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INFINITELY MANY SOLUTIONS FOR SUBLINEAR KIRCHHOFF EQUATIONS IN $\mathbb{R}^N$ WITH SIGN-CHANGING POTENTIALS

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Abstract. In this article we study the Kirchhoff equation

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u+V(x)u=K(x)|u|^{q-1}u,\quad \text{in } \mathbb{R}^N,$$

where $N\geq 3$, $0<q<1$, $a,b>0$ are constants and $K(x),V(x)$ both change sign in $\mathbb{R}^N$. Under appropriate assumptions on $V(x)$ and $K(x)$, the existence of infinitely many solutions is proved by using the symmetric Mountain Pass Theorem.

1. Introduction

In this article, we study the existence of infinitely many solutions of the nonlinear Kirchhoff equation

$$\begin{gathered}-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u+V(x)u=K(x)|u|^{q-1}u,\quad \text{in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N)\cap L^{q+1}(\mathbb{R}^N),\end{gathered}\tag{1.1}$$

where $N\geq 3$, $0<q<1$, a,b are positive constants and $K(x),V(x)\in L^\infty(\mathbb{R}^N)$ both change sign in $\mathbb{R}^N$ and satisfies some conditions specified below.

If in (1.1), we set $V(x)=0$ and replace $\mathbb{R}^N$ by a bounded domain $\Omega\subset\mathbb{R}^N$, then (1.1) reduces to the problem

$$\begin{gathered}-\Big(a+b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u=f(x,u)\quad \text{in } \Omega,\\ u=0\quad \text{on } \partial\Omega,\end{gathered}\tag{1.2}$$

where $f(x,u)\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$. Problem (1.2) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt}-\Big(a+b\int_{\Omega}|\nabla_x u|^2dx\Big)\Delta_x u=f(x,u)$$

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which was proposed by Kirchhoff in 1883 [14] as a generalization of the well-known d'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial^2 t}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial^2 x}=f(x,u)$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Some early classical investigations of Kirchhoff equations can be seen in Bernstein [5] and Pohoẑaev [18]. Problem (1.1) called the attention of several researchers mainly after the work of Lions [15], where a functional analysis approach was proposed. Recently, problems like type (1.1) have been investigated by several authors; see [1, 2, 7, 9, 10, 11, 12, 16, 17, 19].

Ma and Rivera [16] obtained a positive solutions of (1.2) by using variational methods. He and Zou [11] showed that problem (1.2) admits infinitely many solutions by using the local minimum methods and the Fountain Theorem. Very recently, some authors have studied the Kirchhoff equation on the whole space $\mathbb{R}^N$. Since (1.1) is set on $\mathbb{R}^N$, it is well known that the Sobolev embedding $H^1(\mathbb{R}^N)\hookrightarrow L^m(\mathbb{R}^N)$ $(2\le m\le 2^*=\frac{2N}{N-2})$ is not compact and then it is usually difficult to prove that a minimizing sequence or a Palais-Smale sequence is strongly convergent if we seek solution of (1.1) by variational methods. If $V(x)$ is radial, as in [17, 19] we can avoid the lack of compactness of Sobolev embedding by looking for solution of (1.1) in the subspace of radial functions of $D^{1,2}(\mathbb{R}^N)$ since the embedding is compact. Nie and Wu [17] studied a Schrödinger-Kirchhoff-type equation with radial potential and they proved the existence of infinitely many solutions by using a symmetric Mountain Pass Theorem. On the other hand, Alves and Figueiredo [1] studied a periodic Kirchhoff equation in $\mathbb{R}^N$. They proved the existence of a nontrivial solution when the nonlinearity is in subcritical and critical case. Liu and He [12] proved the existence of infinitely many high-energy solutions where $V(x)\ge 0$ and the nonlinearity is superlinear. We remark that when $a=1$ and $b=0$, problem (1.1) can be rewritten as the well-known Schrödinger equation

$$-\Delta u+V(x)u=K(x)|u|^{q-1}u,\quad x\in\mathbb{R}^N. \tag{1.3}$$

For this equation, there is a large body of literature on the existence and multiplicity of solutions; we for example [4, 6, 8, 13, 20] and the references therein. Motivated by the above fact, in this paper our aim is to study the existence of infinitely many nontrivial solutions for (1.1) when $0<q<1$ and $K(x),V(x)$ both change sign. Our tool is the symmetric Mountain Pass Theorem.

To state our main result we require the following assumptions:

- (A1) $V\in L^\infty(\mathbb{R}^N)$ and there exist $\beta,R_0>0$ such that $V(x)\ge\beta$ for all $|x|\ge R_0$;
- (A2) $K\in L^\infty(\mathbb{R}^N)$ and there exist $\alpha,R_1,R_2>0$, $y_0=(y_1,\cdots,y_N)\in\mathbb{R}^N$ such that $K(x)\le-\alpha$ for all $|x|\ge R_1$ and $K(x)>0$ for all $x\in B(y_0,R_2)$.

Our main result reads as follows.

Theorem 1.1. *Under assumptions* (A1), (A2)*, problem* (1.1) *admits infinitely many nontrivial solutions.*

In the next section we give some notation and preliminary results; and in section 3, we prove theorem (1.1).

2. Preliminaries

We will us the following notation: Let

$$\|u\|_m = \Big(\int_{\mathbb{R}^N} |u(x)|^m dx\Big)^{1/m}, \quad 1 \le m < +\infty.$$

Let $2^* = \frac{2N}{N-2}$ for all $N \ge 3$. Let B_R denote the ball centred in zero of radius $R > 0$ in $\mathbb{R}^N$ and $B_R^c = \mathbb{R}^N \backslash B_R$. Let $F'(u)$: the Fréchet derivative of F at u. For s, be the Sobolev constant in

$$\|u\|_{2^*} \le s\|\nabla u\|_2, \quad \forall u \in H^1(\mathbb{R}^N).$$

Let $E = H^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$, $0 < q < 1$, endowed with the norm

$$\|u\| = \|\nabla u\|_2 + \|u\|_{q+1},$$

The space E becomes a reflexive Banach space.

Problem (1.1) has a variational structure. Indeed we consider the functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^2 + \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \frac{1}{q+1}\int_{\mathbb{R}^N} K(x)|u|^{q+1} dx.$$

As is well known, I is of class C^1 on E and any critical point of I is a solution of (1.1).

A functional I is said to satisfy the Palais-Smale condition (PS, for short) if for very sequence (u_n) such that

$$I(u_n) \text{ is bounded, and } \|I'(u_n)\| \to 0,$$

there is a convergent subsequence of (u_n).

Before proving Theorem (1.1), we give the symmetric Mountain Pass Theorem.

Definition 2.1. Let E be a Banach space and A a subset of E. Set A is said to be symmetric if $u \in E$ implies $-u \in E$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k\backslash\{0\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. We set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \ge k$.

Now we give the symmetric Mountain Pass Theorem [2] which improved by Kajikiya [13] to obtain the following Theorem.

Theorem 2.2. *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy:*

(1) *I is even, bounded from below, $I(0) = 0$ and I satisfies the Palais-Smale condition.*
(2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that*

$$\sup_{u \in A_k} I(u) < 0.$$

Then either of the following tow conditions holds:

(i) *There exists a sequence (u_k) such that $I'(u_k) = 0$, $I(u_k) < 0$ and (u_k) converges to zero; or*
(ii) *There exist two sequences (u_k) and (v_k) such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to+\infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k\to+\infty} I(v_k) = 0$ and (v_k) converges to a non-zero limit.*

3. Proof of Theorem (1.1)

Lemma 3.1. *Under assumptions* (A1), (A2)*, the functional* I *is bounded from below.*

Proof. By (A1), (A2) and Hölder inequality, we have

$$\begin{aligned} I(u) &= \frac{b}{4}\Big(\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)^2+\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx-\frac{1}{q+1}\int_{\mathbb{R}^N}K(x)|u|^{q+1}dx \\ &\geq \frac{b}{4}(\int_{\mathbb{R}^N}|\nabla u|^2dx)^2-\frac{1}{2}\int_{\mathbb{R}^N}V^-(x)u^2dx-\frac{1}{q+1}\int_{\mathbb{R}^N}K^+(x)|u|^{q+1}dx \\ &\geq \frac{b}{4}\|\nabla u\|_2^4-\frac{s^2}{2}\|V^-\|_{N/2}\|\nabla u\|_2^2-s^{q+1}\|K^+\|_{\frac{2^*}{2^*-q-1}}\|\nabla u\|_2^{q+1}. \end{aligned}$$

Since $0<q<1$, we conclude the proof. $\square$

Lemma 3.2. *Assume* (A1), (A2) *hold. Then, any* (PS) *sequence* (u_n) *of* I *is bounded in* E*.*

Proof. Let (u_n) be a (PS) sequence of I. Then, there exists a positive constant $c>0$ such that

$$\begin{aligned} &c\geq I(u_n) \\ &= \frac{b}{4}\Big(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\Big)^2+\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u_n^2dx \\ &\quad -\frac{1}{q+1}\int_{\mathbb{R}^N}K(x)|u_n|^{q+1}dx \\ &\geq \frac{b}{4}(\int_{\mathbb{R}^N}|\nabla u_n|^2dx)^2-\frac{1}{2}\int_{\mathbb{R}^N}V^-(x)u_n^2dx-\frac{1}{q+1}\int_{\mathbb{R}^N}K^+(x)|u_n|^{q+1}dx \\ &\geq \frac{b}{4}\|\nabla u_n\|_2^4-\frac{s^2}{2}\|V^-\|_{N/2}\|\nabla u_n\|_2^2-s^{q+1}\|K^+\|_{\frac{2^*}{2^*-q-1}}\|\nabla u_n\|_2^{q+1}. \end{aligned}$$

Hence, there exists $\gamma_0>0$ such that

$$\|\nabla u_n\|_2\leq\gamma_0,\quad \forall n\in\mathbb{N}. \tag{3.1}$$

On the other hand, there exists $c>0$ such that

$$\begin{aligned} c+\frac{\|u_n\|}{4} &\geq -\frac{1}{4}\langle I'(u_n),u_n\rangle+I(u_n) \\ &= \frac{a}{4}\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\frac{1}{4}\int_{\mathbb{R}^N}V(x)u_n^2dx+(\frac{1}{4}-\frac{1}{q+1})\int_{\mathbb{R}^N}K(x)|u_n|^{q+1}dx \\ &\geq (\frac{1}{q+1}-\frac{1}{4})\int_{\mathbb{R}^N}(K^-(x)+\chi_{B_{R_1}}(x))|u_n|^{q+1}dx \\ &\quad -(\frac{1}{q+1}-\frac{1}{4})\int_{\mathbb{R}^N}(K^+(x)+\chi_{B_{R_1}}(x))|u_n|^{q+1}dx-\frac{1}{4}\int_{\mathbb{R}^N}V^-(x)u_n^2dx \\ &\geq (\frac{1}{q+1}-\frac{1}{4})\min(\alpha,1)\int_{\mathbb{R}^N}|u_n|^{q+1}(x)dx-\frac{s^2\|V^-\|_{N/2}}{4}\|\nabla u_n\|^2 \\ &\quad -s^{q+1}(\frac{1}{q+1}-\frac{1}{4})\|K^++\chi_{B_{R_1}}\|_{\frac{2^*}{2^*-q-1}}\|\nabla u_n\|_2^{q+1}. \end{aligned}$$

Therefore, by using (3.1), we obtain

$$\|u_n\|_{q+1}\leq\gamma_1,\quad \text{for some } \gamma_1>0. \tag{3.2}$$

Combining (3.1) and (3.2), we conclude the proof. □

We need the following Lemma to prove that the Palais-Smale condition is satisfied for I on E.

Lemma 3.3. *Let x and y two arbitrary real numbers, then there exists a constant $c > 0$ such that*

$$\big||x+y|^{q+1} - |x|^{q+1} - |y|^{q+1}\big| \le c|x|^q y \tag{3.3}$$

Proof. If $x = 0$, the inequality (3.3) is trivial. Suppose that $x \neq 0$. We consider the continuous function f defined on $\mathbb{R}\backslash\{0\}$ by

$$f(t) = \frac{|1+t|^{q+1} - |t|^{q+1} - 1}{|t|}.$$

Note that $\lim_{|t|\to+\infty} f(t) = 0$ and $\lim_{t\to 0\pm} f(t) = \pm(q+1)$. Then there exists a constant $c > 0$ such that $|f(t)| \le c$, for all $t \in \mathbb{R}\backslash\{0\}$. In particular $|f(\frac{y}{x})| \le c$, so

$$\big||1+\frac{y}{x}|^{q+1} - |\frac{y}{x}|^{q+1} - 1\big| \le c|\frac{y}{x}|,$$

multiplying by $|x|^{q+1}$, we obtain the desired result. □

Lemma 3.4. *Assume that* (a1), (A2) *hold. Then I satisfies the Palais-Smale condition in E.*

Proof. Let (u_n) be a (PS) sequence. By Lemma (3.2), (u_n) is bounded in E. Then there exists a subsequence $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^p_{Loc}(\mathbb{R}^N)$ for all $1 \le p \le 2^*$ and $u_n \to u$ a.e in $\mathbb{R}^N$.

By [8], it is sufficient to prove that for any $\epsilon > 0$, there exist $R_3 > 0$ and $n_0 \in \mathbb{N}^*$ such that

$$\int_{|x|\ge R_3} (|\nabla u_n|^2 + |u_n|^{q+1})dx \le \epsilon, \quad \text{for all } R \ge R_3 \text{ and } n \ge n_0.$$

Let ϕ_R be a cut-off function so that $\phi_R = 0$ on $B_{\frac{R}{2}}$, $\phi_R = 1$ on B^c_R, $0 < \phi_R < 1$ and

$$|\nabla\phi_R|(x) \le \frac{c}{R}, \quad \text{for all } x \in \mathbb{R}^N. \tag{3.4}$$

We can easily remark that for any $u \in E$ and $R \ge 1$,

$$\|\phi_R u\| \le c\|u\|. \tag{3.5}$$

Since $I'(u_n) \to 0$ in E' as $n \to +\infty$, we know that for any $\epsilon > 0$, there exists $n_0 > 0$ such that

$$|\langle I'(u_n), \phi_R u_n\rangle| \le c\|I'(u_n)\|_{E'}\|u_n\| \le \frac{\epsilon}{3}, \quad \forall n \ge n_0;$$

that is, $n \ge n_0$. Then

$$\begin{aligned}&(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx)\int_{\mathbb{R}^N} |\nabla u_n|^2\phi_R(x)dx \\ &+ \int_{\mathbb{R}^N} V(x)|u_n|^2\phi_R(x)dx - \int_{\mathbb{R}^N} K(x)|u_n|^{q+1}\phi_R(x)dx \le \frac{\epsilon}{3}.\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^N}(a|\nabla u_n|^2+(K^-+\chi_{B_{R_1}})(x)|u_n|^{q+1})\phi_R(x)dx\\
&\leq \int_{\mathbb{R}^N}V^-(x)u_n^2\phi_R dx - a\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_R dx\\
&\quad+\int_{\mathbb{R}^N}(K^++\chi_{B_{R_1}})(x)|u_n|^{q+1}\phi_R dx+\frac{\epsilon}{3}.
\end{aligned}\tag{3.6}$$

By Hölder inequality and (3.4), there exists $R_4>0$ such that

$$\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\phi_R dx\leq\frac{c}{R}<\frac{\epsilon}{3},\quad\forall|x|\geq R_4.\tag{3.7}$$

From (A1) and (A2), there exists $R_5>0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N}V^-(x)u_n^2\phi_R dx+\int_{\mathbb{R}^N}(K^++\chi_{B_{R_1}})(x)|u_n|^{q+1}dx\\
&\leq c\|V^-\phi_R\|_{N/2}+c\|(K^++\chi_{B_{R_1}})\phi_R\|_{\frac{2^*}{2^*-q-1}}\\
&\leq\frac{\epsilon}{3}\quad\text{for } |x|\geq R_5.
\end{aligned}\tag{3.8}$$

Put $R_3=\max(R_4,R_5)$. By (3.6), (3.7) and (3.8), we have

$$\min(a,\min(\alpha,1))\int_{\mathbb{R}^N}(|\nabla u_n|^2+|u_n|^{q+1})\phi_R dx\leq\epsilon.$$

The proof is complete. □

Lemma 3.5. *Assume* (A1), (A2) *hold. Then for each* $k\in\mathbb{N}$*, there exists an* $A_k\in\Gamma_k$ *such that*

$$\sup_{u\in A_k}I(u)<0.$$

Proof. We use the following geometric construction introduced by Kajikiya [13]: Let R_2 and y_0 be fixed by assumption (A1) and consider the cube

$$D(R_2)=\{(x_1,\cdots,x_N)\in\mathbb{R}^N:|x_i-y_i|<R_2,\ 1\leq i\leq N\}.$$

Fix $k\in\mathbb{N}$ arbitrarily. Let $n\in\mathbb{N}$ be the smallest integer such that $n^N\geq k$. We divide $D(R_2)$ equally into n^N small cubes, denote them by D_i with $1\leq i\leq n^N$, by planes parallel to each face of $D(R_2)$. The edge of D_i has the length of $z=\frac{R_2}{n}$. We construct a new cubes E_i in D_i such that E_i has the same center as that of D_i. The faces of E_i and D_i are parallel and the edge of E_i has the length of $\frac{z}{2}$. Then, we make a function ψ_i, $1\leq i\leq k$, such that

$$\begin{gathered}
\operatorname{supp}(\psi_i)\subset D_i,\quad\operatorname{supp}(\psi_i)\cap\operatorname{supp}(\psi_j)=\emptyset\quad(i\neq j),\\
\psi_i(x)=1\quad\text{for } x\in E_i,\quad 0\leq\psi_i(x)\leq1,\quad\forall x\in\mathbb{R}^N.
\end{gathered}$$

We denote

$$S^{k-1}=\{(t_1,\cdots,t_k)\in\mathbb{R}^k:\max_{1\leq i\leq k}|t_i|=1\},\tag{3.9}$$

$$W_k=\{\sum_{i=1}^k t_i\psi_i(x):(t_1,\cdots,t_k)\in S^{k-1}\}\subset E.\tag{3.10}$$

Since the mapping $(t_1,\dots,t_k)\to\sum_{i=1}^k t_i\psi_i$ from S^{k-1} to W_k is odd and homeomorphic, then $\gamma(W_k)=\gamma(S^{k-1})=k$. W_k is compact in E, then there is a constant $\alpha_k>0$ such that

$$\|u\|^2\le\alpha_k\quad\text{for all } u\in W_k.$$

we need to recall the inequality

$$\|u\|_2\le c\|\nabla u\|_2^r\|u\|_{q+1}^{1-r}\le c\|u\| \tag{3.11}$$

with $r=\frac{2^*(q-1)}{2(2^*-q-1)}$. Then, there is a constant $c_k>0$ such that

$$\|u\|_2^2\le c_k\quad\text{for all } u\in W_k.$$

Let $z>0$ and $u=\sum_{i=1}^k t_i\psi_i(x)\in W_k$,

$$I(zu)\le z^4b\frac{\alpha_k^2}{4}+\frac{az^2}{2}\alpha_k+z^2\frac{\|V\|_\infty}{2}c_k-\frac{1}{q+1}\sum_{i=1}^k\int_{D_i}K(x)|zt_i\psi_i|^{q+1}dx. \tag{3.12}$$

By (3.9), there exists $j\in[1,k]$ such that $|t_j|=1$ and $|t_i|\le 1$ for $i\ne j$. Then

$$\begin{aligned}&\sum_{i=1}^k\int_{D_i}K(x)|zt_i\psi_i|^{q+1}dx\\&=\int_{E_j}K(x)|zt_j\psi_j|^{q+1}dx+\int_{D_j\setminus E_j}K(x)|zt_j\psi_j|^{q+1}dx+\sum_{i\ne j}\int_{D_i}K(x)|zt_i\psi_i|^{q+1}dx\,.\end{aligned} \tag{3.13}$$

Since $\psi_j(x)=1$ for $x\in E_j$ and $|t_j|=1$, we have

$$\int_{E_j}K(x)|zt_j\psi_j|^{q+1}dx=|z|^{q+1}\int_{E_j}K(x)dx. \tag{3.14}$$

On the other hand by (A1) we obtain

$$\int_{D_j\setminus E_j}K(x)|zt_j\psi_j|^{q+1}dx+\sum_{i\ne j}\int_{D_i}K(x)|zt_i\psi_i|^{q+1}dx\ge 0. \tag{3.15}$$

From (3.12), (3.13), (3.14) and (3.15), we obtain

$$\frac{I(zu)}{z^2}\le z^2\frac{b\alpha_k^2}{4}+\frac{a\alpha_k}{2}+\frac{\|V\|_\infty}{2}c_k-\frac{|z|^{q+1}}{z^2}\inf_{1\le i\le k}\Big(\int_{E_i}K(x)dx\Big).$$

It follows that

$$\lim_{z\to 0}\sup_{u\in W_k}\frac{I(zu)}{z^2}=-\infty.$$

We fix z small such that

$$\sup\{I(u),u\in A_k\}<0,\quad\text{where } A_k=zW_k\in\Gamma_k.$$

The proof is complete. $\square$

Proof of Theorem (1.1). Evidently, $I(0)=0$ and I is an odd functional. Then by Lemmas (3.1), (3.4) and (3.5), conditions (1) and (2) of Theorem 2.2 are satisfied. Then, by Theorem 2.2, problem (1.1) admits an infinitely many solutions $(u_k)\in E$ which converging to 0 and u_0 can be supposed nonnegative since

$$I(u_0)=I(|u_0|).$$

$\square$

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