a particular case of Equation (3.1) below, with $A(t) = -1+\cos(t)$ which generates the evolution semigroup $U(t,s) = e^{-(t-s)+\sin(t)-\sin(s)}$.

Theorem 3.1 (linear evolution equations with almost periodic noise)**.** *Let us consider the stochatic evolution equation*

$$dX_t = A(t)X_t\,dt + g(t)\,dW_t \tag{3.1}$$

where $A(.)$ generates an evolution semigroup $(U(t,s))_{t\geq s}$ on $\mathbb{H}$. We assume that

(a) *(see* [8, Hypothesis 1]*) the Yosida approximations $A_n(t) = nA(t)(nI - A(t))^{-1}$ of $A(t)$, $t\in\mathbb{R}$, generate corresponding evolution operators*
$$(U_n(t,s))_{t\geq s}$$
such that, for every $x\in\mathbb{H}$ and for all $t,s\in\mathbb{R}$ such that $s\leq t$,
$$\lim_{n\to\infty} U_n(t,s)x = U(t,s)x;$$
(b) *A is* uniformly dissipative *(see* [8, Hypothesis 3]*), i.e. there exists $\beta>0$ such that*
$$\langle A(t)x,x\rangle \leq -\beta\|x\|^2,\quad t\in\mathbb{R},\ x\in D(A(t));$$
(c) *U is* exponentially stable *(see* [4, Hypothesis H0]*), i.e.,*
$$\|U(t,s)\|\leq Me^{-\delta(t-s)},\quad t\geq s; \tag{3.2}$$
(d) *$g:\mathbb{R}\to L_2^0$ is almost periodic and satisfies*
$$0<\int_{-\infty}^{+\infty}\|U(t,s)g(s)\|_{L_2^0}^2 ds < +\infty\,. \tag{3.3}$$

Then (3.1) *has no mean square almost periodic solution. However, if the unique* L^2*-bounded solution X to* (3.1) *is such that the family $(X_t)_{t\in\mathbb{R}}$ is tight, then it is almost periodic in distribution.*

Note that, if A and g are T-periodic, then by [8, Theorem 4.1] the L^2-bounded solution is T-periodic in distribution, that is, the mapping $t \mapsto \operatorname{law} X_{t+\cdot}$, $\mathbb{R} \to \mathcal{M}^{1,+}(C(\mathbb{R},\mathbb{E}))$, is periodic.

Proof of Theorem 3.1. The only L^2-bounded (mild) solution to (3.1) is

$$X_t = \int_{-\infty}^{t} U(t,s)g(s)\,dW_s; \tag{3.4}$$

see the proof of [8, Theorem 3.3]. Note that X is Gaussian because the integrand in (3.4) is deterministic. By [9, Theorem 1.4.5], X has a continuous version (actually [9, Theorem 1.4.5] is given for processes defined on the half line $\mathbb{R}^+$, but we can repeat the argument on any interval $[-R,\infty)$). By [8, Theorem 4.3], if the family $(X_t)_{t\in\mathbb{R}}$ is tight, X is almost periodic in distribution.

Let $p > 2$. Applying Burkholder-Davis-Gundy inequalities to the process $t \mapsto \int_{-\infty}^{t} U(t_0,s)g(s)\,dW_s$ for fixed t_0, and then setting $t = t_0$ yields, for some constant c_p,

$$\begin{aligned} E\|X_t\|^p &\le c_p\Big(\int_{-\infty}^{t} \|U(t,s)g(s)\|^2_{L_2^0}\,ds\Big)^{p/2} \\ &\le c_p\Big(\int_{-\infty}^{+\infty} \|U(t,s)g(s)\|^2_{L_2^0}\,ds\Big)^{p/2} < +\infty \end{aligned}$$

(see e.g. [9, Theorems 1.2.1, 1.2.3-(e) and Proposition 1.3.3-(f)]). Thus (X_t) is bounded in L^p, which proves that $(\|X_t\|^2)_{t\in\mathbb{R}}$ is uniformly integrable.

We have $E(X_t) = 0$ for all $t \in \mathbb{R}$. Let $x \in \mathbb{H}$, $t \in \mathbb{R}$ and $\tau \ge 0$, and let us compute the covariance $\operatorname{Cov}\big(\langle x, X_t\rangle, \langle x, X_{t+\tau}\rangle\big)$: We obtain

$$\begin{aligned} &\operatorname{Cov}\big(\langle x, X_t\rangle, \langle x, X_{t+\tau}\rangle\big) \\ &= E\Big(\Big\langle x, \int_{-\infty}^{t} U(t,s)g(s)\,dW_s\Big\rangle \\ &\quad \times \Big\langle x, \Big(\int_{-\infty}^{t} U(t+\tau,s)g(s)\,dW_s + \int_{t}^{t+\tau} U(t+\tau,s)g(s)\,dW_s\Big)\Big\rangle\Big) \\ &= E\Big(\Big\langle x, \int_{-\infty}^{t} U(t,s)g(s)\,dW_s\Big\rangle \Big\langle U(t+\tau,t)^* x, \int_{-\infty}^{t} U(t,s)g(s)\,dW_s\Big\rangle\Big). \end{aligned}$$

Using (3.2) and (3.3), we deduce

$$\begin{aligned} &\lim_{\tau\to+\infty} \big|\operatorname{Cov}\big(\langle x, X_t\rangle, \langle x, X_{t+\tau}\rangle\big)\big| \\ &\le \lim_{\tau\to+\infty} \|U(t+\tau,t)\|\|x\|^2 E\int_{-\infty}^{t} \|(U(t,s)g(s)\|^2_{L_2^0}\,ds = 0. \end{aligned}$$

On the other hand, using (3.3), we have

$$\operatorname{Var}(\|X_t\|) = E\Big(\int_{-\infty}^{t} \|U(t,s)g(s)\|^2_{L_2^0}\,ds\Big) \to \int_{-\infty}^{+\infty} \|U(t,s)g(s)\|^2_{L_2^0}\,ds > 0\,.$$

We conclude by Lemma 2.3 that X is not mean square almost periodic. $\square$

4. Conclusion

A close look at the proofs of [3, 4, 5] shows the same error in each of those papers, which besides are clever at other places. Let us use the notations of the Hilbert setting of Section 3, and assume that all processes are defined on a probability space $(\Omega, \mathcal{F}, P)$. The error lies in the proof of the (untrue) assertion that, if $G : \mathbb{R} \times L^2(P; \mathbb{H}) \to L^2(P; L_2^0)$ is almost periodic in the first variable, uniformly with respect to the second on compact subsets of $L^2(P; \mathbb{H})$, then the stochastic convolution

$$\Psi(Y)_t := \int_{-\infty}^{t} U(t,s)G(s,Y_s)\,dW_s$$

is mean square almost periodic for any continous square integrable stochastic process Y. If this were true, then with $G(t,Y) = g(t)$ an almost periodic function, and assuming the hypothesis of Theorem 3.1, the process $X = \psi(1)$ of Equation (3.4), which is solution of (3.1), would be mean square almost periodic, but we know from Theorem 3.1 that this is not the case. The error consists in a wrong identification between integrals of the form $\int_{-\infty}^{t} Z_s\,dW_s$ and $\int_{-\infty}^{t} Z_s\,d\widetilde{W}_s$, where $\widetilde{W}$ has the same distribution as W.

Actually, mean square almost periodicity appears to be a very strong property for solutions of SDEs. Our counter-examples suggest that there are "very few" examples of SDEs with non trivial mean square almost periodic solutions. The question of their characterization remains open.

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