A GRADIENT ESTIMATE FOR SOLUTIONS TO PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT. Li-Vogelius and Li-Nirenberg gave a gradient estimate for solutions of strongly elliptic equations and systems of divergence forms with piecewise smooth coefficients, respectively. The discontinuities of the coefficients are assumed to be given by manifolds of codimension 1, which we called them *manifolds of discontinuities*. Their gradient estimate is independent of the distances between manifolds of discontinuities. In this paper, we gave a parabolic version of their results. That is, we gave a gradient estimate for parabolic equations of divergence forms with piecewise smooth coefficients. The coefficients are assumed to be independent of time and their discontinuities are likewise the previous elliptic equations. As an application of this estimate, we also gave a pointwise gradient estimate for the fundamental solution of a parabolic operator with piecewise smooth coefficients. Both gradient estimates are independent of the distances between manifolds of discontinuities.

1. INTRODUCTION

For strongly elliptic, second order scalar equations with real coefficients, it is well known that their solutions have the Hölder continuity even in the case that the coefficients are only bounded measurable functions. However, the solutions do not have the Lipschitz continuity in general. For example, Piccinini-Spagnolo [17, p. 396, Example 1] and Meyers [14, p. 204] gave the following case:

Example 1.1 ([14, 17]). Let $B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$ and each $a_{ij} \in L^\infty(B_1)$ be defined as

$$a_{11} = \frac{Mx_1^2 + x_2^2}{|x|^2}, \quad a_{22} = \frac{x_1^2 + Mx_2^2}{|x|^2}, \quad a_{12} = a_{21} = \frac{(M-1)x_1x_2}{|x|^2}$$

with a constant $M > 1$. Then, if we define u as

$$u(x) = \begin{cases} |x|^{1/\sqrt{M}} \frac{x_1}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{1.1}$$

it is easy to see that the Hölder exponent of u is at least less than or equal to $1/\sqrt{M}$ (indeed, for $\overline{x} = (x_1, 0)$ we have $|u(\overline{x}) - u(0)| = |\overline{x}|^{1/\sqrt{M}}$. Hence we have

$$\frac{|u(\overline{x}) - u(0)|}{|\overline{x}|^{(1/\sqrt{M})+\varepsilon}} = |\overline{x}|^{-\varepsilon} \to +\infty \quad \text{as } \overline{x} \to 0$$

for any $\varepsilon > 0$.) and u satisfies the strongly elliptic scalar equation with real coefficients

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = 0. \tag{1.2}$$

The same thing can be said also to the parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = 0, \tag{1.3}$$

because u given by (1.1) satisfies this equation.

This example shows that we cannot expect gradient estimates of solutions to equations (1.2) and (1.3) in the case $a_{ij} \in L^\infty(B_1)$, but we may have the estimates in the case of piecewise C^μ (see (1.5) below) coefficients.

The fact that the gradient estimate of solutions is independent of the distances between manifolds of discontinuities was first observed by Babuška-Andersson-Smith-Levin [2] numerically for certain homogeneous isotropic linear systems of elasticity, that is $|\nabla u|$ is bounded independently of the distances between manifolds of discontinuities. They considered that this numerical property of solutions is mathematically true. This is the so-called Babuška's conjecture. Recently, proofs for this conjecture appeared in [13] and [12]. In elasticity, a small static deformation of an elastic medium with inclusions can be described by an elliptic system of divergence form with piecewise smooth coefficients. The discontinuities of coefficients form the boundaries of inclusions. Similar physical interpretation is also possible for heat conductors. Our main theorem 1.5 given below ensures that this property also holds for parabolic equations of the form (1.3). The details of result given in [13] and [12] for scalar equations will be given below as Theorem 1.2.

To state our main theorem, we begin with introducing several notations which will be used throughout this paper. Let $D \subset \mathbb{R}^n$ be a bounded domain with a $C^{1,\alpha}$ boundary for some $0 < \alpha < 1$, which means that the domain D contains L disjoint subdomains $D_1, \dots, D_L$ with $C^{1,\alpha}$ boundaries, i.e. $D = (\bigcup_{m=1}^{L} \overline{D_m}) \setminus \partial D$, and we also assume that $\overline{D_m} \subset D$ for $1 \le m \le L-1$. Physically, D is a material and D_m ($1 \le m \le L-1$) are considered as inclusions in D. We define the $C^{1,\alpha}$ norm (resp. $C^{1,\alpha}$ seminorm) of $C^{1,\alpha}$ domain D_m in the same way as in [12], that is, as the largest positive number a such that in the a-neighborhood of every point of ∂D_m, identified as 0 after a possible translation and rotation of the coordinates so that $x_n = 0$ is the tangent to ∂D_m at 0, ∂D_m is given by the graph of a $C^{1,\alpha}$ function ψ_m, defined in $|x'| < 2a$ ($x' = (x_1, \dots, x_{n-1})$), the $2a$-neighborhood of 0 in the tangent plane, and it satisfies the estimate $\|\psi_m\|_{C^{1,\alpha}(|x'|<2a)} \le 1/a$ (resp. $[\psi_m]_{C^{1,\alpha}(|x'|<2a)} \le 1/a$), where

$$[\psi]_{C^{1,\alpha}(|x'|<2a)} := \sup_{|x'|,|\xi'|<2a} \frac{|\nabla'\psi(x') - \nabla'\psi(\xi')|}{|x' - \xi'|^\alpha},$$

$$\|\psi\|_{C^{1,\alpha}(|x'|<2a)} := \|\psi\|_{C^1(|x'|<2a)} + [\psi]_{C^{1,\alpha}(|x'|<2a)}.$$

Further, let (a_{ij}) be a symmetric, positive definite matrix-valued function defined on D satisfying

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2. \tag{1.4}$$

Here each a_{ij} is piecewise C^μ in D, $0 < \mu < 1$; that is,

$$a_{ij}(x) = a_{ij}^{(m)}(x) \quad \text{for } x \in D_m,\ 1 \leq m \leq L \tag{1.5}$$

with $a_{ij}^{(m)} \in C^\mu(\overline{D_m})$.

As we have already mentioned above, we will discuss in this paper a gradient estimate for solutions to parabolic equations with piecewise smooth coefficients. Our result is a parabolic version for the results of Li-Vogelius [13] and the scalar equations version of Li-Nirenberg [12]. They showed that solutions $u \in H^1(D)$ to the elliptic equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = h + \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i}, \tag{1.6}$$

where $h \in L^\infty(D)$ and each g_i is defined in D such that $g_i|_{D_m}$ $(1 \leq m \leq L)$ have continuous extensions $\in C^\mu(\overline{D_m})$, $0 < \mu < 1$ up to ∂D_m have global $W^{1,\infty}$ and piecewise $C^{1,\alpha'}$ estimates (see (1.7) below). These estimates are independent of the distances between inclusions when a material has inclusions.

We first give the result of Li-Nirenberg [12] for scalar equations.

Theorem 1.2 ([12, Theorem 1.1]). *For any $\varepsilon > 0$, there exists a constant $C_\sharp > 0$ such that for any α' satisfying*

$$0 < \alpha' < \min\big\{\mu, \frac{\alpha}{2(\alpha+1)}\big\},$$

we have

$$\sum_{m=1}^{L} \|u\|_{C^{1,\alpha'}(\overline{D_m}\cap D_\varepsilon)} \leq C_\sharp\Big(\|u\|_{L^2(D)} + \|h\|_{L^\infty(D)} + \sum_{m=1}^{L}\sum_{i=1}^{n}\|g_i\|_{C^{\alpha'}(\overline{D_m})}\Big), \tag{1.7}$$

where we denote

$$D_\varepsilon := \{x \in D : \operatorname{dist}(x, \partial D) > \varepsilon\}$$

and a positive constant $C_\sharp$ depends only on $n, L, \mu, \alpha, \varepsilon, \lambda, \Lambda, \|a_{ij}\|_{C^{\alpha'}(\overline{D_m})}$ and the $C^{1,\alpha'}$ norms of D_m.

Remark 1.3. The constant $C_\sharp > 0$ is independent of the distances between inclusions D_m. Therefore, the estimate (1.7) holds even in the case that some of inclusions touch another inclusions as in Figure 1.

Now, we consider the parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = f - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} \quad \text{in } Q, \tag{1.8}$$

where

$$f \in L^\infty(Q),\ \frac{\partial f}{\partial t} \in L^\kappa(Q),$$

$$f_i \in L^p(Q),\ \frac{\partial f_i}{\partial t} \in L^p(Q) \quad \text{and} \quad f_i = f_i^{(m)} \quad \text{on } D_m \times (0,T],$$

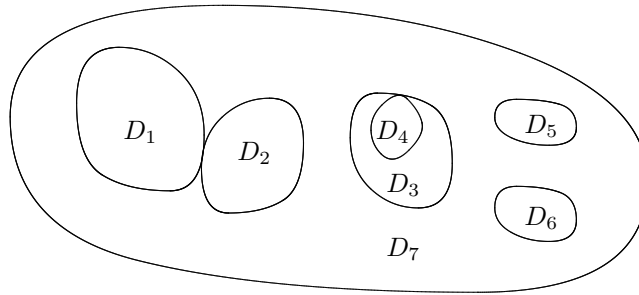FIGURE 1. The case that an inclusion touches another inclusion. $(L = 7)$

with $p > n+2$, $\kappa = p(n+2)/(n+2+p)$, $Q := D \times (0,T]$, $f_i^{(m)} \in L^\infty(0,T; C^\mu(\overline{D_m}))$.
   Now we define a weak solution to the equation (1.8).

Definition 1.4. We call $u \in V_2^{1,0}(Q) := L^2(0,T; H^1(D)) \cap C([0,T]; L^2(D))$ a weak solution to the equation (1.8) when

$$\int_D u(x,t')\, \varphi(x,t')\, dx - \int_0^{t'} \int_D u(x,t)\, \frac{\partial \varphi}{\partial t}(x,t)\, dx\, dt \tag{1.9}$$

$$+ \int_0^{t'} \int_D \sum_{i,j=1}^n a_{ij}(x)\, \frac{\partial u}{\partial x_j}(x,t)\, \frac{\partial \varphi}{\partial x_i}(x,t)\, dx\, dt \tag{1.10}$$

$$= \int_0^{t'} \int_D f(x,t)\, \varphi(x,t)\, dx\, dt + \int_0^{t'} \int_D \sum_{i=1}^n f_i(x,t)\, \frac{\partial \varphi}{\partial x_i}(x,t)\, dx\, dt \tag{1.11}$$

for any $\varphi \in L^2(0,T; H_0^1(D)) \cap H^1(0,T; L^2(D))$ with $\varphi(\cdot,0) = 0$ and $0 < t' \le T$, where $H_0^1(D)$ is the usual L^2-Sobolev space with supports in $\overline{D}$.

   Our main result is as follows.

Theorem 1.5 (Main theorem). *Any weak solutions $u \in V_2^{1,0}(Q)$ to (1.8) have the following up to the inclusion boundary regularity estimate: For any $\varepsilon > 0$, there exists a constant $C'_\sharp > 0$ such that for any α' satisfying*

$$0 < \alpha' < \min\left\{\mu, \frac{\alpha}{2(\alpha+1)}\right\}, \tag{1.12}$$

we have

$$\sum_{m=1}^L \sup_{\varepsilon^2 < t \le T} \|u(\cdot,t)\|_{C^{1,\alpha'}(\overline{D_m} \cap D_\varepsilon)} \le C'_\sharp \left(\|u\|_{L^2(Q)} + F_* + F_{**}\right),$$

where

$$F_* := \|f\|_{L^\kappa(Q)} + \|f\|_{L^{\max\{2,\kappa\}}(Q)} + \|f\|_{L^\infty(Q)} + \|\frac{\partial f}{\partial t}\|_{L^\kappa(Q)},$$

$$F_{**} := \sum_{i=1}^n \left(\|f_i\|_{L^p(Q)} + \|\frac{\partial f_i}{\partial t}\|_{L^2(Q)} + \|\frac{\partial f_i}{\partial t}\|_{L^p(Q)} + \sum_{m=1}^L \sup_{0 < t \le T} \|f_i(\cdot,t)\|_{C^{\alpha'}(\overline{D_m})}\right)$$

and $C'_\sharp$ depends only on $n, L, \mu, \alpha, \varepsilon, \lambda, \Lambda, p, \|a_{ij}\|_{C^{\alpha'}(\overline{D_m})}$ and the $C^{1,\alpha'}$ norms of D_m.

Remark 1.6. (i) Again, the constant $C'_\sharp > 0$ is independent of the distances between inclusions D_m. Then Theorem 1.5 holds even in the case that an inclusion touches another inclusion as Figure 1.

(ii) It is easy to obtain

$$F_* \le C^* \Big(\|f\|_{L^\infty(Q)} + \|\frac{\partial f}{\partial t}\|_{L^\kappa(Q)} \Big),$$

$$F_{**} \le C^* \sum_{i=1}^n \Big(\sum_{m=1}^L \sup_{0<t\le T} \|f_i(\cdot, t)\|_{C^{\alpha'}(\overline{D_m})} + \|\frac{\partial f_i}{\partial t}\|_{L^p(Q)} \Big).$$

However, a constant $C^* > 0$ depends on T and D, unfortunately.

For heat conductive materials with inclusions, the solution u of the initial boundary value problem for (1.8) with heat flux given on $\partial D \times (0, T]$ and zero initial temperature at $t = 0$ in D describes the temperature distribution in D. Injecting the heat flux and measuring the temperature distribution on $\partial D \times (0, T]$ is the measurement of the so called active thermography. This is an non-destructive testing to identify unknown cracks, cavities and inclusions inside a heat conductor from the measurement. As a mathematical study of the active thermography, a method called the dynamical probe method has been given ([7]). It can approximately identify for instance inclusions by one measurement. For identifying the inclusions precisely, it needs infinitely many measurements. Further, it uses the gradient estimate of the fundamental solution of the heat equation with discontinuous conductivities.

The dynamical probe method has been developed only for the case that the inclusions do not touch another inclusions. So, it is interesting to consider the case when some of them touch. For the first task to handle this case, we need to have the gradient estimate of the fundamental solution. Our main result has given an answer to this. Similar situation can be considered for active thermography and non-destructive testing using acoustic waves. For example, [16] and [18] effectively used a result of Li-Vogelius [13] to give a procedure of enclosing the inclusions by the enclosure method (see [6], for example). What is interested about their arguments is that, by adding further arguments, we can even enclose the inclusions in the case that they can touch another inclusions [15]. More precisely for an increasing sets of inclusions, these inclusions can touch at a point of the boundary of the largest inclusion. Therefore, we believe that our gradient estimates will be useful for inverse problems identifying unknown inclusions.

This kind of gradient estimates stated in Theorem 1.5 for solutions of parabolic equations was initiated by Li [10] in his doctor thesis written in Chinese and was completed recently in Li-Li [11]. In [11], they even discussed the interior gradient estimates of solutions of a second order parabolic system of divergence form with inclusions which can touch another inclusions. They also allowed that the coefficients can depend on the time. However, it should be noted that they could not allow the inclusions to depend on the time. Hence, showing the interior gradient estimate for this case is still opened.

We have to emphasize the following two things. The one is that we independently obtained our results. After we finished our paper and posted our paper in the

preprint server arXiv, Li sent the paper [11] to one of us. But we still did not know the paper [10] until very recently by a chance. The results of [11] was also posted in arXiv after us.

The another is that our proof of Theorem 1.5 is totally different from the proofs given in [10] and [11]. We first reduce the problem of the interior gradient estimates for solutions of parabolic equations to that of elliptic parts of the equations using by the idea in [8], and directly apply the result [12] for elliptic equations. By this method, we estimate not only the L^∞-norm of ∇u but also $\sup_{\varepsilon^2 < t \leq T} \|u(\cdot, t)\|_{C^{1,\alpha'}(\overline{D_m} \cap D_\varepsilon)}$ more easily. We also remark that the constant $C'_\sharp$ in Theorem 1.5 is independent of T. In [10], this property is not obvious at least even for the case when the right-hand side f and f_i on (1.8) are identically equal zero.

The rest of this paper is organized as follows. In Section 2, we prove our main theorem, i.e. Theorem 1.5 by applying Lemma 2.1. We prove Lemma 2.1 in Section 3. In Section 4, we consider a pointwise gradient estimate for the fundamental solution of parabolic operators with piecewise smooth coefficients by applying Theorem 1.5.

2. Proof of main result

In this section, we prove our main theorem. We first state some estimates in Lemma 2.1 which we need to prove our main theorem. We prove Lemma 2.1 in Section 3.

Lemma 2.1. *Let (a_{ij}) be a matrix-valued function defined on D. Assume that (a_{ij}) is symmetric, positive definite, and satisfies the condition* (1.4)*. Let Q as before and $\widehat{Q}_\varepsilon := D_\varepsilon \times (\varepsilon^2, T]$. Then for $p > n + 2$, a weak solution $u \in V_2^{1,0}(Q)$ to* (1.8) *satisfies the following estimates:*

$$\sup_{\varepsilon^2 < t \leq T} \|u(\cdot, t)\|_{L^2(D_\varepsilon)} \leq C \left(\|u\|_{L^2(Q)} + F_0 \right), \tag{2.1}$$

$$\|u\|_{L^\infty(\widehat{Q}_\varepsilon)} \leq C \left(\|u\|_{L^2(Q)} + F_0 \right), \tag{2.2}$$

$$\|\frac{\partial u}{\partial t}\|_{L^2(\widehat{Q}_\varepsilon)} \leq C \left(\|u\|_{L^2(Q)} + F_1 \right), \tag{2.3}$$

where we set

$$F_0 := \|f\|_{L^{\frac{p(n+2)}{n+2+p}}(Q)} + \sum_{i=1}^n \|f_i\|_{L^p(Q)}, \tag{2.4}$$

$$F_1 := \|f\|_{L^{\max\{2, \frac{p(n+2)}{n+2+p}\}}(Q)} + \sum_{i=1}^n \left(\|f_i\|_{L^p(Q)} + \|\frac{\partial f_i}{\partial t}\|_{L^2(Q)} \right), \tag{2.5}$$

and $C > 0$ depends only on n, λ, Λ, p and ε.

Now we prove our main theorem by applying Lemma 2.1. This proof is inspired by [8].

Proof of Theorem 1.5. Before going into the proof, we remark that a general constant C which we used below in our estimates depends only on n, λ, Λ, p and ε_j ($j = 1, 2, 3$). To begin with the proof, let $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3$. Then we have

$$\sup_{\varepsilon_2^2 < t \leq T} \|u(\cdot, t)\|_{L^2(D_{\varepsilon_2})} \leq C \left(\|u\|_{L^2(Q)} + F_0 \right) \tag{2.6}$$

and

$$\|\frac{\partial u}{\partial t}\|_{L^2(\widehat{Q}_{\varepsilon_1})} \leq C\left(\|u\|_{L^2(Q)} + F_1\right) \tag{2.7}$$

by (2.1) and (2.3) in Lemma 2.1, where F_0, F_1 are defined by (2.4) and (2.5). On the other hand, $u_t = \partial u/\partial t$ satisfies the equation

$$\frac{\partial u_t}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_t}{\partial x_j}\right) = \frac{\partial f}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f_i}{\partial t}\right)$$

by applying $\partial/\partial t$ to (1.8) (also see Remark 2.2). Hence we have

$$\|u_t\|_{L^\infty(\widehat{Q}_{\varepsilon_2})} \leq C\left(\|u_t\|_{L^2(\widehat{Q}_{\varepsilon_1})} + F_0'\right) \tag{2.8}$$

by Lemma 2.1 (2.2), where we define

$$F_0' := \|\frac{\partial f}{\partial t}\|_{L^{\frac{p(n+2)}{n+2+p}}(Q)} + \sum_{i=1}^{n} \|\frac{\partial f_i}{\partial t}\|_{L^p(Q)}.$$

In particular, $u_t(\cdot, t) \in L^\infty(D_{\varepsilon_2})$ holds for a.e. $t \in (\varepsilon_2^2, T]$. Now we regard the equation (1.8) as the elliptic equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = \frac{\partial u}{\partial t} - f + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \tag{2.9}$$

by fixing $t \in (\varepsilon_2^2, T]$. We remark that $\partial u/\partial t - f \in L^\infty(D_{\varepsilon_2})$. Then, for any α' with the condition (1.12), we have the estimate

$$\sum_{m=1}^{L} \|u(\cdot, t)\|_{C^{1,\alpha'}(\overline{D_m} \cap D_{\varepsilon_3})} \leq C_\sharp\Big(\|u(\cdot, t)\|_{L^2(D_{\varepsilon_2})} + \|\frac{\partial u}{\partial t}(\cdot, t)\|_{L^\infty(D_{\varepsilon_2})}$$

$$+ \|f(\cdot, t)\|_{L^\infty(D_{\varepsilon_2})} + \sum_{m=1}^{L}\sum_{i=1}^{n} \|f_i(\cdot, t)\|_{C^{\alpha'}(\overline{D_m})}\Big) \tag{2.10}$$

by Theorem 1.2, where $C_\sharp > 0$ depends only on $n, L, \mu, \alpha, \varepsilon, \lambda, \Lambda, \|a_{ij}\|_{C^{\alpha'}(\overline{D_m})}$ and the $C^{1,\alpha'}$ norms of D_m. Taking the supremum of the inequality (2.10) over $(\varepsilon_2^2, T]$ with respect to t, and using (2.6), (2.7) and (2.8), we have

$$\sum_{m=1}^{L} \sup_{\varepsilon_2^2 < t \leq T} \|u(\cdot, t)\|_{C^{1,\alpha'}(\overline{D_m} \cap D_{\varepsilon_3})}$$

$$\leq C_\sharp\Big(\sup_{\varepsilon_2^2 < t \leq T} \|u(\cdot, t)\|_{L^2(D_{\varepsilon_2})} + \|\frac{\partial u}{\partial t}\|_{L^\infty(\widehat{Q}_{\varepsilon_2})} + \|f\|_{L^\infty(\widehat{Q}_{\varepsilon_2})}$$

$$+ \sum_{m=1}^{L}\sum_{i=1}^{n} \sup_{\varepsilon_2^2 < t \leq T} \|f_i(\cdot, t)\|_{C^{\alpha'}(\overline{D_m})}\Big)$$

$$\leq C_\sharp C\Big(\|u\|_{L^2(Q)} + F_0 + F_1 + F_0' + \|f\|_{L^\infty(\widehat{Q}_{\varepsilon_2})}$$

$$+ \sum_{m=1}^{L}\sum_{i=1}^{n} \sup_{\varepsilon_2^2 < t \leq T} \|f_i(\cdot, t)\|_{C^{\alpha'}(\overline{D_m})}\Big),$$

which is the estimate we want to obtain. $\qquad\square$

Remark 2.2. Since we assume that u belongs only in $V_2^{1,0}(Q)$ with respect to the regularity of a weak solution, one may think that we cannot apply $\partial/\partial t$ directly. However, it is enough to consider the Steklov mean function and to make h tend to 0, where we define the Steklov mean function v_h of v by

$$v_h(x,t) = \frac{1}{h} \int_t^{t+h} v(x,\tau)\, d\tau.$$

Hereafter we omit the detail with respect to this remark although we often apply this argument. Also see [9, III §2 p. 141] and (62) in [8, p. 152], for example.

3. Some estimates

In this section, we prove Lemma 2.1. The estimates (2.1) and (2.2) are well known, but we give these proofs in Appendix for readers' convenience. To show the estimate (2.3), we prepare some necessary lemmas for its proof.

Throughout this section, $C > 0$ denotes a general constant depending only on n, λ, Λ. Also, we assume that the coefficient (a_{ij}) is a matrix-value function defined on D, symmetric, positive definite, and satisfies the condition (1.4). Moreover, we set $Q_r := B_r(x_0) \times (t_0 - r^2, t_0]$, and assume that $Q_{2\rho} \subset D \times (0, T]$ with $0 < \rho \le 1$.

The following two lemmas are essentially shown in [8]. We give their proofs here for the sake of completeness.

Lemma 3.1 ([8, Lemma 3]). *Let $1 < r < \infty$ and $1/r + 1/r' = 1$. Then a solution u to (1.8) satisfies the estimate*

$$\|\nabla u\|_{L^2(Q_\rho)} \le C\Big[(\rho^{n/2} + \rho^{(n+2)/r'})\operatorname{osc}_{Q_{2\rho}} u + \|f\|_{L^r(Q_{2\rho})} + \sum_{i=1}^n \|f_i\|_{L^2(Q_{2\rho})}\Big]. \quad (3.1)$$

Proof. Let ζ be a smooth cut-off function on $Q_{2\rho}$ satisfying $\zeta \equiv 1$ on Q_ρ, $\zeta \equiv 0$ on $Q_{2\rho} \setminus Q_{3\rho/2}$, $0 \le \zeta \le 1$ on $Q_{2\rho}$, and $|\partial\zeta/\partial t| + |\nabla\zeta|^2 \le C\rho^{-2}$ on $Q_{2\rho}$. Let u_0 be the average value of u in $Q_{2\rho}$:

$$u_0 := \frac{1}{|Q_{2\rho}|} \iint_{Q_{2\rho}} u(x,t)\, dx\, dt,$$

where $|Q_{2\rho}|$ denotes the measure of $Q_{2\rho}$. Testing (1.8) by $(u-u_0)\zeta^2$ and integrating by parts (i.e. taking $\varphi = (u - u_0)\zeta^2$ for (1.11). Also see Remark 2.2), we have

$$\frac{1}{2}\int_{B_{2\rho}(x_0)} \big((u-u_0)^2\zeta^2\big)(x,t_0)\, dx - \iint_{Q_{2\rho}} (u-u_0)^2\zeta\frac{\partial\zeta}{\partial t}\, dx\, dt$$

$$+ \iint_{Q_{2\rho}} \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\zeta^2\, dx\, dt + 2\iint_{Q_{2\rho}} \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}(u-u_0)\zeta\frac{\partial\zeta}{\partial x_i}\, dx\, dt$$

$$= \iint_{Q_{2\rho}} f(u-u_0)\zeta^2\, dx\, dt + \sum_{i=1}^n \iint_{Q_{2\rho}} \Big[f_i\frac{\partial u}{\partial x_i}\zeta^2 + 2f_i(u-u_0)\zeta\frac{\partial\zeta}{\partial x_i}\Big] dx\, dt.$$

Hence we have

$$\frac{1}{2}\int_{B_{2\rho}(x_0)} \big((u-u_0)^2\zeta^2\big)(x,t_0)\, dx + \lambda\iint_{Q_{2\rho}} |\nabla u|^2\zeta^2\, dx\, dt$$

$$\le \frac{1}{2}\int_{B_{2\rho}(x_0)} \big((u-u_0)^2\zeta^2\big)(x,t_0)\, dx + \iint_{Q_{2\rho}} \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\zeta^2\, dx\, dt$$

$$
\begin{aligned}
= &\iint_{Q_{2\rho}} (u-u_0)^2 \zeta \frac{\partial \zeta}{\partial t} dx\, dt - 2 \iint_{Q_{2\rho}} \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} (u-u_0)\zeta \frac{\partial \zeta}{\partial x_i} dx\, dt \\
&+ \iint_{Q_{2\rho}} f(u-u_0)\zeta^2\, dx\, dt \\
&+ \sum_{i=1}^n \iint_{Q_{2\rho}} \Big[f_i \frac{\partial u}{\partial x_i} \zeta^2 + 2 f_i (u-u_0)\zeta \frac{\partial \zeta}{\partial x_i} \Big] dx\, dt \\
\leq &\iint_{Q_{2\rho}} (u-u_0)^2 \zeta \big| \frac{\partial \zeta}{\partial t} \big| dx\, dt + \varepsilon_1 \iint_{Q_{2\rho}} |\nabla u|^2 \zeta^2\, dx\, dt \\
&+ \frac{C}{\varepsilon_1} \iint_{Q_{2\rho}} |u-u_0|^2 |\nabla \zeta|^2\, dx\, dt + \frac{1}{2} \Big(\iint_{Q_{2\rho}} |f\zeta|^r\, dx\, dt \Big)^{2/r} \\
&+ \frac{1}{2} \Big(\iint_{Q_{2\rho}} |(u-u_0)\zeta|^{r'}\, dx\, dt \Big)^{2/r'} + \varepsilon_1 \iint_{Q_{2\rho}} |\nabla u|^2 \zeta^2\, dx\, dt \\
&+ \big(\frac{1}{\varepsilon_1} + 1 \big) \iint_{Q_{2\rho}} \sum_{i=1}^n |f_i|^2 \zeta^2\, dx\, dt + \iint_{Q_{2\rho}} |u-u_0|^2 |\nabla \zeta|^2\, dx\, dt.
\end{aligned}
$$

We now take $\varepsilon_1 > 0$ small enough. Then, we have

$$
\begin{aligned}
&\iint_{Q_\rho} |\nabla u|^2\, dx\, dt \\
\leq &\iint_{Q_{2\rho}} |\nabla u|^2 \zeta^2\, dx\, dt \\
\leq &C \iint_{Q_{2\rho}} (u-u_0)^2 \Big[\zeta \big| \frac{\partial \zeta}{\partial t} \big| + |\nabla \zeta|^2 \Big] dx\, dt + C \Big(\iint_{Q_{2\rho}} |(u-u_0)\zeta|^{r'}\, dx\, dt \Big)^{2/r'} \\
&+ C \Big(\iint_{Q_{2\rho}} |f\zeta|^r\, dx\, dt \Big)^{2/r} + C \iint_{Q_{2\rho}} \sum_{i=1}^n |f_i|^2 \zeta^2\, dx\, dt \\
\leq &C \Big[\big(\rho^n + \rho^{2(n+2)/r'} \big) \big(\operatorname{osc}_{Q_{2\rho}} u \big)^2 + \|f\|_{L^r(Q_\rho)}^2 + \sum_{i=1}^n \|f_i\|_{L^2(Q_\rho)}^2 \Big],
\end{aligned}
$$

because $|u(x,t) - u_0| \leq \operatorname{osc}_{Q_{2\rho}} u$ holds for any $(x,t) \in Q_{2\rho}$. This completes the proof. $\qquad\square$

Lemma 3.2 ([8, Lemma 5]). *A solution u to (1.8) satisfies the estimate*

$$
\begin{aligned}
\|\frac{\partial u}{\partial t}\|_{L^2(Q_\rho)} \leq C \Big[&\rho^{-1} \|\nabla u\|_{L^2(Q_{2\rho})} + \|f\|_{L^2(Q_{2\rho})} \\
&+ \sum_{i=1}^n \Big(\rho^{-1} \|f_i\|_{L^2(Q_{2\rho})} + \|\frac{\partial f_i}{\partial t}\|_{L^2(Q_{2\rho})} \Big) \Big]
\end{aligned} \tag{3.2}
$$

Proof. We first take the same smooth cut-off function ζ as in the proof of Lemma 3.1. Testing (1.8) by $(\partial u/\partial t)\zeta^2$ and integrating by parts (also see Remark 2.2), we have

$$
\begin{aligned}
&\frac{1}{2} \int_{B_{2\rho}(x_0)} \sum_{i,j=1}^n \Big(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \zeta^2 \Big)(x,t_0)\, dx \\
&+ \iint_{Q_{2\rho}} \Big[|\frac{\partial u}{\partial t}|^2 \zeta^2 - \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \zeta \frac{\partial \zeta}{\partial t} + 2 \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} \zeta \frac{\partial \zeta}{\partial x_i} \Big] dx\, dt
\end{aligned}
$$

$$= \iint_{Q_{2\rho}} f \frac{\partial u}{\partial t} \zeta^2 \, dx \, dt + \sum_{i=1}^n \Big[\int_{B_{2\rho}(x_0)} \Big(f_i \frac{\partial u}{\partial x_i} \zeta^2 \Big)(x, t_0) \, dx$$

$$+ \iint_{Q_{2\rho}} \Big(- \frac{\partial f_i}{\partial t} \frac{\partial u}{\partial x_i} \zeta^2 - 2 f_i \frac{\partial u}{\partial x_i} \zeta \frac{\partial \zeta}{\partial t} + 2 f_i \frac{\partial u}{\partial t} \zeta \frac{\partial \zeta}{\partial x_i} \Big) \Big] dx \, dt$$

due to

$$\sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial t \partial x_i} \frac{\partial u}{\partial x_j} \zeta^2 = \frac{1}{2} \frac{\partial}{\partial t} \Big(\sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \zeta^2 \Big) - \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \zeta \frac{\partial \zeta}{\partial t}$$

and

$$f_i \frac{\partial^2 u}{\partial t \partial x_i} \zeta^2 = \frac{\partial}{\partial t} \left(f_i \frac{\partial u}{\partial x_i} \zeta^2 \right) - \frac{\partial u}{\partial x_i} \frac{\partial}{\partial t} (f_i \zeta^2).$$

Hence we have

$$\frac{\lambda}{2} \int_{B_{2\rho}(x_0)} \big(|\nabla u|^2 \zeta^2 \big)(x, t_0) \, dx + \iint_{Q_{2\rho}} |\frac{\partial u}{\partial t}|^2 \zeta^2 \, dx \, dt$$

$$\leq \frac{1}{2} \int_{B_{2\rho}(x_0)} \Big(\sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \zeta^2 \Big)(x, t_0) \, dx + \iint_{Q_{2\rho}} |\frac{\partial u}{\partial t}|^2 \zeta^2 \, dx \, dt$$

$$= \iint_{Q_{2\rho}} \Big[\sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \zeta \frac{\partial \zeta}{\partial t} - 2 \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t} \zeta \frac{\partial \zeta}{\partial x_i} \Big] dx \, dt$$

$$+ \iint_{Q_{2\rho}} f \frac{\partial u}{\partial t} \zeta^2 \, dx \, dt + \sum_{i=1}^n \Big[\int_{B_{2\rho}(x_0)} \Big(f_i \frac{\partial u}{\partial x_i} \zeta^2 \Big)(x, t_0) \, dx$$

$$+ \iint_{Q_{2\rho}} \Big(- \frac{\partial f_i}{\partial t} \frac{\partial u}{\partial x_i} \zeta^2 - 2 f_i \frac{\partial u}{\partial x_i} \zeta \frac{\partial \zeta}{\partial t} + 2 f_i \frac{\partial u}{\partial t} \zeta \frac{\partial \zeta}{\partial x_i} \Big) \Big] dx \, dt$$

$$\leq C \iint_{Q_{2\rho}} |\nabla u|^2 \zeta |\frac{\partial \zeta}{\partial t}| dx \, dt + \varepsilon_2 \iint_{Q_{2\rho}} |\frac{\partial u}{\partial t}|^2 \zeta^2 \, dx \, dt$$

$$+ \frac{C}{\varepsilon_2} \iint_{Q_{2\rho}} |\nabla u|^2 |\nabla \zeta|^2 \, dx \, dt + \varepsilon_2 \iint_{Q_{2\rho}} |\frac{\partial u}{\partial t}|^2 \zeta^2 \, dx \, dt$$

$$+ \frac{C}{\varepsilon_2} \iint_{Q_{2\rho}} |f|^2 \zeta^2 \, dx \, dt + \varepsilon_2 \int_{B_{2\rho}(x_0)} \big(|\nabla u|^2 \zeta^2 \big)(x, t_0) \, dx$$

$$+ \frac{C}{\varepsilon_2} \int_{B_{2\rho}(x_0)} \Big(\sum_{i=1}^n |f_i|^2 \zeta^2 \Big)(x, t_0) \, dx$$

$$+ C \iint_{Q_{2\rho}} |\nabla u|^2 \zeta^2 \, dx \, dt + C \iint_{Q_{2\rho}} \sum_{i=1}^n |\frac{\partial f_i}{\partial t}|^2 \zeta^2 \, dx \, dt$$

$$+ C \iint_{Q_{2\rho}} |\nabla u|^2 \zeta |\frac{\partial \zeta}{\partial t}| dx \, dt + C \iint_{Q_{2\rho}} \sum_{i=1}^n |f_i|^2 \zeta |\frac{\partial \zeta}{\partial t}| dx \, dt$$

$$+ \varepsilon_2 \iint_{Q_{2\rho}} |\frac{\partial u}{\partial t}|^2 \zeta^2 \, dx \, dt + \frac{C}{\varepsilon_2} \iint_{Q_{2\rho}} \sum_{i=1}^n |f_i|^2 |\nabla \zeta|^2 \, dx \, dt.$$

We remark that

$$\int_{B_{2\rho}(x_0)} (f_i \zeta)^2 (x, t_0) \, dx = \int_{B_{2\rho}(x_0)} \int_{t_0 - (2\rho)^2}^{t_0} \frac{\partial}{\partial t} \big((f_i \zeta)^2 \big)(x, t) \, dt \, dx$$

$$\leq C \iint_{Q_{2\rho}} \Big[|f_i|^2 \Big(\zeta^2 + \zeta |\frac{\partial \zeta}{\partial t}| \Big) + |\frac{\partial f_i}{\partial t}|^2 \zeta^2 \Big] dx \, dt.$$

Therefore, by taking $\varepsilon_2 > 0$ small enough, we have

$$\iint_{Q_\rho} |\frac{\partial u}{\partial t}|^2 dx \, dt \leq \int_{B_{2\rho}(x_0)} \big(|\nabla u|^2 \zeta^2 \big)(x, t_0) \, dx + \iint_{Q_{2\rho}} |\frac{\partial u}{\partial t}|^2 \zeta^2 \, dx \, dt$$

$$\leq C \iint_{Q_{2\rho}} |\nabla u|^2 \Big(\zeta^2 + \zeta |\frac{\partial \zeta}{\partial t}| + |\nabla \zeta|^2 \Big) dx \, dt + C \iint_{Q_{2\rho}} |f|^2 \zeta^2 \, dx \, dt$$

$$+ C \iint_{Q_{2\rho}} \sum_{i=1}^n \Big[|f_i|^2 \Big(\zeta^2 + \zeta |\frac{\partial \zeta}{\partial t}| + |\nabla \zeta|^2 \Big) + |\frac{\partial f_i}{\partial t}|^2 \zeta^2 \Big] dx \, dt$$

$$\leq C\rho^{-2} \|\nabla u\|_{L^2(Q_{2\rho})}^2 + C\|f\|_{L^2(Q_{2\rho})}^2 + C\rho^{-2} \sum_{i=1}^n \|f_i\|_{L^2(Q_{2\rho})}^2$$

$$+ C \sum_{i=1}^n \|\frac{\partial f_i}{\partial t}\|_{L^2(Q_{2\rho})}^2.$$

$\square$

We obtain the estimate (2.3) from Lemmas 5.6 (given in Appendix), 3.1 and 3.2.

4. A GRADIENT ESTIMATE OF THE FUNDAMENTAL SOLUTION

In this section, we consider a gradient estimate of the fundametal solution of parabolic operators. We first state some facts. It is known that if coefficient (a_{ij}) is a symmetric and positive definite matrix-value $L^\infty(\mathbb{R}^n)$ function satisfying (1.4), then there exists a fundamental solution $\Gamma(x, t; y, s)$ of the parabolic operator

$$\frac{\partial}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \Big(a_{ij} \frac{\partial}{\partial x_j} \Big) \tag{4.1}$$

with the estimate

$$|\Gamma(x, t; y, s)| \leq \frac{C_*}{(t-s)^{n/2}} \exp\Big(-\frac{c_*|x-y|^2}{t-s} \Big) \chi_{[s,\infty)}(t) \tag{4.2}$$

for all $t, s \in \mathbb{R}$, and a.e. $x, y \in \mathbb{R}^n$, where $C_*, c_* > 0$ depend only on n, λ, Λ (see [1] or [4], for example). In particular, the constants C_* and c_* are independent of the distance between inclusions. If the coefficients (a_{ij}) is not piecewise smooth but Hölder continuous in the whole space $\mathbb{R}^n$, then the pointwise gradient estimate

$$|\nabla_x \Gamma(x, t; y, s)| \leq \frac{C_*}{(t-s)^{(n+1)/2}} \exp\Big(-\frac{c_*|x-y|^2}{t-s} \Big) \chi_{[s,\infty)}(t)$$

holds for $t, s \in \mathbb{R}$, a.e. $x, y \in \mathbb{R}^n$ (see [9, Chapter IV §11–13], for example).

Now, the aim of this section is to show the gradient estimate (4.8) in Theorem 4.3 even if the coefficients are piecewise C^μ in D. We assume that (a_{ij}) defined in D satisfies the conditions (1.4) and (1.5), and extend it to the whole $\mathbb{R}^n$ by defining $(a_{ij}) \equiv \Lambda I$ in $\mathbb{R}^n \setminus D$, where I is the identity matrix. We remark that this extension does not destroy the conditions (1.4) and (1.5). Then there exists a fundamental solution $\Gamma(x, t; y, s)$ of the parabolic operator (4.1) with the estimate (4.2) as we stated above.

To prove our gradient estimate of the fundamental solution, we apply the following corollary from Theorem 1.5.

Corollary 4.1. *Let $0 < \rho \leq 1$. Then a solution u to the parabolic equation*

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = 0 \quad \text{in } B_\rho(x_0) \times (t_0 - \rho^2, t_0] \tag{4.3}$$

has the estimate

$$\|\nabla u\|_{L^\infty(B_{\rho/2}(x_0) \times (t_0 - (\rho/2)^2, t_0]))} \leq \frac{C'_\sharp}{\rho^{n/2+2}}\|u\|_{L^2(B_\rho(x_0) \times (t_0 - \rho^2, t_0])}, \tag{4.4}$$

where $C'_\sharp > 0$ depends only on $n, L, \mu, \alpha, \lambda, \Lambda$, and $\|a_{ij}\|_{C^{\alpha'}(\overline{D_m})}$ and the $C^{1,\alpha'}$ norms of D_m for some α' with (1.12).

Proof. It is sufficient to apply the scaling argument. Let $\rho y = x - x_0$, $\rho^2(s-1) = t - t_0$ and

$$\widetilde{u}(y,s) := u(x,t) = u\big(\rho y + x_0, \rho^2(s-1) + t_0\big),$$
$$\widetilde{a}_{ij}(y) := a_{ij}(x) = a_{ij}(\rho y + x_0), \tag{4.5}$$
$$\widetilde{D}_m := \big\{\frac{1}{\rho}(x - x_0) : x \in D_m\big\}.$$

Then we have

$$\frac{\partial \widetilde{u}}{\partial s} - \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\Big(\widetilde{a}_{ij}\frac{\partial \widetilde{u}}{\partial y_j}\Big) = 0 \quad \text{in } B_1(0) \times (0,1]. \tag{4.6}$$

Therefore, by noting Remark 4.2, we have

$$\|\nabla \widetilde{u}\|_{L^\infty(B_{1/2}(0) \times (3/4,1])} \leq C'_\sharp \|\widetilde{u}\|_{L^2(B_1(0) \times (0,1))}$$

by Theorem 1.5, where $C'_\sharp$ depends only on $n, L, \mu, \alpha, \lambda, \Lambda, \|a_{ij}\|_{C^{\alpha'}(\overline{D_m})}$, and the $C^{1,\alpha'}$ seminorms of D_m. By this estimate and the definition (4.5), we obtain the estimate (4.4). $\square$

Remark 4.2. One may think that $C'_\sharp$ depends also on ρ since $\|\widetilde{a}_{ij}\|_{C^{\alpha'}(\overline{\widetilde{D}_m})}$ and the $C^{1,\alpha'}$ norms of $\widetilde{D}_m$ depend on ρ. However, we can take $C'_\sharp$ independent of ρ by taking the following into consideration.

First we consider

$$\|\widetilde{a}_{ij}\|_{C^{\alpha'}(\overline{\widetilde{D}_m})} = \|\widetilde{a}_{ij}\|_{C^0(\overline{\widetilde{D}_m})} + [\widetilde{a}_{ij}]_{C^{\alpha'}(\overline{\widetilde{D}_m})}$$
$$:= \sup_{y \in \overline{\widetilde{D}_m}} |\widetilde{a}_{ij}(y)| + \sup_{y,\eta \in \overline{\widetilde{D}_m}} \frac{|\widetilde{a}_{ij}(y) - \widetilde{a}_{ij}(\eta)|}{|y - \eta|^{\alpha'}}.$$

It is easy to show

$$\|\widetilde{a}_{ij}\|_{C^0(\overline{\widetilde{D}_m})} = \|a_{ij}\|_{C^0(\overline{D_m})}$$

and

$$[\widetilde{a}_{ij}]_{C^{\alpha'}(\overline{\widetilde{D}_m})} = \rho^{\alpha'}[a_{ij}]_{C^{\alpha'}(\overline{D_m})} \leq [a_{ij}]_{C^{\alpha'}(\overline{D_m})}.$$

Then we have

$$\|\widetilde{a}_{ij}\|_{C^{\alpha'}(\overline{\widetilde{D}_m})} \leq \|a_{ij}\|_{C^{\alpha'}(\overline{D_m})}.$$

Next we consider the $C^{1,\alpha'}$ norms of $\widetilde{D}_m$. We need to recall the proofs of the results of [12] and [13] more carefully. In the case when we consider the L^∞-norm of $\nabla\widetilde{u}$ for a solution $\widetilde{u}$ to the equation (4.6), the influence of the $C^{1,\alpha'}$ norms of

subdomains $\widetilde{D}_m$ appears only in the following constant C in (4.7): We estimate $O\big(|x'|^{1+\alpha}\big)$ in the equation (49) in [13, p. 118], i.e.

$$f_m(x') = f_m(0') + \nabla f_m(0')x' + O\big(|x'|^{1+\alpha}\big) \tag{49}$$

as

$$\big|O\big(|x'|^{1+\alpha}\big)\big| \leq C|x|^{1+\alpha} \tag{4.7}$$

(See also [12, Lemma 4.3]). Here $C^{1,\alpha}$ functions f_m are defined in the cube $(-1,1)^n$, and the graphs of f_m describe ∂D_m. Now we remark that the constant C in (4.7) depends only on the $C^{1,\alpha}$ seminorms of f_m. We consider the variable change $\rho y = x$. Then the graph $x_n = f_m(x')$ is changed to $y_n = \widetilde{f}_m(y')$, where $\widetilde{f}_m(y') := \rho^{-1} f_m(\rho y')$, and we have

$$\begin{aligned}
[\widetilde{f}_m]_{C^{1,\alpha}((-1,1)^n)} &\leq [\widetilde{f}_m]_{C^{1,\alpha}((-1/\rho,1/\rho)^n)} \\
&= \rho^\alpha [f_m]_{C^{1,\alpha}((-1,1)^n)} \leq [f_m]_{C^{1,\alpha}((-1,1)^n)}.
\end{aligned}$$

Hence, even when we consider the variable change $\rho y = x$, we can take the constant C in (4.7) independent of ρ.

Considering the circumstances mentioned above, we can take $C'_\sharp > 0$ independent of ρ.

Now we state the estimate of $\nabla_x \Gamma(x,t;y,s)$.

Theorem 4.3. *We have*

$$|\nabla_x \Gamma(x,t;y,s)| \leq \frac{C}{(t-s)^{(n+1)/2}} \exp\Big(-\frac{c|x-y|^2}{t-s}\Big) \tag{4.8}$$

for a.e. $x, y \in \mathbb{R}^n$ and $t > s$ with $|x-y|^2 + t - s \leq 16$, where $C, c > 0$ depend only on $n, L, \mu, \alpha, \lambda, \Lambda, \|a_{ij}\|_{C^{\alpha'}(\overline{D_m})}$ and the $C^{1,\alpha'}$ seminorms of D_m for some α' with (1.12).

We prove Theorem 4.3 in the same way as the proof of [3, Proposition 3.6]. We first show the following lemmas.

Lemma 4.4. *Let $\rho := (|x_0 - \xi|^2 + t_0 - \tau)^{1/2}/4$. Then*

$$\int_{t_0-\rho^2}^{t_0} \int_{B_\rho(x_0)} |\Gamma(x,t;\xi,\tau)|^2 \, dx \, dt \leq \frac{(C'_*)^2 \rho^n}{(t_0-\tau)^{n-1}} \exp\Big(-\frac{2c'_*|x_0-\xi|^2}{t_0-\tau}\Big)$$

for $t_0 > \tau$, where $C'_*, c'_* > 0$ depend only on n, λ, Λ.

Proof. By (4.2), it is sufficient to obtain the estimate

$$\begin{aligned}
I_0 &:= \int_{t_0-\rho^2}^{t_0} \int_{B_\rho(x_0)} \frac{1}{(t-\tau)^n} \exp\Big(-\frac{2c_*|x-\xi|^2}{t-\tau}\Big) \chi_{[\tau,\infty)}(t) \, dx \, dt \\
&\leq \frac{(C'_*)^2 \rho^n}{(t_0-\tau)^{n-1}} \exp\Big(-\frac{2c'_*|x_0-\xi|^2}{t_0-\tau}\Big).
\end{aligned} \tag{4.9}$$

We consider the following three cases:

(i) $t_0 - \rho^2 \leq \tau < t_0$,
(i) $t_0 - 2\rho^2 \leq \tau \leq t_0 - \rho^2$,
(i) $\tau \leq t_0 - 2\rho^2$.

Now we consider the case (i). Then we have $(\sqrt{15}-1)\rho \le |x-\xi|$ for any $x \in B_\rho(x_0)$, because $|x_0 - \xi| \ge \sqrt{15}\,\rho$. Hence we have

$$I_0 \le \int_\tau^{t_0} \int_{B_\rho(x_0)} \frac{1}{(t-\tau)^n} \exp\Big(-\frac{c_1\rho^2}{t-\tau}\Big) dx\,dt = |B_1(0)|\rho^n \int_0^{t_0-\tau} \varphi_1(s)\,ds,$$

where $\varphi_1(s) := s^{-n}\exp(-c_1\rho^2/s)$ and $c_1 := 2(\sqrt{15}-1)^2 c_*$. If $0 < t_0 - \tau \le c_1\rho^2/n$, then we have

$$\int_0^{t_0-\tau} \varphi_1(s)\,ds \le \int_0^{t_0-\tau} \varphi_1(t_0-\tau)\,ds = (t_0-\tau)^{-n+1}\exp\Big(-\frac{c_1\rho^2}{t_0-\tau}\Big),$$

because $\varphi_1(s) \le \varphi_1(t_0-\tau)$ holds for any $s \in [0,\,t_0-\tau]$. On the other hand, if $c_1\rho^2/n \le t_0 - \tau \le \rho^2$, then we have

$$\int_0^{t_0-\tau} \varphi_1(s)\,ds \le \int_0^{t_0-\tau} \varphi_1\Big(\frac{c_1\rho^2}{n}\Big) ds = \Big(\frac{n}{c_1}\Big)^n (t_0-\tau)\rho^{-2n}\exp(-n)$$

$$\le \Big(\frac{n}{c_1}\Big)^n (t_0-\tau)^{1-n}\exp\Big(-\frac{c_1\rho^2}{t_0-\tau}\Big),$$

where we used the properties that

$$\varphi_1(s) \le \varphi_1\Big(\frac{c_1\rho^2}{n}\Big) \quad \text{for any } 0 < s \le t_0 - \tau;$$

$$n \ge \frac{c_1\rho^2}{t_0-\tau} \quad \text{and} \quad \rho^2 \ge t_0 - \tau.$$

Summing up, we have

$$I_0 \le \max\Big\{1,\ \Big(\frac{n}{c_1}\Big)^n\Big\}|B_1(0)|\rho^n(t_0-\tau)^{1-n}\exp\Big(-\frac{c_1\rho^2}{t_0-\tau}\Big).$$

Let us consider the case (ii). Then we have $(\sqrt{14}-1)\rho \le |x-\xi|$ for all $x \in B_\rho(x_0)$, because $|x_0 - \xi| \ge \sqrt{14}\,\rho$. Hence we have

$$I_0 \le \int_{t_0-\rho^2}^{t_0} \int_{B_\rho(x_0)} \frac{1}{(t-\tau)^n} \exp\Big(-\frac{c_2\rho^2}{t-\tau}\Big) dx\,dt = |B_1(0)|\rho^n \int_{t_0-\rho^2-\tau}^{t_0-\tau} \varphi_2(s)\,ds,$$

where $\varphi_2(s) := s^{-n}\exp(-c_2\rho^2/s)$ and $c_2 := 2(\sqrt{14}-1)^2 c_*$. In a similarly way as the case (i), if $\rho^2 \le t_0 - \tau \le c_2\rho^2/n$, then we have

$$\int_{t_0-\rho^2-\tau}^{t_0-\tau} \varphi_2(s)\,ds \le \int_{t_0-\rho^2-\tau}^{t_0-\tau} \varphi_2(t_0-\tau)\,ds = \rho^2(t_0-\tau)^{-n}\exp\Big(-\frac{c_2\rho^2}{t_0-\tau}\Big)$$

$$\le (t_0-\tau)^{-n+1}\exp\Big(-\frac{c_2\rho^2}{t_0-\tau}\Big),$$

because $\varphi_2(s) \le \varphi(t_0-\tau)$ for any $s \in [t_0-\rho^2-\tau,\,t_0-\tau]$, and we have $\rho^2 \le t_0 - \tau$. On the other hand, if $c_2\rho^2/n \le t_0 - \tau \le 2\rho^2$, then we have

$$\int_{t_0-\rho^2-\tau}^{t_0-\tau} \varphi_2(s)\,ds \le \int_{t_0-\rho^2-\tau}^{t_0-\tau} \varphi_2\Big(\frac{c_2\rho^2}{n}\Big) ds = \Big(\frac{n}{c_2}\Big)^n \rho^{-2n+2}\exp(-n)$$

$$\le 2^{n-1}\Big(\frac{n}{c_2}\Big)^n (t_0-\tau)^{1-n}\exp\Big(-\frac{c_2\rho^2}{t_0-\tau}\Big),$$

where we used the properties that

$$\varphi_2(s) \le \varphi_2\Big(\frac{c_2\rho^2}{n}\Big) \quad \text{for any } t_0 - \rho^2 - \tau \le s \le t_0 - \tau;$$

$$n \geq \frac{c_2 \rho^2}{t_0 - \tau}, \text{ and } \rho^2 \geq \frac{t_0 - \tau}{2}.$$

Summing up, we have

$$I_0 \leq |B_1(0)| \max \left\{ 1, 2^{n-1} \left(\frac{n}{c_2} \right)^n \right\} \rho^n (t_0 - \tau)^{1-n} \exp \left(-\frac{c_2 \rho^2}{t_0 - \tau} \right).$$

Finally we consider the case (iii). We first remark that

$$\int_{t_0 - \rho^2}^{t_0} (t - \tau)^{-n} \, dt \leq \begin{cases} \frac{1}{n-1} (t_0 - \rho^2 - \tau)^{-n+1} & \text{if } n \geq 2, \\ \log 2 & \text{if } n = 1, \end{cases}$$

because $t_0 - \tau \leq 2(t_0 - \rho^2 - \tau)$. In particular, we have

$$\int_{t_0 - \rho^2}^{t_0} (t - \tau)^{-n} \, dt \leq (t_0 - \rho^2 - \tau)^{-n+1} \leq 2^{n-1}(t_0 - \tau)^{-n+1}.$$

Hence we have

$$I_0 \leq |B_1(0)| \rho^n \int_{t_0 - \rho^2}^{t_0} (t - \tau)^{-n} \, dt \leq 2^{n-1} |B_1(0)| \rho^n (t_0 - \tau)^{-n+1}$$

$$\leq 2^{n-1} \exp(8) |B_1(0)| \rho^n (t_0 - \tau)^{-n+1} \exp \left(-\frac{|x_0 - \xi|^2}{t_0 - \tau} \right),$$

because $|x_0 - \xi|^2 / (t_0 - \tau) \leq (4\rho)^2 / 2\rho^2 = 8$. Therefore we have the estimate (4.9) in every case. $\square$

Proof of Theorem 4.3. Let $x_0, \xi \in \mathbb{R}^n$ and $t_0 > \tau$. Let $\rho := (|x_0 - \xi|^2 + t_0 - \tau)^{1/2}/4 \leq 1$. Then, by Corollary 4.1, we have

$$\|\nabla_x \Gamma(\cdot, \cdot; \xi, \tau)\|_{L^\infty(B_{\rho/2}(x_0) \times (t_0 - (\rho/2)^2, t_0))}$$

$$\leq \frac{C'_\sharp}{\rho^{n/2+2}} \|\Gamma(\cdot, \cdot; \xi, \tau)\|_{L^2(B_\rho(x_0) \times (t_0 - \rho^2, t_0))},$$

because we have

$$\frac{\partial \Gamma}{\partial t}(x, t; \xi, \tau) - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial \Gamma}{\partial x_j}(x, t; \xi, \tau) \right) = 0 \quad \text{in } B_\rho(x_0) \times (t_0 - \rho^2, t_0].$$

By this estimate and Lemma 4.4, we have

$$\|\nabla_x \Gamma(\cdot, \cdot; \xi, \tau)\|_{L^\infty(B_{\rho/2}(x_0) \times (t_0 - (\rho/2)^2, t_0])}$$

$$\leq \frac{C'_\sharp}{\rho^{n/2+2}} \|\Gamma(\cdot, \cdot; \xi, \tau)\|_{L^2(B_\rho(x_0) \times (t_0 - \rho^2, t_0])}$$

$$\leq \frac{C'_\sharp C'_*}{\rho^2} \frac{1}{(t_0 - \tau)^{(n-1)/2}} \exp \left(-\frac{c'_* |x_0 - \xi|^2}{t_0 - \tau} \right)$$

$$\leq \frac{16 C'_\sharp C'_*}{(t_0 - \tau)^{(n+1)/2}} \exp \left(-\frac{c'_* |x_0 - \xi|^2}{t_0 - \tau} \right),$$

because we have $\rho^{-2} \leq 16(t_0 - \tau)^{-1}$. Hence the proof is complete. $\square$

5. APPENDIX

In Appendix, we show the estimates (2.1) and (2.2) in Lemma 2.1 for the sake of completeness. To begin with, we give some embedding lemma which is necessary to show the estimates (2.1) and (2.2). First, the following Gagliardo-Nirenberg's inequality is well known (see [5, p. 24, Theorem 9.3], for example).

Lemma 5.1 (Gagliardo-Nirenberg's inequality). *Let r, s be any numbers satisfying $1 \le r, s \le \infty$, and let j, k be any integers satisfying $0 \le j < k$. If u is any function in $W_s^k(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, then*

$$\|D^j u\|_{L^q(\mathbb{R}^n)} \le C_1 \|D^k u\|_{L^s(\mathbb{R}^n)}^\gamma \|u\|_{L^r(\mathbb{R}^n)}^{1-\gamma}, \tag{5.1}$$

where

$$\frac{1}{q} = \frac{j}{n} + \gamma\Big(\frac{1}{s} - \frac{k}{n}\Big) + \frac{1-\gamma}{r} \tag{5.2}$$

for all γ in the interval

$$\frac{j}{k} \le \gamma \le 1,$$

where a positive constant C_1 depends only on n, k, j, r, s, γ, with the following exception: If $k - j - n/s$ is a nonnegative integer, then (5.1) holds only for $j/k \le \gamma < 1$.

Then, as an application of Lemma 5.1, we have the following embedding lemma.

Lemma 5.2 (Embedding lemma). *Let $v \in L^\infty\big(0, T; L^2(D)\big) \cap L^2\big(0, T; H_0^1(D)\big)$. Then $v \in L^{2(n+2)/n}(Q)$ holds. Moreover, we have the estimate*

$$\begin{aligned}
\|v\|_{L^{2(n+2)/n}(Q)} &\le C_1 \|v\|_{L^\infty(0,T;L^2(D))}^{2/(n+2)} \|\nabla v\|_{L^2(Q)}^{n/(n+2)} \\
&\le C_1 \big(\|v\|_{L^\infty(0,T;L^2(D))} + \|\nabla v\|_{L^2(Q)}\big),
\end{aligned} \tag{5.3}$$

where a positive constant C_1 depends only on n, and we denote $Q := D \times (0, T]$.

Proof. We apply Lemma 5.1 with $q = 2(n+2)/n$, $r = 2$, $s = 2$, $k = 1$ and $j = 0$. Then the equation (5.2) yields $\gamma = n/(n+2)$. Hence we have

$$\|v(\cdot, t)\|_{L^{2(n+2)/n}(D)} \le C_1 \|\nabla v(\cdot, t)\|_{L^2(D)}^{n/(n+2)} \|v(\cdot, t)\|_{L^2(D)}^{2/(n+2)}.$$

Therefore, we have

$$\begin{aligned}
\|v\|_{L^{2(n+2)/n}(Q)}^{2(n+2)/n} &= \int_0^T \|v(\cdot, t)\|_{L^{2(n+2)/n}(D)}^{2(n+2)/n} \, dt \\
&\le \int_0^T \Big(C_1 \|\nabla v(\cdot, t)\|_{L^2(D)}^{n/(n+2)} \|v(\cdot, t)\|_{L^2(D)}^{2/(n+2)}\Big)^{2(n+2)/n} \, dt \\
&\le C_1^{2(n+2)/n} \|v\|_{L^\infty(0,T;L^2(D))}^{4/n} \|\nabla v\|_{L^2(Q)}^2.
\end{aligned}$$

By this inequality and Young's inequality, we have the estimate (5.3). $\qquad\square$

Based on Di Giorgi's famous argument, we start to estimate solutions to the parabolic equation (1.8). By testing $(u - k)_+ \zeta^2$ to (1.8) we have the following lemma, where $v_+(x) := \max\{v(x), 0\}$.

Lemma 5.3. *Let $p > 2$. Let $Q_\rho := B_\rho(x_0) \times (t_0 - \rho^2, t_0] \subset Q$ and $\zeta \in C^\infty\big([t_0 - \rho^2, t_0]; C_0^\infty(B_\rho(x_0))\big)$ satisfy $0 \le \zeta \le 1$ and $\zeta(\cdot, t_0 - \rho^2) = 0$. Then a solution u to the parabolic equation (1.8) satisfies*

$$\|(u - k)_+\zeta\|_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}^2 + \big\|\nabla\big((u-k)_+\zeta\big)\big\|_{L^2(Q_\rho)}^2$$
$$\le C_2\Big[\Big(\|\frac{\partial\zeta}{\partial t}\|_{L^\infty(Q_\rho)} + \|\nabla\zeta\|_{L^\infty(Q_\rho)}^2\Big)\|(u-k)_+\|_{L^2(Q_\rho)}^2 \tag{5.4}$$
$$+ F_{0,\rho}^2 \big|Q_\rho \cap \{u(x,t) > k\}\big|^{1-2/p}\Big]$$

for any $k \in \mathbb{R}$, where

$$F_{0,r} := \|f\|_{L^{\frac{p(n+2)}{n+2+p}}(Q_r)} + \sum_{i=1}^n \|f_i\|_{L^p(Q_r)} \quad \text{for } r > 0 \tag{5.5}$$

and $C_2 > 0$ depends only on n, Λ and λ.

Proof. Multiplying (1.8) by $(u-k)_+\zeta^2$ and integrating it over $Q_\rho' := B_\rho(x_0) \times (t_0 - \rho^2, t')$ (also see Remark 2.2), we have

$$\text{(LHS)} = \iint_{Q_\rho'} \Big(\frac{\partial}{\partial t}(u-k)_+\Big)(u-k)_+\zeta^2 \, dx \, dt$$
$$- \sum_{i,j=1}^n \iint_{Q_\rho'} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}(u-k)_+\Big)(u-k)_+\zeta^2 \, dx \, dt$$
$$= \frac{1}{2} \iint_{Q_\rho'} \Big(\frac{\partial}{\partial t}(u-k)_+^2\Big)\zeta^2 \, dx \, dt$$
$$+ \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij}\frac{\partial}{\partial x_j}(u-k)_+\frac{\partial}{\partial x_i}\big((u-k)_+\zeta^2\big) \, dx \, dt$$
$$= \frac{1}{2} \iint_{Q_\rho'} \Big[\frac{\partial}{\partial t}\big((u-k)_+^2\zeta^2\big) - 2(u-k)_+^2\zeta\frac{\partial\zeta}{\partial t}\Big] dx \, dt$$
$$+ \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij}\frac{\partial}{\partial x_j}\big((u-k)_+\zeta\big)\frac{\partial}{\partial x_i}\big((u-k)_+\zeta\big) \, dx \, dt$$
$$- \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij}(u-k)_+^2\frac{\partial\zeta}{\partial x_j}\frac{\partial\zeta}{\partial x_i} \, dx \, dt$$
$$= \frac{1}{2} \int_{B_\rho(x_0)} (u-k)_+^2\zeta^2 \, dx \Big|_{t=t'} - \iint_{Q_\rho'} (u-k)_+^2\zeta\frac{\partial\zeta}{\partial t} \, dx \, dt$$
$$+ \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij}\frac{\partial}{\partial x_j}\big((u-k)_+\zeta\big)\frac{\partial}{\partial x_i}\big((u-k)_+\zeta\big) \, dx \, dt$$
$$- \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij}(u-k)_+^2\frac{\partial\zeta}{\partial x_j}\frac{\partial\zeta}{\partial x_i} \, dx \, dt.$$

Hence we have

$$
\begin{aligned}
&\frac{1}{2} \int_{B_\rho(x_0)} (u-k)_+^2 \zeta^2 \, dx \Big|_{t=t'} \\
&+ \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij} \frac{\partial}{\partial x_j} \big((u-k)_+ \zeta\big) \frac{\partial}{\partial x_i} \big((u-k)_+ \zeta\big) \, dx \, dt \\
&= \iint_{Q_\rho'} (u-k)_+^2 \zeta \frac{\partial \zeta}{\partial t} \, dx \, dt + \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij} (u-k)_+^2 \frac{\partial \zeta}{\partial x_j} \frac{\partial \zeta}{\partial x_i} \, dx \, dt \\
&\quad + \iint_{Q_\rho'} f(u-k)_+ \zeta^2 \, dx \, dt + \sum_{i=1}^n \iint_{Q_\rho'} f_i \frac{\partial}{\partial x_i} \big((u-k)_+ \zeta^2\big) \, dx \, dt.
\end{aligned}
\tag{5.6}
$$

We remark that

$$
\begin{aligned}
&\Big| \iint_{Q_\rho'} f_i \frac{\partial}{\partial x_i} \big((u-k)_+ \zeta^2\big) \, dx \, dt \Big| \\
&= \Big| \iint_{Q_\rho' \cap \{u(x,t)>k\}} f_i \zeta \frac{\partial}{\partial x_i} \big((u-k)_+ \zeta\big) \, dx \, dt \\
&\quad + \iint_{Q_\rho' \cap \{u(x,t)>k\}} f_i (u-k)_+ \zeta \frac{\partial \zeta}{\partial x_i} \, dx \, dt \Big| \\
&\leq \varepsilon_1 \iint_{Q_\rho' \cap \{u(x,t)>k\}} \Big| \frac{\partial}{\partial x_i} \big((u-k)_+ \zeta\big) \Big|^2 \, dx \, dt + \frac{1}{\varepsilon_1} \iint_{Q_\rho' \cap \{u(x,t)>k\}} |f_i \zeta|^2 \, dx \, dt \\
&\quad + \iint_{Q_\rho' \cap \{u(x,t)>k\}} |f_i \zeta|^2 \, dx \, dt + \iint_{Q_\rho' \cap \{u(x,t)>k\}} (u-k)_+^2 \Big| \frac{\partial \zeta}{\partial x_i} \Big|^2 \, dx \, dt.
\end{aligned}
$$

Hence, by (1.4) and (5.6), we have

$$
\begin{aligned}
&\frac{1}{2} \int_{B_\rho(x_0)} (u-k)_+^2 \zeta^2 \, dx \Big|_{t=t'} + \lambda \iint_{Q_\rho'} \big| \nabla\big((u-k)_+ \zeta\big) \big|^2 \, dx \, dt \\
&\leq \frac{1}{2} \int_{B_\rho(x_0)} (u-k)_+^2 \zeta^2 \, dx \Big|_{t=t'} \\
&\quad + \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij} \frac{\partial}{\partial x_j} \big((u-k)_+ \zeta\big) \frac{\partial}{\partial x_i} \big((u-k)_+ \zeta\big) \, dx \, dt \\
&= \iint_{Q_\rho'} (u-k)_+^2 \zeta \frac{\partial \zeta}{\partial t} \, dx \, dt + \sum_{i,j=1}^n \iint_{Q_\rho'} a_{ij} (u-k)_+^2 \frac{\partial \zeta}{\partial x_j} \frac{\partial \zeta}{\partial x_i} \, dx \, dt \\
&\quad + \iint_{Q_\rho'} f(u-k)_+ \zeta^2 \, dx \, dt + \sum_{i=1}^n \iint_{Q_\rho'} f_i \frac{\partial}{\partial x_i} \big((u-k)_+ \zeta^2\big) \, dx \, dt \\
&\leq \Big\| \frac{\partial \zeta}{\partial t} \Big\|_{L^\infty(Q_\rho)} \iint_{Q_\rho'} (u-k)_+^2 \, dx \, dt + \Lambda \|\nabla \zeta\|_{L^\infty(Q_\rho)}^2 \iint_{Q_\rho'} (u-k)_+^2 \, dx \, dt \\
&\quad + \iint_{Q_\rho'} f(u-k)_+ \zeta^2 \, dx \, dt + \varepsilon_1 \iint_{Q_\rho'} \big| \nabla\big((u-k)_+ \zeta\big) \big|^2 \, dx \, dt \\
&\quad + \Big(\frac{1}{\varepsilon_1} + 1 \Big) \iint_{Q_\rho' \cap \{u(x,t)>k\}} \sum_{i=1}^n |f_i|^2 \, dx \, dt
\end{aligned}
$$

$$+ n\|\nabla\zeta\|^2_{L^\infty(Q_\rho)} \iint_{Q'_\rho} (u-k)^2_+ \, dx\, dt;$$

that is,

$$\frac{1}{2}\int_{B_\rho(x_0)} (u-k)^2_+\zeta^2 \, dx\Big|_{t=t'} + (\lambda - \varepsilon_1)\iint_{Q'_\rho} \big|\nabla\big((u-k)_+\zeta\big)\big|^2 \, dx\, dt$$

$$\leq (\Lambda + n)\Big(\|\frac{\partial\zeta}{\partial t}\|_{L^\infty(Q_\rho)} + \|\nabla\zeta\|^2_{L^\infty(Q_\rho)}\Big) \iint_{Q'_\rho} (u-k)^2_+ \, dx\, dt$$

$$+ \big(\frac{1}{\varepsilon_1}+1\big)\iint_{Q'_\rho\cap\{u(x,t)>k\}} \sum_{i=1}^n |f_i|^2 \, dx\, dt + \iint_{Q'_\rho} f(u-k)_+\zeta^2 \, dx\, dt.$$

Taking the supremum of the inequality over $(t_0-\rho^2, t_0]$ with respect to t', we have

$$\max\Big\{\frac{1}{2}\|(u-k)_+\zeta\|^2_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}, \ (\lambda-\varepsilon_1)\big\|\nabla\big((u-k)_+\zeta\big)\big\|^2_{L^2(Q_\rho)}\Big\}$$

$$\leq (\Lambda + n)\big(\|\frac{\partial\zeta}{\partial t}\|_{L^\infty(Q_\rho)} + \|\nabla\zeta\|^2_{L^\infty(Q_\rho)}\big) \iint_{Q_\rho} (u-k)^2_+ \, dx\, dt \qquad (5.7)$$

$$+ \big(\frac{1}{\varepsilon_1}+1\big)\iint_{Q_\rho\cap\{u(x,t)>k\}} \sum_{i=1}^n |f_i|^2 \, dx\, dt + \iint_{Q_\rho} f(u-k)_+\zeta^2 \, dx\, dt.$$

Now we estimate the last two terms in the right-hand side of (5.7). First we obtain

$$\iint_{Q_\rho\cap\{u(x,t)>k\}} |f_i|^2 \, dx\, dt \leq \big|Q_\rho\cap\{u(x,t)>k\}\big|^{1-2/p}\|f_i\|^2_{L^p(Q_\rho)} \qquad (5.8)$$

by Hölder's inequality. Now we estimate $\iint_{Q_\rho} f(u-k)_+\zeta^2 \, dx\, dt$. We first recall

$$\|(u-k)_+\zeta\|_{L^{2(n+2)/n}(Q_\rho)}$$

$$\leq C_1 \Big(\|(u-k)_+\zeta\|_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))} + \big\|\nabla\big((u-k)_+\zeta\big)\big\|_{L^2(Q_\rho)}\Big)$$

by Lemma 5.2, where $C_1 > 0$ depends only on n. Then, by this inequality, Hölder's inequality and Young's inequality, we have

$$\iint_{Q_\rho} f(u-k)_+\zeta^2 \, dx\, dt$$

$$\leq \|f\zeta\|_{L^{2(n+2)/(n+4)}(Q_\rho\cap\{u(x,t)>k\})}\|(u-k)_+\zeta\|_{L^{2(n+2)/n}(Q_\rho)}$$

$$\leq \varepsilon_2\|(u-k)_+\zeta\|^2_{L^{2(n+2)/n}(Q_\rho)} + \frac{1}{\varepsilon_2}\|f\zeta\|^2_{L^{2(n+2)/(n+4)}(Q_\rho\cap\{u(x,t)>k\})} \qquad (5.9)$$

$$\leq 2\varepsilon_2 C_1^2 \max\Big\{\|(u-k)_+\zeta\|^2_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))}, \ \big\|\nabla\big((u-k)_+\zeta\big)\big\|^2_{L^2(Q_\rho)}\Big\}$$

$$+ \frac{1}{\varepsilon_2}\|f\|^2_{L^{\frac{p(n+2)}{n+2+p}}(Q_\rho)} \big|Q_\rho\cap\{u(x,t)>k\}\big|^{1-2/p}$$

because $2(n+2)/(n+4) < p(n+2)/(n+2+p)$. By (5.7), (5.8) and (5.9), we obtain the estimate (5.4). □

By the same argument, we obtain the following lemma for the function $v_-(x) := \max\{-v(x), 0\}$.

Lemma 5.4. *Under the same assumption as in Lemma 5.3, a solution u to the parabolic equation* (1.8) *satisfies*

$$\|(u-k)_-\zeta\|^2_{L^\infty(t_0-\rho^2,t_0;L^2(B_\rho(x_0)))} + \left\|\nabla\big((u-k)_-\zeta\big)\right\|^2_{L^2(Q_\rho)}$$

$$\leq C_2\Big[\Big(\|\frac{\partial\zeta}{\partial t}\|_{L^\infty(Q_\rho)} + \|\nabla\zeta\|^2_{L^\infty(Q_\rho)}\Big)\|(u-k)_-\|^2_{L^2(Q_\rho)} \tag{5.10}$$

$$+ F^2_{0,\rho}|Q_\rho \cap \{u(x,t) < k\}|^{1-2/p}\Big]$$

for any $k \in \mathbb{R}$, where we define $F_{0,\rho}$ as (5.5), *and $C_2 > 0$ depends only on n, Λ and λ.*

The estimate (2.1) easily follows from Lemmas 5.3 and 5.3′. Our next task is to prove the estimate (2.2). We start by giving a technical lemma which will be used later.

Lemma 5.5. *Let $\widetilde{C} > 0$, $b > 1$ and $\varepsilon > 0$. If a sequence $\{y_m\}_{m=0}^\infty$ satisfies*

$$y_0 \leq \theta_0 := \widetilde{C}^{-1/\varepsilon}b^{-1/\varepsilon^2} \quad \text{and} \quad 0 \leq y_{m+1} \leq \widetilde{C}b^m y_m^{1+\varepsilon}, \tag{5.11}$$

then

$$\lim_{m\to\infty} y_m = 0$$

holds.

Proof. We show

$$y_m \leq \frac{\theta_0}{r^m}, \quad m = 0, 1, 2, \dots \tag{5.12}$$

by inductive method, where we will determine $r > 1$ later. By assumption, (5.12) with $m = 0$ holds. Hence we now assume (5.12) holds, and show (5.12) for $m + 1$. By the assumption (5.11) and the induction hypothesis, we have

$$y_{m+1} \leq \widetilde{C}b^m y_m^{1+\varepsilon} \leq \widetilde{C}b^m\Big(\frac{\theta_0}{r^m}\Big)^{1+\varepsilon} = \frac{\theta_0}{r^{m+1}}\widetilde{C}b^m\frac{\theta_0^\varepsilon}{r^{m\varepsilon-1}}.$$

Now we take $r = b^{1/\varepsilon}$. Then we have

$$y_{m+1} \leq \frac{\theta_0}{r^{m+1}}\widetilde{C}b^m\frac{\theta_0^\varepsilon}{r^{m\varepsilon-1}} = \frac{\theta_0}{r^{m+1}}\widetilde{C}r\theta_0^\varepsilon = \frac{\theta_0}{r^{m+1}},$$

which is (5.12) for $m + 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we are now ready to show the estimate (2.2). The estimate easily follows if we have the following lemma.

Lemma 5.6. *Let $p > n + 2$. Then a solution u to* (1.8) *satisfies the estimate*

$$\|u\|_{L^\infty(Q_\rho)} \leq C_\rho\left(\|u\|_{L^2(Q_{2\rho})} + F_{0,2\rho}\right),$$

where we define $F_{0,2\rho}$ by (5.5), *and $C_\rho > 0$ depends only on n, λ, Λ, p and ρ.*

Proof. First of all a letter C denotes a general constant depending only on n, Λ, λ and p. Now, let $\rho_m := (1 + 2^{-m})\rho$ and $k_m = k(2 - 2^{-m})$ for $m = 0, 1, 2, \dots$, where we will determine $k > 0$ later. For $m = 0, 1, 2, \dots$, we take cut-off functions $\zeta_m \in C^\infty(Q_{\rho_m})$ which satisfy

$$0 \leq \zeta_m \leq 1 \quad \text{in } Q_{\rho_m},$$

$$\zeta_m = \begin{cases} 1 & \text{in } Q_{\rho_{m+1}}, \\ 0 & \text{in } Q_{\rho_m} \setminus Q_{(\rho_m+\rho_{m+1})/2}, \end{cases}$$

$$\|\frac{\partial \zeta_m}{\partial t}\|_{L^\infty(Q_{\rho_m})} + \|\nabla \zeta_m\|_{L^\infty(Q_{\rho_m})}^2 \leq \frac{C}{(\rho_m - \rho_{m+1})^2}.$$

We remark that $\zeta_m = 0$ near $B_{\rho_m}(x_0) \times \{t_0 - \rho_m^2\} \cup \partial B_{\rho_m}(x_0) \times (t_0 - \rho_m^2, t_0)$ in particular. By Lemmas 5.2 and 5.3, we have

$$
\begin{aligned}
&\|(u - k_{m+1})_+ \zeta_m\|_{L^{2(n+2)/n}(Q_{\rho_m})}^2 \\
&\leq C\Big(\|(u - k_{m+1})_+ \zeta_m\|_{L^\infty(t_0-\rho_m^2,t_0;L^2(B_{\rho_m}(x_0)))}^2 \\
&\qquad + \big\|\nabla\big((u - k_{m+1})_+ \zeta_m\big)\big\|_{L^2(Q_{\rho_m})}^2\Big) \\
&\leq C\Big[\Big(\|\frac{\partial \zeta_m}{\partial t}\|_{L^\infty(Q_{\rho_m})} + \|\nabla \zeta_m\|_{L^\infty(Q_{\rho_m})}^2\Big)\|(u - k_{m+1})_+\|_{L^2(Q_{\rho_m})}^2 \\
&\quad + F_{0,\rho_m}^2 \big|Q_{\rho_m} \cap \{u(x,t) > k_{m+1}\}\big|^{1-2/p}\Big] \\
&\leq C\Big[\frac{2^{2m}}{\rho^2}\|(u - k_{m+1})_+\|_{L^2(Q_{\rho_m})}^2 + F_{0,2\rho}^2 \big|A_m(k_{m+1})\big|^{1-2/p}\Big],
\end{aligned}
\tag{5.13}
$$

where $A_m(l) := Q_{\rho_m} \cap \{u(x,t) > l\}$ for $l \in \mathbb{R}$. Now we take $k > 0$ as

$$k \geq \rho^{1-(n+2)/p} F_{0,2\rho}. \tag{5.14}$$

Then we have

$$
\begin{aligned}
&\|(u - k_{m+1})_+ \zeta_m\|_{L^{2(n+2)/n}(Q_{\rho_m})}^2 \\
&\leq C\Big[\frac{2^{2m}}{\rho^2}\|(u - k_{m+1})_+\|_{L^2(Q_{\rho_m})}^2 + \frac{k^2}{\rho^{2(1-(n+2)/p)}}\big|A_m(k_{m+1})\big|^{1-2/p}\Big]
\end{aligned}
$$

by the estimate (5.13). By defining $\varphi_m := \|(u - k_m)_+\|_{L^2(Q_{\rho_m})}^2$, we have

$$
\begin{aligned}
&\varphi_{m+1} \\
&= \|(u - k_{m+1})_+ \zeta_m\|_{L^2(Q_{\rho_{m+1}})}^2 \leq \|(u - k_{m+1})_+ \zeta_m\|_{L^2(Q_{\rho_m})}^2 \\
&\leq |A_m(k_{m+1})|^{2/(n+2)}\|(u - k_{m+1})_+ \zeta_m\|_{L^{2(n+2)/n}(Q_{\rho_m})}^2 \\
&\leq C|A_m(k_{m+1})|^{2/(n+2)} \\
&\quad \times \Big[\frac{2^{2m}}{\rho^2}\|(u - k_{m+1})_+\|_{L^2(Q_{\rho_m})}^2 + \frac{k^2}{\rho^{2(1-(n+2)/p)}}\big|A_m(k_{m+1})\big|^{1-2/p}\Big] \\
&\leq C|A_m(k_{m+1})|^{2/(n+2)}\Big[\frac{2^{2m}}{\rho^2}\varphi_m + \frac{k^2}{\rho^{2(1-(n+2)/p)}}\big|A_m(k_{m+1})\big|^{1-2/p}\Big],
\end{aligned}
\tag{5.15}
$$

where we used Hölder's inequality and the estimate

$$\|(u - k_{m+1})_+\|_{L^2(Q_{\rho_m})}^2 \leq \|(u - k_m)_+\|_{L^2(Q_{\rho_m})}^2 = \varphi_m.$$

On the other hand, we have

$$
\begin{aligned}
\varphi_m = \|(u - k_m)_+\|_{L^2(Q_{\rho_m})}^2 &\geq \iint_{A_m(k_{m+1})} (u - k_m)_+^2 \, dx \, dt \\
&\geq \iint_{A_m(k_{m+1})} (k_{m+1} - k_m)_+^2 \, dx \, dt = \frac{k^2}{2^{2m+2}}|A_m(k_{m+1})|;
\end{aligned}
$$

that is,

$$|A_m(k_{m+1})| \leq \frac{2^{2m+2}}{k^2}\varphi_m. \tag{5.16}$$

By (5.15) and (5.16), we have

$$\varphi_{m+1} \leq C2^{2m\left(1+\frac{2}{n+2}\right)}$$
$$\times \left[\rho^{-2}k^{-\frac{4}{n+2}}\varphi_m^{1+\frac{2}{n+2}} + \rho^{-2\left(1-\frac{n+2}{p}\right)}k^{-\left(\frac{4}{n+2}-\frac{4}{p}\right)}\varphi_m^{1+\frac{2}{n+2}-\frac{2}{p}}\right]. \tag{5.17}$$

We now take k as

$$k \geq \left(\frac{1}{|Q_{2\rho}|}\iint_{Q_{2\rho}} u^2\, dx\, dt\right)^{1/2}. \tag{5.18}$$

Then we have

$$\varphi_m \leq \iint_{Q_{\rho_m}} u^2\, dx\, dt \leq \iint_{Q_{2\rho}} u^2\, dx\, dt \leq |Q_{2\rho}|k^2;$$

that is,

$$\varphi_m^{2/p} \leq |Q_{2\rho}|^{2/p}k^{4/p}.$$

By this inequality and (5.17), we have

$$\varphi_{m+1} \leq C2^{2m\left(1+\frac{2}{n+2}\right)}\varphi_m^{1+\frac{2}{n+2}-\frac{2}{p}}\left[\rho^{-2}k^{-\frac{4}{n+2}}\varphi_m^{2/p} + \rho^{-2\left(1-\frac{n+2}{p}\right)}k^{-\left(\frac{4}{n+2}-\frac{4}{p}\right)}\right]$$

$$\leq C2^{2m\left(1+\frac{2}{n+2}\right)}\varphi_m^{1+\frac{2}{n+2}-\frac{2}{p}}$$

$$\times \left[\rho^{-2}k^{-\frac{4}{n+2}}|Q_{2\rho}|^{2/p}k^{4/p} + \rho^{-2\left(1-\frac{n+2}{p}\right)}k^{-\left(\frac{4}{n+2}-\frac{4}{p}\right)}\right]$$

$$\leq C2^{2m\left(1+\frac{2}{n+2}\right)}\rho^{-2\left(1-\frac{n+2}{p}\right)}k^{-\frac{4}{n+2}\left(1-\frac{n+2}{p}\right)}\varphi_m^{1+\frac{2}{n+2}-\frac{2}{p}}. \tag{5.19}$$

Now we denote $y_m := k^{-2}|Q_{2\rho}|^{-1}\varphi_m$. Then by (5.19), we have

$$y_{m+1} \leq C2^{2m\left(1+\frac{2}{n+2}\right)}y_m^{1+\left(\frac{2}{n+2}-\frac{2}{p}\right)}, \tag{5.20}$$

which is the second condition of (5.11) with

$$\widetilde{C} = C, \quad b = 2^{2(1+\frac{2}{n+2})}, \quad \varepsilon = \frac{2}{n+2} - \frac{2}{p}. \tag{5.21}$$

Then $\lim_{m\to\infty} y_m = 0$ if

$$y_0 \leq C^{-1/\varepsilon}b^{-1/\varepsilon^2} =: \theta_0 \tag{5.22}$$

by Lemma 5.5, where b and ε are defined by (5.21) and C is the constant C in (5.20). We remark that the condition (5.22) is equivalent to

$$\|(u-k)_+\|_{L^2(Q_{2\rho})}^2 \leq \theta_0 k^2 |Q_{2\rho}|. \tag{5.23}$$

Now we take k as

$$k^2 \geq \frac{1}{\theta_0 |Q_{2\rho}|}\|u\|_{L^2(Q_{2\rho})}^2. \tag{5.24}$$

Then the condition (5.23); i.e., the condition (5.22) is satisfied.

Summing up, if we take k such that the conditions (5.14), (5.18) and (5.24) are satisfied, then we have $\lim_{m\to\infty} y_m = 0$. On the other hand, since

$$y_m = \frac{1}{k^2|Q_{2\rho}|}\varphi_m = \frac{1}{k^2|Q_{2\rho}|}\|(u-k_m)_+\|_{L^2(Q_{\rho_m})}^2$$

$$\to \frac{1}{k^2|Q_{2\rho}|}\|(u-2k)_+\|_{L^2(Q_\rho)}^2 \quad \text{as } m \to \infty.$$

Then we have $\|(u-2k)_+\|_{L^2(Q_\rho)}^2 = 0$; that is,

$$u \leq 2k \quad \text{a.e. in } Q_\rho. \tag{5.25}$$

Now we take k as

$$k = \frac{1}{\sqrt{\theta_0 |Q_{2\rho}|}} \|u\|_{L^2(Q_{2\rho})} + \rho^{1-(n+2)/p} F_{0,2\rho},$$

which satisfies the conditions (5.14), (5.18) and (5.24). Hence we have (5.25), which is

$$\sup_{Q_\rho} u \le C_\rho \left(\|u\|_{L^2(Q_{2\rho})} + F_{0,2\rho} \right).$$

Replacing Lemma 5.3 by Lemma 5.3′ and doing the same argument, we can obtain

$$-u \le C_\rho \left(\|u\|_{L^2(Q_{2\rho})} + F_{0,2\rho} \right) \quad \text{in } Q_\rho$$

and thus the proof has been completed. $\square$

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