Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 96, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

FIRST INTEGRAL METHOD FOR AN OSCILLATOR SYSTEM

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Abstract. In this article, we consider the nonlinear Duffing-van der Pol-type oscillator system by means of the first integral method. This system has physical relevance as a model in certain flow-induced structural vibration problems, which includes the van der Pol oscillator and the damped Duffing oscillator etc as particular cases. Firstly, we apply the Division Theorem for two variables in the complex domain, which is based on the ring theory of commutative algebra, to explore a quasi-polynomial first integral to an equivalent autonomous system. Then, through solving an algebraic system we derive the first integral of the Duffing-van der Pol-type oscillator system under certain parametric condition.

1. Introduction

As we know, the van der Pol oscillator was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol whilst he was working at Philips. van der Pol found stable oscillations, which he called relaxation-oscillations and are now known as limit cycles, in electrical circuits employing vacuum tubes [25]. When these circuits were driven near the limit cycle they become entrained, i.e. the driving signal pulls the current along with it. Van der Pol and his colleague, van der Mark, reported in [26] that at certain drive frequencies an irregular noise was heard. This irregular noise was always heard near the natural entrainment frequencies. This was one of the first discovered instances of deterministic chaos. The van der Pol equation has wide applications, especially in the physical and biological sciences. The typical example lies in biology, where Fitzhugh [12] and Nagumo etc [24] extended the equation in a planar field as a model for action potentials of neurons.

In this article, we consider a general Duffing-van der Pol-type oscillator system of the form

$$\ddot{u} + (\delta + \beta u^n)\dot{u} - \mu u + \alpha u^{n+1} = 0, \tag{1.1}$$

where an over-dot represents differentiation with respect to the independent variable ξ, and all coefficients δ, β, μ, α are real constants with $\delta \cdot \beta \cdot \mu \cdot \alpha \neq 0$. It can also be regarded as a general combination of the van der Pol oscillator and damped

2000 *Mathematics Subject Classification.* 34A25, 34L30.
Key words and phrases. First integral; Duffing oscillator; van der Pol oscillator; autonomous system; division theorem.

Submitted December 10, 2012. Published April 16, 2013.

Duffing equation, since the choices $\delta \neq 0$, $\beta \neq 0$, $\mu \neq 0$, $\alpha = 0$ and $n = 0$ leads equation (1.1) to the van der Pol oscillator [25, 26]

$$\ddot{u} + (\delta + \beta u^2)\dot{u} - \mu u = 0. \tag{1.2}$$

The choices $\delta \neq 0$, $\mu \neq 0$, $\alpha \neq 0$, $\beta = 0$ and $n = 2$ leads equation (1.1) to the damped Duffing equation [6, 15]

$$\ddot{u} + \delta\dot{u} - \mu u + \alpha u^3 = 0, \tag{1.3}$$

which describes the motion of a damped oscillator with a more complicated potential than in simple harmonic motion without the driving force. When we choose $\delta \neq 0$, $\mu \neq 0$, $\alpha \neq 0$, $\beta = 0$ and $n = 1$, equation (1.1) becomes the damped Helmholtz oscillator [1, 16]

$$\ddot{u} + \delta\dot{u} - \mu u + \alpha u^2 = 0. \tag{1.4}$$

Furthermore, if we take $\delta \neq 0$, $\mu \neq 0$, $\alpha \neq 0$, $\beta \neq 0$ and $n = 2$, equation (1.1) becomes the standard form of the Duffing-van der Pol oscillator, whose autonomous version (force free) takes the form [15, 17]

$$\ddot{u} + (\delta + \beta u^2)\dot{u} - \mu u + \alpha u^3 = 0. \tag{1.5}$$

As we see, the nonlinear differential equation (1.1) is widely used in physics, engineering, electronics, biology, neurology and many other disciplines [18, 20, 21, 27, 28]. Therefore, it is one of the most intensively studied systems in nonlinear dynamics [15, 21]. It is well known that there are a great number of theoretical works dealing with equations (1.2)-(1.5) [4, 8, 9, 10, 11, 13, 17, 19, 23] and references therein, and applications of these four equations and related systems can be seen in quite a few scientific areas [1, 3, 14]. For example, In 1980, Holmes and Rand applied equation (1.5) to the study of the local and global bifurcation of the Duffing-van der Pol-system [17]. In 1998, Maccari investigated the main resonance of the Duffing-van der Pol-system using asymptotic perturbation method and obtained the sufficient conditions for period-doubling motion of the system [23].

In this present paper, we apply the first integral method to study the nonlinear Duffing-van der Pol-type oscillator system (1.1) and obtain its first integrals under certain parametric condition. The main idea of this method is to use the Division Theorem for two variables in the complex domain based on the ring theory of commutative algebra. The paper is organized as follows. In the next section, we construct the first integral for equation (1.1) by means of the first integral method. In Section 3, we present a brief discussion.

2. Main Results

In this section, we consider the Duffing-van der Pol-type oscillator system (1.1) for arbitrary degree n by applying the first integral method [7].

Consider the oscillator equation (1.1) in the form

$$\ddot{u} = -(\delta + \beta u^n)\dot{u} + \mu u - \alpha u^{n+1}, \tag{2.1}$$

where u' denotes differentiation with respect to ξ. Let $x = u$ and $y = u_\xi$, then equation (2.1) is equivalent to an autonomous system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(\delta + \beta x^n)y + \mu x - \alpha x^{n+1}. \end{aligned} \tag{2.2}$$

By the qualitative theory of ordinary differential equations [5], if we can find two first integrals to system (2.2) under the same conditions, the general solution to equation (2.1) can be expressed explicitly. However, generally, it is difficult for us to realize this, even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way to tell us what these first integrals are.

As we know, Hilbert-Nullstellensatz Theorem (Zero-locus-Theorem) is a theorem which makes a fundamental relationship between the geometric and algebraic geometry. That is, it relates algebraic sets to ideals in polynomial rings over algebraically closed fields [2].

Theorem 2.1 (Hilbert-Nullstellensatz Theorem)**.** *Let k be a field and L be an algebraic closure of k.*

(i) *Every ideal γ of $k[X_1, X_2, \dots, X_n]$ not containing 1 admits at least one zero in L^n.*
(ii) *Let $x = (x_1, x_2, \dots x_n)$, $y = (y_1, y_2, \dots, y_n)$ be two elements of L^n; for the set of polynomials of $k[X_1, X_2, \dots, X_n]$ zero at x to be identical with the set of polynomials of $k[X_1, X_2, \dots, X_n]$ zero at y, it is necessary and sufficient that there exists a k-automorphism s of L such that $y_i = s(x_i)$ for $1 \le i \le n$.*
(iii) *For an ideal α of $k[X_1, X_2, \dots, X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $k[X_1, X_2, \dots, X_n]$ zero at x.*
(iv) *For a polynomial Q of $k[X_1, X_2, \dots, X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $k[X_1, X_2, \dots, X_n]$, it is necessary and sufficient that there exist an integral $m > 0$ such that $Q^m \in \gamma$.*

Following immediately from the Hilbert-Nullstellensatz Theorem, we obtain the Division Theorem for two variables in the complex domain $\mathbb{C}$.

Theorem 2.2 (Division Theorem)**.** *Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $\mathbb{C}[w, z]$, and $P(w, z)$ is irreducible in $\mathbb{C}[w, z]$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $\mathbb{C}[w, z]$ such that*

$$Q(w, z) = P(w, z) \cdot G(w, z). \tag{2.3}$$

Now, we apply the above Division Theorem to seek the first integral of polynomial form to system (2.2).

Suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to system (2.2), and $p(x, y) = \sum_{i=0}^{i=m} a_i(x) y^i$ is an irreducible polynomial in $\mathbb{C}[x, y]$ such that

$$p[x(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(x) y^i = 0, \tag{2.4}$$

where $a_i(x)(i = 0, 1, \cdot, m)$ are polynomials of x and are all relatively prime in $\mathbb{C}[x, y]$, and $a_m(x) \neq 0$. Equation (2.4) is also called the first integral of polynomial form to equation (2.2). We start our study by considering $m = 3$ in equation (2.4).

From system (2.2), we have

$$
\begin{aligned}
&\frac{dp(x,y)}{d\xi}\\
&= \Big(\frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial \xi}\Big)\\
&= [a_0'(x)+a_1'(x)y+a_2'(x)y^2+a_3'(x)y^3]y\\
&\quad + [a_1(x)+2a_2(x)y+3a_3(x)y^2][-(\delta+\beta x^n)y+\mu x-\alpha x^{n+1}]\\
&= a_1(x)(\mu x-\alpha x^{n+1})+[a_0'(x)+2a_2(x)(\mu x-\alpha x^{n+1})-a_1(x)(\delta+\beta x^n)]y\\
&\quad + [a_1'(x)+3a_3(x)(\mu x-\alpha x^{n+1})-2a_2(x)(\delta+\beta x^n)]y^2\\
&\quad + [a_2'(x)-3a_3(x)(\delta+\beta x^n)]y^3+a_3'(x)y^4.
\end{aligned}
\tag{2.5}
$$

Note that $\frac{dp}{d\xi}$ is a polynomial in x and y, and $p[x(\xi),y(\xi)]=0$ always implies $\frac{dp}{d\xi}=0$. By the Division Theorem, there exists a polynomial $H(x,y)=\rho(x)+\eta(x)y$ in $\mathbb{C}[x,y]$ such that

$$
\begin{aligned}
\frac{dp(x,y)}{d\xi} &= H(x,y)\cdot p(x,y)\\
&= [\rho(x)+\eta(x)y][a_0(x)+a_1(x)y+a_2(x)y^2+a_3(x)y^3]\\
&= \rho(x)a_0(x)+(\rho(x)a_1(x)+\eta(x)a_0(x))y+(\rho(x)a_2(x)+\eta(x)a_1(x))y^2\\
&\quad + (\rho(x)a_3(x)+\eta(x)a_2(x))y^3+\eta(x)a_3(x)y^4.
\end{aligned}
\tag{2.6}
$$

By (2.5) and (2.6), on equating the coefficients of y^i ($i=0,1,2,3,4$) on both sides of above equation (2.6), we have

$$
\begin{gathered}
a_1(x)(\mu x-\alpha x^{n+1})=\rho(x)a_0(x),\\
a_0'(x)+2a_2(x)(\mu x-\alpha x^{n+1})-a_1(x)(\delta+\beta x^n)=\rho(x)a_1(x)+\eta(x)a_0(x),\\
a_1'(x)+3a_3(x)(\mu x-\alpha x^{n+1})-2a_2(x)(\delta+\beta x^n)=\rho(x)a_2(x)+\eta(x)a_1(x),\\
a_2'(x)-3a_3(x)(\delta+\beta x^n)=\rho(x)a_3(x)+\eta(x)a_2(x),\\
a_3'(x)=\eta(x)a_3(x).
\end{gathered}
$$

That is,

$$
\begin{gathered}
a'(x)=A(x)\cdot\mathbf{a(x)},\\
\big[0,0,\alpha x^{n+1}-\mu x,\rho(x)\big]\cdot\mathbf{a(x)}=0,
\end{gathered}
\tag{2.7}
$$

where

$$
\mathbf{a(x)}=\begin{bmatrix} a_3(x)\\ a_2(x)\\ a_1(x)\\ a_0(x)\end{bmatrix},
\tag{2.8}
$$

and

$$
A(x)=\begin{bmatrix}
\eta(x) & 0 & 0 & 0\\
\rho(x)+3(\delta+\beta x^n) & \eta(x) & 0 & 0\\
3(\alpha x^{n+1}-\mu x) & \rho(x)+2(\delta+\beta x^n) & \eta(x) & 0\\
0 & 2(\alpha x^{n+1}-\mu x) & \rho(x)+\delta+\beta x^n & \eta(x)
\end{bmatrix}.
\tag{2.9}
$$

Since $a_i(x)$ are polynomials, from the first equation of system (2.7), we deduce that $a_3(x)$ is a nonzero constant and $\eta(x)=0$. For simplicity, we take $a_3(x)=1$. Solving system (2.7) for $\mathbf{a}(\mathbf{x})$ yields

$$\mathbf{a}(\mathbf{x})=\begin{bmatrix}1\\ \int[\rho(x)+3\delta+3\beta x^n]dx\\ \int[3(\alpha x^{n+1}-\mu x)+\rho(x)a_2(x)+2(\delta+\beta x^n)a_2(x)]dx\\ \int[2(\alpha x^{n+1}-\mu x)a_2(x)+\rho(x)a_1(x)+\delta a_1(x)+\beta x^n a_1(x)]dx\end{bmatrix}. \tag{2.10}$$

We need to determine the degree of polynomials $\rho(x)$ and $a_2(x)$ based on system (2.7) and formula (2.10).

Step 1. If $\deg\rho(x)=k>n>0$, we have

$$\deg a_2(x)=k+1,\quad \deg a_1(x)=2k+2,\quad \deg a_0(x)=3k+3.$$

From (2.7), we derive that

$$\begin{gathered}\deg\,[a_1(x)(\mu x-\alpha x^{n+1})]=2k+2+n+1,\\ \deg\,[\rho(x)\cdot a_0(x)]=k+3k+3.\end{gathered}$$

This gives $2k+n+3=4k+3$; i.e., $k=n/2$. Apparently, it yields a contradiction.

Step 2. If $\deg\rho(x)=k$ and $n>k>0$, we have

$$\deg a_2(x)=n+1,\quad \deg a_1(x)=2n+2,\quad \deg a_0(x)=3n+3.$$

From (2.7) again, we deduce that

$$\begin{gathered}\deg\,[a_1(x)(\mu x-\alpha x^{n+1})]=2n+2+n+1,\\ \deg\,[\rho(x)\cdot a_0(x)]=k+3n+3,\end{gathered}$$

which gives $k=0$. This yields another contradiction.

Step 3. If $\deg\rho(x)=k=0$, which implies $\deg a_2(x)=n+1$, we assume that

$$a_2(x)=B_2x^{n+1}+B_1x+B_0. \tag{2.11}$$

Through formula (2.10), we find

$$a_2(x)=\int[\rho(x)+3\delta+3\beta x^n]dx=(\rho(x)+3\delta)x+\frac{3}{n+1}\beta x^{n+1}+B_0. \tag{2.12}$$

From (2.11) and (2.12), we obtain

$$\begin{gathered}B_1=3\delta+\rho(x),\\ B_2=\frac{3}{n+1}\beta.\end{gathered} \tag{2.13}$$

Furthermore, substituting (2.11)–(2.13) into (2.10), we can also deduce that

$$\begin{aligned}a_1(x)&=\int\big[3(\alpha x^{n+1}-\mu x)+\rho(x)a_2(x)+2(\delta+\beta x^n)a_2(x)\big]\,dx\\ &=\int\big[3(\alpha x^{n+1}-\mu x)+\rho(x)(B_2x^{n+1}+B_1x+B_0)\big]\\ &\quad+\big[2(\delta+\beta x^n)(B_2x^{n+1}+B_1x+B_0)\big]\,dx\\ &=\frac{2\beta B_2}{2n+2}x^{2n+2}+\frac{2\delta B_2+2\beta B_1+(B_1-3\delta)B_2+3\alpha}{n+2}x^{n+2}\\ &\quad+\frac{2\beta B_0}{n+1}x^{n+1}+\frac{B_1^2-\delta B_1-3\mu}{2}x^2+(2\delta B_0+B_0(B_1-3\delta))x+D,\end{aligned} \tag{2.14}$$

where D is an arbitrary integration constant.

Substituting (2.11) and (2.14) into (2.10), where $\rho(x) = B_1 - 3\delta$, we have

$$\begin{aligned}
a_0(x) &= \int \big[2(\alpha x^{n+1} - \mu x)a_2(x) + \rho(x)a_1(x) + \delta a_1(x) + \beta x^n a_1(x)\big]\,dx \\
&= \frac{\beta^2 B_2}{(3n+3)(n+1)}x^{3n+3} + \Big(\frac{2\alpha B_2}{2n+3} + \frac{(B_1-2\delta)\beta B_2}{(n+1)(2n+3)} \\
&\quad + \frac{2\delta\beta B_2 + 2\beta^2 B_1 + (B_1-3\delta)B_2\beta + 3\alpha\beta}{(n+2)(2n+3)}\Big)x^{2n+3} \\
&\quad + \frac{2\beta^2 B_0}{(n+1)(2n+2)}x^{2n+2} + \Big(\frac{2\alpha B_1}{n+3} + \frac{\beta(B_1^2 - \delta B_1 - 3\mu)}{2(n+3)} - \frac{2\mu B_2}{n+3} \\
&\quad + \frac{(B_1-2\delta)(2\delta B_2 + 2\beta B_1 + (B_1-3\delta)B_2 + 3\alpha)}{(n+2)(n+3)}\Big)x^{n+3} \\
&\quad + \Big(\frac{2\alpha B_0}{n+2} + \frac{2\beta B_0(B_1-2\delta)}{(n+1)(n+2)} + \frac{\beta(B_0B_1 - \delta B_0)}{n+2}\Big)x^{n+2} + \frac{\beta D}{n+1}x^{n+1} \\
&\quad + \Big(\frac{-2\mu B_1}{3} + \frac{(B_1^2 - \delta B_1 - 3\mu)(B_1-2\delta)}{6}\Big)x^3 \\
&\quad + \Big(-\mu B_0 + \frac{(B_1-2\delta)(B_0B_1 - \delta B_0)}{2}\Big)x^2 + (B_1-2\delta)Dx + D'.
\end{aligned} \tag{2.15}$$

Note that

$$\begin{aligned}
&a_1(x)(\mu x - \alpha x^{n+1}) \\
&= \Big(\frac{2\beta B_2}{2n+2}x^{2n+2} + \frac{2\delta B_2 + 2\beta B_1 + (B_1-3\delta)B_2 + 3\alpha}{n+2}x^{n+2} + \frac{2\beta B_0}{n+1}x^{n+1} \\
&\quad + \frac{B_1^2 - \delta B_1 - 3\mu}{2}x^2 + (2\delta B_0 + B_0(B_1-3\delta))x + D\Big)(\mu x - \alpha x^{n+1}) \\
&= -\frac{2\alpha\beta B_2}{2n+2}x^{3n+3} + \Big(-\frac{2\alpha\delta B_2 + 2\alpha\beta B_1 + \alpha(B_1-3\delta)B_2 + 3\alpha^2}{n+2} + \frac{2\mu\beta B_2}{2n+2}\Big)x^{2n+3} \\
&\quad - \frac{2\alpha\beta B_0}{n+1}x^{2n+2} + \Big(\frac{2\mu\delta B_2 + 2\mu\beta B_1 + \mu(B_1-3\delta)B_2 + 3\mu\alpha}{n+2} \\
&\quad - \frac{\alpha(B_1^2 - \delta B_1 - 3\mu}{2}\Big)x^{n+3} + \Big(\frac{2\mu\beta B_0}{n+1} - \big(2\alpha\delta B_0 + \alpha B_0(B_1-3\delta)\big)\Big)x^{n+2} \\
&\quad - D\alpha x^{n+1} + \frac{(B_1^2 - \delta B_1 - 3\mu)\mu}{2}x^3 + \big(2\mu\delta B_0 + \mu B_0(B_1-3\delta)\big)x^2 + D\mu x,
\end{aligned}$$

and

$$\begin{aligned}
&\rho(x)a_0(x) \\
&= \frac{(B_1-3\delta)\beta^2 B_2}{(3n+3)(n+1)}x^{3n+3} + (B_1-3\delta)\Big(\frac{2\alpha B_2}{2n+3} + \frac{(B_1-2\delta)\beta B_2}{(n+1)(2n+3)} \\
&\quad + \frac{2\delta\beta B_2 + 2\beta^2 B_1 + (B_1-3\delta)B_2\beta + 3\alpha\beta}{(n+2)(2n+3)}\Big)x^{2n+3} \\
&\quad + \frac{(B_1-3\delta)2\beta^2 B_0}{(n+1)(2n+2)}x^{2n+2} + (B_1-3\delta)\Big(\frac{2\alpha B_1}{n+3} + \frac{\beta(B_1^2 - \delta B_1 - 3\mu)}{2(n+3)} \\
&\quad - \frac{2\mu B_2}{n+3} + \frac{(B_1-2\delta)(2\delta B_2 + 2\beta B_1 + (B_1-3\delta)B_2 + 3\alpha)}{(n+2)(n+3)}\Big)x^{n+3}
\end{aligned}$$

$$
\begin{aligned}
&+ (B_1-3\delta)\Big(\frac{2\alpha B_0}{n+2}+\frac{2\beta B_0(B_1-2\delta)}{(n+1)(n+2)}+\frac{\beta(B_0B_1-\delta B_0)}{n+2}\Big)x^{n+2}\\
&+ \frac{(B_1-3\delta)\beta D}{n+1}x^{n+1}+(B_1-3\delta)\Big(\frac{-2\mu B_1}{3}+\frac{(B_1^2-\delta B_1-3\mu)(B_1-2\delta)}{6}\Big)x^3\\
&+ (B_1-3\delta)\Big(-\mu B_0+\frac{(B_1-2\delta)(B_0B_1-\delta B_0)}{2}\Big)x^2\\
&+ (B_1-3\delta)(B_1-2\delta)Dx+(B_1-3\delta)D'.
\end{aligned}
$$

Using system (2.7) again, we have

$$
a_1(x)(\mu x-\alpha x^{n+1})=\rho(x)a_0(x).
$$

Taking integration constants $D=0$ leads to

$$
-\frac{2\alpha\beta B_2}{2n+2}=\frac{(B_1-3\delta)\beta^2B_2}{(3n+3)(n+1)}, \tag{I}
$$

$$
\left.
\begin{aligned}
&-\frac{2\alpha\delta B_2+2\alpha\beta B_1+\alpha(B_1-3\delta)B_2+3\alpha^2}{n+2}+\frac{2\mu\beta B_2}{2n+2}\\
&=(B_1-3\delta)\Big(\frac{2\alpha B_2}{2n+3}+\frac{(B_1-2\delta)\beta B_2}{(n+1)(2n+3)}\\
&\quad+\frac{2\delta\beta B_2+2\beta^2B_1+(B_1-3\delta)B_2\beta+3\alpha\beta}{(n+2)(2n+3)}\Big),
\end{aligned}
\right\} \tag{II}
$$

$$
-\frac{2\alpha\beta B_0}{n+1}=\frac{(B_1-3\delta)2\beta^2B_0}{(n+1)(2n+2)}, \tag{III}
$$

$$
\begin{aligned}
&\frac{2\mu\delta B_2+2\mu\beta B_1+\mu(B_1-3\delta)B_2+3\mu\alpha}{n+2}-\frac{\alpha(B_1^2-\delta B_1-3\mu)}{2}\\
&=(B_1-3\delta)\frac{(B_1-2\delta)(2\delta B_2+2\beta B_1+(B_1-3\delta)B_2+3\alpha)}{(n+2)(n+3)}\\
&\quad+(B_1-3\delta)\Big(\frac{2\alpha B_1}{n+3}+\frac{\beta(B_1^2-\delta B_1-3\mu)}{2(n+3)}-\frac{2\mu B_2}{n+3}\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2\mu\beta B_0}{n+1}-(2\alpha\delta B_0+\alpha B_0(B_1-3\delta)\\
&=(B_1-3\delta)\Big(\frac{2\alpha B_0}{n+2}+\frac{2\beta B_0(B_1-2\delta)}{(n+1)(n+2)}+\frac{\beta(B_0B_1-\delta B_0)}{n+2}\Big),
\end{aligned}
$$

$$
\frac{(B_1^2-\delta B_1-3\mu)\mu}{2}=(B_1-3\delta)\Big(\frac{-2\mu B_1}{3}+\frac{(B_1^2-\delta B_1-3\mu)(B_1-2\delta)}{6}\Big),
$$

$$
2\mu\delta B_0+\mu B_0(B_1-3\delta)=(B_1-3\delta)\Big(-\mu B_0+\frac{(B_1-2\delta)(B_0B_1-\delta B_0)}{2}\Big).
$$

From (I), we deduce

$$
B_2[(B_1-3\delta)\beta+3\alpha(n+1)]=0;
$$

that is,

$$
B_2=0 \quad\text{or}\quad B_1=\frac{-3\alpha(n+1)}{\beta}+3\delta.
$$

Since $B_2 = \frac{3}{n+1}\beta \neq 0$, it gives

$$B_1 = \frac{-3\alpha(n+1)}{\beta} + 3\delta.$$

From (III), we have

$$B_0[(B_1 - 3\delta)\beta + 2\alpha(n+1)] = 0;$$

that is,

$$B_0 = 0 \quad \text{or} \quad B_1 = \frac{-2\alpha(n+1)}{\beta} + 3\delta.$$

Note the fact that $B_1 = \frac{-3\alpha(n+1)}{\beta} + 3\delta$ from (I) and $\alpha \neq 0$, so the only possibility is

$$B_1 \neq \frac{-2\alpha(n+1)}{\beta} + 3\delta, \quad B_0 = 0.$$

That is,

$$B_0 = 0, \quad B_1 = \frac{-3\alpha(n+1)}{\beta} + 3\delta, \quad B_2 = \frac{3\beta}{n+1}. \tag{2.16}$$

Substituting B_1, B_2 back into equation (II) leads to

$$\begin{aligned} &-\frac{2\alpha\delta B_2 + 2\alpha\beta B_1 + \alpha(B_1 - 3\delta)B_2 + 3\alpha^2}{n+2} + \frac{2\mu\beta B_2}{2n+2} \\ &= (B_1 - 3\delta)\Big(\frac{2\alpha B_2}{2n+3} + \frac{(B_1 - 2\delta)\beta B_2}{(n+1)(2n+3)} \\ &\quad + \frac{2\delta\beta B_2 + 2\beta^2 B_1 + (B_1 - 3\delta)B_2\beta + 3\alpha\beta}{(n+2)(2n+3)}\Big). \end{aligned}$$

That is,

$$-\frac{\frac{6\alpha\beta\delta}{n+1} - 6\alpha^2(n+1) + 6\alpha\beta\delta - 6\alpha^2}{n+2} + \frac{3\mu\beta^2}{(n+1)^2} = 9\alpha^2 - \frac{9\alpha\beta\delta}{n+1}.$$

A straightforward calculation gives

$$\alpha^2(n+1)^2 - \alpha\beta\delta(n+1) - \mu\beta^2 = 0. \tag{2.17}$$

Substituting (2.16) in (2.11), we obtain

$$a_2(x) = \frac{3\beta}{n+1}x^{n+1} + \Big(-\frac{3\alpha(n+1)}{\beta} + 3\delta\Big)x.$$

Substituting (2.16) in (2.14) gives

$$\begin{aligned} a_1(x) &= \frac{3\beta^2}{(n+1)^2}x^{2n+2} + \frac{6\beta\delta - 6\alpha(n+1)}{(n+1)}x^{n+2} \\ &\quad + \frac{9\alpha^2(n+1)^2 + 6\beta^2\delta^2 - 15\alpha\beta\delta(n+1) - 3\mu\beta^2}{2\beta^2}x^2. \end{aligned}$$

Substituting (2.16) in (2.15) gives

$$\begin{aligned} &a_0(x) \\ &= \frac{\beta^3}{(n+1)^3}x^{3n+3} \\ &\quad + \frac{6\alpha\beta(n+1)(-n) - 9\alpha\beta(n+1) + 3\beta^2\delta + 6\beta^2\delta(n+1)}{(n+1)^2(2n+3)}x^{2n+3} \end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{24\alpha^2(n+1)^2-36\alpha\beta\delta(n+1)+9\alpha^2(n+1)^3}{2(n+1)(n+3)\beta}\Big)x^{n+3}\\
&+\Big(\frac{6\beta^2\delta^2(n+1)-15\alpha\beta\delta(n+1)^2-3\beta^2\mu(n+1)-12\mu\beta^2+12\beta^2\delta^2}{2(n+1)(n+3)\beta}\Big)x^{n+3}\\
&+\frac{21\alpha\beta^2\mu(n+1)-15\beta^3\mu\delta-27\alpha^3(n+1)^3+54\alpha^2\beta\delta(n+1)^2}{6\beta^3}x^3\\
&+\frac{-33\alpha\beta^2\delta^2(n+1)+6\beta^3\delta^3}{6\beta^3}x^3.
\end{aligned}
$$

Note that the coefficient of x^{2n+3} in the formula $a_0(x)$ can be simplified as

$$
\frac{6\alpha\beta(n+1)(-n)-9\alpha\beta(n+1)+3\beta^2\delta+6\beta^2\delta(n+1)}{(n+1)^2(2n+3)}=\frac{-3\alpha\beta(n+1)+3\beta^2\delta}{(n+1)^2}. \tag{2.18}
$$

Let C_{n+3} be the coefficient of x^{n+3} in the formula $a_0(x)$. By condition (2.17), we have

$$
C_{n+3}=\frac{3\alpha\beta\mu(n+1)+6\alpha\beta\mu+6\alpha\beta\delta^2-6\alpha^2\delta(n+1)-3\beta^2\delta\mu}{\alpha(n+1)(n+3)}. \tag{2.19}
$$

Similarly, let C_3 be the coefficient of x^3 in the formula $a_0(x)$. Using the condition (2.17) again gives

$$
C_3=\frac{-\alpha\mu(n+1)+2\beta\mu\delta-\alpha\delta^2(n+1)+\beta\delta^3}{\beta}. \tag{2.20}
$$

Hence, from (2.18)–(2.20), the formula $a_0(x)$ can be rewritten as

$$
\begin{aligned}
a_0(x)=\;&\frac{\beta^3}{(n+1)^3}x^{3n+3}+\frac{-3\alpha\beta(n+1)+3\beta^2\delta}{(n+1)^2}x^{2n+3}\\
&+\frac{3\alpha\beta\mu(n+1)+6\alpha\beta\mu+6\alpha\beta\delta^2-6\alpha^2\delta(n+1)-3\beta^2\delta\mu}{\alpha(n+1)(n+3)}x^{n+3}\\
&+\frac{-\alpha\mu(n+1)+2\beta\mu\delta-\alpha\delta^2(n+1)+\beta\delta^3}{\beta}x^3.
\end{aligned}
$$

Substituting $a_0(x)$, $a_1(x)$, $a_2(x)$ and $a_3(x)=1$ into (2.4), we obtain the first integral of equation (1.1) as follows

$$
\begin{aligned}
&y^3+\Big(\frac{3\beta}{n+1}x^{n+1}+\Big(-\frac{3\alpha(n+1)}{\beta}+3\delta\Big)x\Big)y^2\\
&+\Big(\frac{3\beta^2}{(n+1)^2}x^{2n+2}+\frac{6\beta\delta-6\alpha(n+1)}{(n+1)}x^{n+2}\\
&+\frac{9\alpha^2(n+1)^2+6\beta^2\delta^2-15\alpha\beta\delta(n+1)-3\mu\beta^2}{2\beta^2}x^2\Big)y\\
&+\frac{\beta^3}{(n+1)^3}x^{3n+3}+\frac{6\alpha\beta(n+1)(-n)-9\alpha\beta(n+1)+3\beta^2\delta+6\beta^2\delta(n+1)}{(n+1)^2(2n+3)}x^{2n+3}\\
&+\frac{3\alpha\beta\mu(n+1)+6\alpha\beta\mu+6\alpha\beta\delta^2-6\alpha^2\delta(n+1)-3\beta^2\delta\mu}{\alpha(n+1)(n+3)}x^{n+3}\\
&+\frac{-\alpha\mu(n+1)+2\beta\mu\delta-\alpha\delta^2(n+1)+\beta\delta^3}{\beta}x^3=0.
\end{aligned} \tag{2.21}
$$

It is notable that under the parametric condition (2.17), equation (2.21) can be simplified as

$$\Big[y + \frac{\beta}{n+1}x^{n+1} + \Big(-\frac{\alpha(n+1)}{\beta} + \delta\Big)x\Big]^3 = 0.$$

Consequently, under the parametric condition (2.17), we have

$$y = \Big(\frac{\alpha(n+1)}{\beta} - \delta\Big)x - \frac{\beta}{n+1}x^{n+1}.$$

By (2.6), namely,

$$\frac{dp(x,y)}{d\xi} = [-\frac{3\alpha(n+1)}{\beta}] \cdot p(x,y), \tag{2.22}$$

solving equation (2.22) yields the first integral

$$\Big[u' + \frac{\beta}{n+1}u^{n+1} + \Big(-\frac{\alpha(n+1)}{\beta} + \delta\Big)u\Big] \cdot \exp\Big[\frac{\alpha(n+1)}{\beta}\Big]\xi = I, \tag{2.23}$$

where I is an arbitrary constant.

3. Discussion

Since the first integral method was introduced in [7] for studying traveling wave phenomena of Burgers-KdV equation, it has become a very useful method to deal with exact solutions of a rather diverse classes of nonlinear differential equations. One of advantages of this method is that it is not only efficient to find the first integral, but also has the merit of being widely applicable. As described in [9, 10, 11, 22], one can apply this technique to some nonlinear models arising in physical and biological phenomena, such as the nonlinear Schrödinger equation, the Klein-Gordon equation, and the higher order KdV-like equation.

In this paper, we applied the first integral method to investigate a nonlinear Duffing-van der Pol-type oscillator system for its first integral. Under certain parametric conditions, we obtained a first integral of the Duffing-van der Pol-type oscillator system. Note that formula (2.23) is in agreement with the main result presented in [8] by using the Lie symmetry reduction method, but our parametric condition (2.17) appear weaker.

In [4], when $n = 2$, the first integral of the Duffing-van der Pol oscillator system with the parameter $\alpha = 1$ is considered, namely

$$\ddot{y} + (\delta + \beta y^2)\dot{y} - \mu y + y^3 = 0. \tag{3.1}$$

Following the parametric condition (2.17); that is,

$$\delta = \frac{3}{\beta} - \frac{\mu\beta}{3},$$

and using formula (2.23), we can obtain immediately that the Duffing-van der Pol equation (3.1) has one first integral of the form

$$[\dot{y} + \big(\delta - \frac{3}{\beta}\big)y + \frac{\beta}{3}y^3]e^{3x/\beta} = I_1, \tag{3.2}$$

where I_1 is an arbitrary constant. Note that the first integral (3.2) is identical to formula (17) described in [9].

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