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STABILITY FOR TRAJECTORIES OF PERIODIC EVOLUTION FAMILIES IN HILBERT SPACES

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Abstract. Let q be a positive real number and let $A(\cdot)$ be a q-periodic linear operator valued function on a complex Hilbert space H, and let D be a dense linear subspace of H. Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be the evolution family generated by the family $\{A(t)\}$. We prove that if the solution of the well-posed inhomogeneous Cauchy Problem

$$\dot{u}(t) = A(t)u(t) + e^{i\mu t}y, \quad t > 0$$
$$u(0) = 0,$$

is bounded on $\mathbb{R}_+$, for every $y \in D$, and every $\mu \in \mathbb{R}$, by the positive constant $K\|y\|$, K being an absolute constant, and if, in addition, for some $x \in D$, the trajectory $U(\cdot,0)x$ satisfies a Lipschitz condition on the interval $(0,q)$, then

$$\sup_{z\in\mathbb{C},|z|=1}\ \sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} z^k U(q,0)^k x\| := N(x) < \infty.$$

The latter discrete boundedness condition has a lot of consequences concerning the stability of solutions of the abstract nonautonomous system $\dot{u}(t) = A(t)u(t)$. To our knowledge, these results are new. In the special case, when $D = H$ and for every $x \in H$, the map $U(\cdot,0)x$ satisfies a Lipschitz condition on the interval $(0,q)$, the evolution family $\mathcal{U}$ is uniformly exponentially stable. In the autonomous case, (i.e. when $U(t,s) = U(t-s,0)$ for every pair (t,s) with $t \geq s \geq 0$), the latter assumption is too restrictive. More exactly, in this case, the semigroup $\mathbf{T} := \{U(t,0)\}_{t\geq 0}$, is uniformly continuous.

1. Introduction

In his famous article [16], Prüss showed, concerning a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ acting on a Hilbert space H, that the following two statements are equivalent:

(1) For every $f \in L^1([0,1],H)$, the equation $\dot{u}(t) = Au(t) + f(t)$, has a unique 1-periodic mild solution.
(2) The resolvent set of A contains $2\pi i\mathbb{Z}$ and

$$\sup_{n\in\mathbb{Z}} \|R(2\pi i n, A)\| = M < \infty.$$

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Earlier, such result has been obtained by Haraux ([10]), who additionally had to assume that

$$\sup_{n\in\mathbb{Z}} \|nR(2\pi in, A)\| \le C < \infty.$$

By contrast with the autonomous case, the spectral criterion does not work in the nonautonomous one. See, for example, [17], for further details and counterexamples.

One typical assumption, concerning uniform exponential stability results (exponential stability results) for strongly continuous semigroups acting in Banach spaces, is the existence of a bounded and holomorphic continuation of the resolvent (or of the local resolvent) to the right half-plane of the complex plane. See [3, 4, 12, 20, 21, 22], and the references therein.

Going on the similar way, the boundedness assumption of our announced result, may by written as:

$$\sup_{\mu\in\mathbb{R}} \sup_{t\ge 0} \Big\| \int_0^t e^{i\mu s} U(t,s)x ds \Big\| := M(x) < \infty, \quad \forall x \in H. \tag{1.1}$$

Boundedness conditions, like (1.1), with $e^{(t-s)A}$ instead of $U(t,s)$, A being a bounded linear operator acting on a Banach space X, seems to go back to the work of Krein. A history of this problem may be followed in [6] or [5].

This paper is motivated by a recent result in [7], where the same assertion, referred to uniformly bounded evolution families, is obtained under the following stronger assumption,

$$\sup_{\mu\in\mathbb{R}} \sup_{t\ge\tau\ge 0} \Big\| \int_\tau^t e^{i\mu s} U(t,s)x\, ds \Big\| := M(x) < \infty, \quad \forall x \in X,$$

where X is a complex Banach space.

It is interesting to compare the results from the present paper with those in [7].

First of all, the assertions in Theorem 2.1 and Corollary 2.4 below, refer only to one trajectory, hence they have an individual character, while those in [7] have a global character. Moreover, even Theorem 2.2 below could be stated as an individual result.

Mention that the state space in [7] is a Banach space, while in the present paper, all results are formulated in the framework of Hilbert spaces. It seems that the results in [7] could be applied to partial differential equations (cf. [7, Remark 5.2]) while the autonomous version of the Corollary 2.7 below, cannot be applied to such equations (cf. Remark 2.8 below).

However, Corollaries 2.4, 2.5 and 2.6 below could be applied to evolution equations with (possible) unbounded coefficients.

Another difference is that the boundedness assumptions (2.3) and (2.6) are written along the solutions $u(\cdot)$ satisfying the initial condition $u(0) = 0$, while in [7], are considered all solutions which verify $u(s) = 0$ for every $s \ge 0$.

The assertions of Corollaries 2.4, 2.5 and 2.6 below, are stated in terms of strong stability. In the end of this paper we provide an example which shows that uniform exponential stability of a strongly continuous and uniformly bounded semigroup acting on a Hilbert space, is not a consequence of boundedness assumption (2.6) below. A famous result, known as ABLV theorem, see [1, 13], provides a sufficient condition for strong stability of semigroups (acting on Banach spaces) in terms of countability of the boundary spectrum $\sigma(A)\cap i\mathbb{R}$. In particular, lack of the spectrum of the infinitesimal generator of an uniformly bounded semigroup which acts on a Banach space on the imaginary axis implies the strong stability of semigroup. However, it is not always easy to manipulate these criteria in specific examples, while the assumption (2.6) can be checked easily, as shown by the two examples presented in the last section of the article.

An interesting question may be risen when we connect the above nonautonomous problem with the similar one in the semigroup case. This latter one, is completely solved by van Neerven ([20]) and Vu Phong ([15]). In these papers, the assumption refereing to Lipschitz condition did not appear explicitly, but, in this case, it is automatically verified for all x in the domain of the infinitesimal generator. Then, a natural question in the nonautonomous case is: can we replace in the Corollary 2.7 the assumption referred to the Lipschitz condition for every $x \in H$, with the similar one, but only for x in a dense subset D, of H? We have not an answer to this question yet.

The proof of the main result consists by an estimation of the integral in (1.1), with x replaced by $f_x(s)$, f_x being a H-valued and q-periodic function, which is smooth in some sense. The proof of Corollary 2.7 is completed by using a well-known discrete criterion for exponential stability of periodic evolution families.

2. Notations and statement of the result

Let H be a complex separable Hilbert space and let $\mathcal{L}(H)$ be the Banach algebra of all bounded linear operators acting on H. The inner product in H is denoted by $\langle\cdot,\cdot\rangle$, while the norms in H and in $\mathcal{L}(H)$ are denoted by the same symbol, namely by $\|\cdot\|$. As usual, $\sigma(T)$ denotes the spectrum of a linear operator T. When T is bounded its spectral radius, denoted by $r(T)$, is given by the Gelfand formula

$$r(T) := \sup\{|z| : z \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Recall that the set $\{u_k\}_{k\in\mathbb{Z}_+}$, with $u_k \in H$, is a basis in H if the linear span of $\{u_k : k \in \mathbb{Z}_+\}$ is dense in H. Such basis is called orthogonal if $\langle u_k, u_j\rangle = 0$, for every pair (k, j) of different nonnegative integers. As is well-known, the orthonormal set $\{u_k\}_{k\in\mathbb{Z}_+}$ is a basis in H if and only if for each $f \in H$ one has $\sum_{k\in\mathbb{Z}_+} |\langle f, u_k\rangle|^2 = \|f\|^2$. In this case, every element $f \in H$ may be represented as

$$f = \sum_{k\in\mathbb{Z}_+} \langle f, u_k\rangle u_k.$$

As usual, by $L^2([0, q], \mathbb{C})$, we denote the Hilbert space consisting of all $\mathbb{C}$-valued square integrable functions defined on the interval $[0, q]$, endowed with the usual inner product and norm. The set of functions $\{\frac{e^{2i\pi(n/q)\cdot}}{\sqrt{q}}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2([0, q], \mathbb{C})$, so any function $z \in L^2([0, q], \mathbb{C})$, may be represented as:

$$z(t) = \frac{1}{q} \sum_{n\in\mathbb{Z}} \Big(\int_0^q z(s)e^{-2in\pi s/q} ds\Big) e^{2in\pi t/q}, \quad t \in [0, q]. \tag{2.1}$$

By $\mathcal{H} := L^2([0, q], H)$ we denote the set of all H-valued measurable functions f defined on $[0, q]$ satisfying the condition

$$\Big(\int_0^q \|f(t)\|^2 dt\Big)^{1/2} := \|f\|^2_{L^2([0,q],H)} < \infty.$$

In $\mathcal{H}$ the functions equal almost everywhere are identified. Endowed with the inner product $\langle f, g\rangle_{\mathcal{H}} = \int_0^q \langle f(t), g(t)\rangle dt$, $\mathcal{H}$ becomes a Hilbert space. In as follows, by $\varphi\otimes u_k$ we denote the tensor product between the scalar-valued function φ defined on $[0,q]$ and the vector $u_k\in H$, i.e. the map defined for $t\in[0,q]$, by: $(\varphi\otimes u_k)(t)=\varphi(t)u_k$. The system of vectors

$$\mathcal{B}:=\Big\{\frac{1}{\sqrt{q}}e^{2i\pi(n/q)\cdot}\otimes u_k : n\in\mathbb{Z}, k\in\mathbb{Z}_+\Big\}$$

is an orthonormal basis in $\mathcal{H}$. In fact, for $k,p\in\mathbb{Z}_+$ and $m,n\in\mathbb{Z}$, one has:

$$\begin{aligned}
\Big\langle\frac{1}{\sqrt{q}}e^{2in\pi\cdot/q}\otimes u_k, \frac{1}{\sqrt{q}}e^{2im\pi\cdot/q}\otimes u_p\Big\rangle_{\mathcal{H}} &= \frac{1}{q}\int_0^q\langle e^{2in\pi t/q}u_k, e^{2im\pi t/q}u_p\rangle dt\\
&= \frac{\delta_{kp}}{q}\int_0^q e^{2i(n-m)\pi t/q}dt\\
&= \begin{cases}1, & k=p \text{ and } n=m\\ 0, & k\neq p \text{ or } n\neq m\,.\end{cases}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= \int_0^q\Big\langle\sum_{k\in\mathbb{Z}_+}\langle f(t),u_k\rangle u_k, \sum_{p\in\mathbb{Z}_+}\langle f(t),u_p\rangle u_p\Big\rangle dt\\
&= \sum_{k,p\in\mathbb{Z}_+}\int_0^q\langle f(t),u_k\rangle\overline{\langle f(t),u_p\rangle}\langle u_k,u_p\rangle dt\\
&= \sum_{k\in\mathbb{Z}_+}\int_0^q\langle f(t),u_k\rangle\overline{\langle f(t),u_k\rangle}dt.
\end{aligned}$$

In view of (2.1), for $z(t)=\langle f(t),u_k\rangle$, obtain

$$\begin{aligned}
\|f\|_{\mathcal{H}}^2 &= \frac{1}{q^2}\sum_{k\in\mathbb{Z}_+}\Big[\sum_{n,m\in\mathbb{Z}}\Big(\int_0^q\langle f(s),u_k\rangle e^{-2in\pi s/q}ds\Big)\\
&\quad\times\Big(\int_0^q\overline{\langle f(s),u_k\rangle e^{-2im\pi s/q}}ds\Big)\int_0^q e^{2i(n-m)\pi t/q}dt\Big]\\
&= \frac{1}{q}\sum_{(k,n)\in\mathbb{Z}_+\times\mathbb{Z}}\int_0^q\langle f(s),e^{2in\pi s/q}u_k\rangle ds\overline{\int_0^q\langle f(s),e^{2in\pi s/q}u_k\rangle ds}\\
&= \frac{1}{q}\sum_{(k,n)\in\mathbb{Z}_+\times\mathbb{Z}}\Big|\int_0^q\langle f(t),e^{2in\pi t/q}u_k\rangle dt\Big|^2\\
&= \sum_{(k,n)\in\mathbb{Z}_+\times\mathbb{Z}}\Big|\langle f,\frac{1}{\sqrt{q}}e^{2i\pi(n/q)\cdot}\otimes u_k\rangle_{\mathcal{H}}\Big|^2.
\end{aligned}$$

As a consequence, every function $f\in\mathcal{H}$ may be represented as

$$f(\cdot)=\frac{1}{q}\sum_{n\in\mathbb{Z}}e^{2in\pi\cdot/q}c_n(f),$$

where $c_n(f)\in H$, the n^{th} Fourier coefficient associated to f, is given by

$$c_n(f)=\sum_{k\in\mathbb{Z}_+}\Big(\int_0^q\langle f(s),e^{2in\pi s/q}u_k\rangle ds\Big)u_k. \tag{2.2}$$

A family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(H)$ is called evolution family on H if $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r \geq 0$ and $U(t,t) = I$ for $t \geq 0$. Here I is the identity operator of $\mathcal{L}(H)$. An evolution family $\mathcal{U}$ on H is called *strongly continuous* if for each $x \in H$, the map

$$(t,s) \mapsto U(t,s)x : \{(t,s) \in \mathbb{R}^2 : t \geq s \geq 0\} \to X$$

is continuous. We say that the evolution family $\mathcal{U}$ *has exponential growth* or that it is *exponentially bounded* if there exist the constants $M \geq 1$ and $\omega > 0$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)}$, for all $t \geq s$. The evolution family $\mathcal{U}$ is q-periodic, for some positive q, if $U(t+q, s+q) = U(t,s)$ for all pairs (t,s) with $t \geq s \geq 0$. Every strongly continuous and q-periodic evolution family acting on a Banach space has an exponential growth, [8]. Let $x \in H$ be fixed.

The following assumptions, concerning the evolution family, and the trajectory $U(\cdot,0)x$, are referred to several times. For this reason, we state them separately.

(A1) The map $t \mapsto u_x(t) := U(t,0)x$ satisfies a Lipschitz condition on the interval $(0,q)$.

The main result of this article reads as follows.

Theorem 2.1. *Let D be a dense linear subspace of H, and let $\boldsymbol{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be a strongly continuous and q-periodic evolution family of bounded linear operators acting on H. If*

$$\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \| \int_0^t e^{i\mu s} U(t,s)y \, ds\| \leq K\|y\| < \infty, \quad \forall y \in D, \tag{2.3}$$

where K is a positive absolute constant, and if, in addition, for a given $x \in D$, the assumption (A1) *is fulfilled, then*

$$\sup_{z \in \mathbb{C}, |z|=1} \sup_{n \in \mathbb{Z}_+} \| \sum_{k=0}^{n} z^k U(q,0)^k x\| := N(x) < \infty. \tag{2.4}$$

In view of the denseness of D in H and by using the Dominated Convergence Theorem, the condition (2.3) is equivalent with the same one, but with H instead of D.

Recall that if A is an infinitesimal generator of a strongly continuous semigroup acting on a Hilbert space H the its maximal domain $D(A)$ becomes a Hilbert space when endow it with the norm

$$\|x\|_{D(A)} := \sqrt{\|x\|_H^2 + \|Ax\|_H^2}, \quad x \in D(A).$$

In the framework of semigroups, the assumptions of the previous theorem may be relaxed as follows:

Theorem 2.2. *Let $\boldsymbol{T} = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators acting on H, and let $(A, D(A))$ be its infinitesimal generator. If there exists a positive absolute constant R_1 such that*

$$\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \| \int_0^t e^{i\mu s} T(t-s)y \, ds\| := R(y) \leq R_1\|y\|_{D(A)} < \infty, \quad \forall y \in D(A),$$

then, for every $x \in D := D(A^2)$, one has:

$$\sup_{z \in \mathbb{C}, |z|=1} \sup_{n \in \mathbb{Z}_+} \| \sum_{k=0}^{n} z^k T(q)^k x\| := N(x) < \infty.$$

To present some consequences of Theorem 2.1, we state and prove the following useful Lemma.

Lemma 2.3. *Let T be a bounded linear operator acting on H and let $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T^n\| \leq Me^{\omega n}$ for all $n \in \mathbb{Z}_+$. If for some $x \in H$, one has*

$$\sup_{z\in\mathbb{C},|z|=1} \sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} \frac{T^k x}{z^{k+1}}\| := N(x) < \infty, \tag{2.5}$$

then $\lim_{k\to\infty} T^k x = 0$.

Proof. The proof is modeled after [20, Theorem 4]. Obviously, the map $z \mapsto R_{n,x}(z) := \sum_{k=0}^{n} \frac{T^k x}{z^{k+1}}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Let $z_0 \in \mathbb{C}$ such that $|z_0| = e^{\omega}$. By assumption, the sequence of functions $(R_{n,x}(\cdot))$, is uniformly bounded on the unit circle. On the other hand, for $|z| \geq |z_0| + 1$ we have

$$\|R_{n,x}(z)\| \leq \sum_{k=0}^{n} \frac{M|z_0|^k\|x\|}{(|z_0|+1)^{k+1}} \leq M(1+|z_0|)\|x\|.$$

Thus, by the Phragmen-Lindelöf theorem, the sequence $(R_{n,x}(\cdot))$, is uniformly bounded on the circular crown $1 \leq |z| \leq |z_0| + 1$, and then it is uniformly bounded on the set $\{z \in \mathbb{C} : |z| \geq 1\}$, as well.

Since, for $|z| > |z_0|$, one have that $R_{n,x}(z) \to R(z,T)x$, as $n \to \infty$, the Vitali theorem [11, Theorem 3.14.1] assures us that the limit $\lim_{n\to\infty} R_{n,x}(z)$, exists for all z in the circular crown $1 \leq |z| \leq |z_0| + 1$. This yields, $\lim_{k\to\infty} T^k x = 0$. □

Corollary 2.4. *Let $\mathcal{U}$ be a strongly continuous and q-periodic evolution family acting on H, such that all assumptions in Theorem 2.1 are fulfilled. Then the trajectory $U(\cdot,0)x$ is strongly stable, i.e.,* $\lim_{t\to\infty} U(t,0)x = 0$.

Proof. Let $t > 0$ and k be the integer part of $\frac{t}{q}$, and let $\rho \in [0,q]$ such that $t = kq+\rho$. Obviously, $t \to \infty$ if and only if $k \to \infty$. Since the family $\{U(t,s) : t \geq s \geq 0\}$ is exponentially bounded, this yields

$$\begin{aligned} \|U(t,0)x\| &= \|U(t,kq)U(kq,0)x\| \leq \|U(t,kq)\| \cdot \|U(q,0)^k x\| \\ &\leq Me^{\omega q}\|U(q,0)^k x\| \to 0, \end{aligned}$$

where Lemma 2.3 with $U(q,0)$ instead of T was used. □

Corollary 2.5. *Let $\mathcal{U}$ be a strongly continuous and q-periodic evolution family acting on H. Assume the following:*

1. *The condition* (2.3) *is fulfilled.*
2. *The map $U(\cdot,0)y$ satisfies a Lipschitz condition for every $y \in D$.*
3. *The evolution family $\mathcal{U}$ acts properly on the linear subspace D, i.e. $U(t,s)(D) \subset D$ for every $t \geq s \geq 0$.*

Under these assumptions, the trajectory $t \mapsto U(t,s)x : [s,\infty) \to H$, is strongly stable for every $s \geq 0$ and every $x \in D$.

Proof. Let $t \geq s \geq 0$ and N be any positive integer such that $t \geq Nq \geq s$. Such N exists for t large enough. Then $U(t,s)x = U(t-Nq,0)U(Nq,s)x$. Since $y_s := U(Nq,s)x \in D$, we may apply the previous Corollary to finish the proof. □

Corollary 2.6. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a uniformly bounded and strongly continuous semigroup acting on a Hilbert space H, and let $D(A)$ the maximal domain of its infinitesimal generator. If there exists a positive constant K_1 such that*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\|\int_0^t e^{i\mu s}T(t-s)y\,ds\| := K(y) \leq K_1\|y\|_{D(A)} < \infty, \quad \forall y \in D(A), \tag{2.6}$$

then, the semigroup $\mathbf{T}$ is strongly stable.

Proof. Since, for every $x \in D(A)$, the map $T(\cdot)x$ is differentiable, it satisfies a Lipschitz condition on $(0, q)$. Then, by applying the above Theorem 2.2 it follows that $T(\cdot)x$ is strongly stable for every $x \in D(A)$. Let now, $y \in H$ and $x_n \in D(A)$ such that $x_n \to y$, as $n \to \infty$, in the norm of H. Then,

$$\begin{aligned}\|T(t)y\| &\leq \|T(t)(y - x_n)\| + \|T(t)x_n\| \\ &\leq \sup\{\|T(t)\| : t \geq 0\}\|y - x_n\| + \|T(t)x_n\| \to 0, \quad \text{as } t, n \to \infty.\end{aligned}$$

□

Corollary 2.7. *If an evolution family $\boldsymbol{U} = \{U(t, s)\}_{t\geq s\geq 0}$, as in Theorem 2.1, verify* (2.3) *for every $y \in H$, and also* (A1) *with v instead of x, for every $v \in H$, then it is uniformly exponentially stable, i.e. there are two positive constants N and ν such that*

$$\|U(t, s)\| \leq Ne^{-\nu(t-s)}, \quad \textit{for all } t \geq s \geq 0.$$

The proof of the above corollary follows by applying Theorem 2.1 and Lemma 2.11, below.

Remark 2.8. The assumption that (A1), with v instead of x, is satisfied for every $v \in H$, is too restrictive. For example, in the particular case when $U(t, s) = T(t-s)$ for every $t \geq s \geq 0$, where $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup acting on H, the family $\mathcal{U}$ verify the above Lipschitz assumption if and only if the semigroup is uniformly continuous, i.e. if and only if there exists a bounded linear operator acting on H such that $T(t) = e^{tA}$ for all $t \geq 0$.

Proof. Indeed, under assumption (A1), the map $t \mapsto u_x(t) := T(t)x$ belongs to $W^{1,\infty}((0, q), H)$, therefore $u_x(\cdot)$ is differentiable a.e. on $(0, q), \|u_x'(t)\| \leq L_x$ (L_x being the constant of Lipschtitz of $u_x(\cdot)$) and $u_x(t)$ belongs in the domain $D(A)$ of the infinitesimal generator a.e. on $(0, q)$. Then, there exists a sequence (t_n) of positive real numbers, with $t_n \to 0$ as $n \to \infty$ such that $u_x(t_n) \in D(A), u_x(t_n) = T(t_n)x \to x$, as $n \to \infty$ and the sequence $(u_x'(t_n)) = (AT(t_n))$ converges in the weak topology of H. This latter fact is an obvious consequence of the Banach-Steinhaus theorem by using the classical fact that the adjoint A^*, of A, is the infinitesimal generator of the semigroup $\{T(t)^*\}_{t\geq 0}$. Since, A is closed in the weak topology of H, it follows $x \in D(A)$. Then $D(A) = X$ and A is bounded. □

The following lemmas are useful in the proof of Theorem 2.1.

Lemma 2.9 ([2, Lemma 2.2])**.** *Let us consider the functions h_1, $h_2 : [0, q] \to \mathbb{C}$, defined by:*

$$h_1(s) = \begin{cases} s, & s \in [0, q/2) \\ q - s, & s \in [q/2, q] \end{cases} \quad \textit{and} \quad h_2(s) = s(q - s), \quad s \in [0, q].$$

Denote

$$H_1(\mu) := \int_0^q h_1(s)e^{i\mu s}ds, \quad H_2(\mu) := \int_0^q h_2(s)e^{i\mu s}ds.$$

Then, $|H_1(\mu)| + |H_2(\mu)| \neq 0$ *for all* $\mu \in \mathbb{R}$.

The continuation by periodicity on the real axis of the function h_j, for $j \in \{1,2\}$, will be denoted by the same symbol.

Lemma 2.10. *Let* $x \in X$ *such that the map* $s \mapsto U(s,0)x$ *satisfies a Lipschitz condition on* $(0,q)$*. For each* $j \in \{1,2\}$*, let consider the* q*-periodic function* $f_j : \mathbb{R} \to H$*, given on* $[0,q]$*, by:*

$$f_j(t) := h_j(t)U(t,0)x.$$

Then the following two statements hold:

1. *Each function* f_j *satisfies a Lipschitz condition on* $\mathbb{R}$.
2. *The Fourier series associated to* f_j *is absolutely and uniformly convergent on* $\mathbb{R}$.

Lemma 2.11 ([6, Lemma 1])**.** *If*

$$\sup_{n\geq 1} \| \sum_{k=0}^{n} e^{-i\mu k}U(q,0)^k \| := M(\mu) < \infty \tag{2.7}$$

then $e^{i\mu}$ *belongs to the resolvent set of* $U(q,0)$*. Moreover, if* (2.7) *holds for every* $\mu \in \mathbb{R}$*, then* $r(U(q,0)) < 1$*, i.e. the family* $\mathcal{U}$ *is uniformly exponentially stable.*

3. Proof of Theorem 2.1

Proof of Lemma 2.10. (1) Let $x \in H$, $K_x := \sup_{s\in[0,q]} ||U(s,0)x||$ and $L_x > 0$ such that $\|U(t,0)x - U(s,0)x\| \leq L_x|t-s|$ for all $t, s \in (0,q)$. One has

$$\begin{aligned}
\|f_j(t) - f_j(s)\| &= \|h_j(t)U(t,0)x - h_j(s)U(s,0)x\| \\
&\leq |h_j(t)|\|U(t,0)x - U(s,0)x\| + |h_j(t) - h_j(s)|\|U(s,0)x\| \\
&\leq (\max\{\frac{q}{2}, \frac{q^2}{4}\}L_x + K_x \cdot \max\{1,q\})|t-s|.
\end{aligned}$$

Using the continuity of the map $U(\cdot,0)x$, the previous inequality may be extended first for $t, s \in [0,q]$ and then, by using the periodicity of the map f_j, to the entire axis.

(2) An argument of this type for scalar valued functions may be found in [18, Exercise 16, pp. 92-93]. For sake of completeness we present the details. For each $t \in \mathbb{R}$ and each positive number ρ, which will be chosen later, denote $g_j(t) := f_j(t+\rho) - f_j(t-\rho)$. Using (2.2), we obtain

$$\begin{aligned}
c_n(g_j) &= \sum_{k\in\mathbb{Z}_+} \Big(\int_0^q \langle g_j(s), e^{2in\pi s/q}u_k\rangle ds \Big) u_k \\
&= \sum_{k\in\mathbb{Z}_+} \Big(\int_0^q \langle f_j(s+\rho), e^{2i\pi ns/q}u_k\rangle ds - \int_0^q \langle f_j(s-\rho), e^{2i\pi ns/q}u_k\rangle ds \Big) u_k \\
&= \sum_{k\in\mathbb{Z}_+} \Big(\int_\rho^{q+\rho} \langle f_j(\tau), e^{2i\pi n(\tau-\rho)/q}u_k\rangle d\tau - \int_{-\rho}^{q-\rho} \langle f_j(\tau), e^{2i\pi n(\tau+\rho)/q}u_k\rangle d\tau \Big) u_k
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}_+} \Big[\int_0^q \big(e^{2\pi i n\rho/q} - e^{-2i\pi n\rho/q}\big)\langle f_j(\tau), e^{2i\pi n\tau/q}u_k\rangle d\tau \Big] u_k \\
&= \sum_{k\in\mathbb{Z}_+} \Big(\int_0^q 2i\sin(2n\pi\rho/q)\langle f_j(\tau), e^{2i\pi n\tau/q}u_k\rangle d\tau \Big) u_k \\
&= 2i\sin(2n\pi\rho/q)c_n(f_j).
\end{aligned}
$$

In view of the Bessel inequality and taking into account that the function f_j satisfies a Lipschitz condition on $\mathbb{R}$ with a constant $L(x)$, follows

$$
4q\rho^2L^2(x) \geq \int_0^q \|g_j(t)\|^2 dt \geq \sum_{n\in\mathbb{Z}} \|c_n(g_j)\|^2 = \sum_{n\in\mathbb{Z}} 4\|c_n(f_j)\|^2 |\sin(2n\pi\rho/q)|^2.
$$

Let p be a positive integer and $\rho := \frac{q}{2^{p+2}}$. Set $\mathcal{A}_p := \{n \in \mathbb{Z} | 2^{p-1} < |n| \leq 2^p\}$. Obviously, $|\mathcal{A}_p| = 2^p$ and $\frac{\sqrt{2}}{2} < |\sin(2n\pi\rho/q)|$ for each $n \in \mathcal{A}_p$. Furthermore, $\cup_{p\geq 1}\mathcal{A}_p = \mathbb{Z}\setminus\{-1,0,1\}$. Using the Schwartz inequality, we obtain

$$
\begin{aligned}
\Big(\sum_{n\in\mathcal{A}_p} \|c_n(f_j)\| \Big)^2 &\leq 2^p \sum_{n\in\mathcal{A}_p} \|c_n(f_j)\|^2 \\
&< 2^{p+1} \sum_{n\in\mathcal{A}_p} \|c_n(f_j)\|^2 \sin^2(2n\pi\rho/q) \\
&\leq \frac{q^3L^2(x)}{2^{p+3}},
\end{aligned}
$$

and

$$
\sum_{n\in\mathbb{Z}\setminus\{-1,0,1\}} \|c_n(f_j)\| \leq (\sqrt{2}+1)\frac{q\sqrt{q}L(x)}{2^{3/2}} := L_1(x).
$$

The restriction of f_j to the interval $[0,q]$ belongs to $L^2([0,q],H)$ and, in addition, $f_j = (1/q)\sum_{n\in\mathbb{Z}} e^{2in\pi\cdot/q}c_n(f_j)$. Set

$$
s_{N,j}(t) := \sum_{n=-N}^{N} e^{2int\pi/q}c_n(f_j).
$$

Clearly, $(\int_0^q \|f_j(t) - s_{N,j}(t)\|^2 dt)^{\frac{1}{2}})$, decays to 0 when $N \to \infty$. As is already shown, the series $(\sum_{n\in\mathbb{Z}} e^{2in\pi t/q}c_n(f_j))$, is uniformly convergent on $\mathbb{R}$, hence there exists a continuous function $s_j : \mathbb{R} \to H$, such that

$$
\sup_{t\in\mathbb{R}} \|s_j(t) - s_{N,j}(t)\| \to 0 \quad \text{as } N \to \infty.
$$

Since

$$
\|s_{N,j} - s_j\|_{L^2([0,q],H)} \leq \sqrt{q}\|s_{N,j} - s_j\|_\infty \to 0,
$$

$s_j = f_j$ in $L^2([0,q],H)$. The functions s_j and f_j are continuous and equal almost everywhere on $[0,q]$, so $f_j(t) = s_j(t)$ for each $t \in [0,q]$. Taking into account that both functions are q-periodic, they are equal on $\mathbb{R}$. To conclude, the Fourier series associated to f_j, is absolutely and uniformly convergent on $\mathbb{R}$ to f_j. □

Proof of Theorem 2.1. For every $n \in \mathbb{Z}_+$ and every $x \in X$, one has

$$
\int_0^{nq} e^{i\mu s}U(nq,s)f_j(s)ds = \sum_{k=0}^{n-1} \int_{kq}^{(k+1)q} e^{i\mu s}U(nq,s)f_j(s)ds
$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \int_0^q e^{i\mu(kq+\rho)} U(nq, kq+\rho) f_j(kq+\rho) d\rho \\
&= H_j(\mu) \sum_{k=0}^{n-1} e^{i\mu kq} U((n-k)q, 0)x.
\end{aligned}
$$

Let $\mathcal{A}$ be the set of all real μ for which $H_1(\mu) = 0$. In view of Lemma 2.9, we obtain

$$
\| \sum_{k=0}^{n-1} e^{i\mu kq} U(q,0)^{n-k} x \| = \begin{cases} \frac{1}{|H_1(\mu)|} \| \int_0^{nq} e^{i\mu s} U(nq, s) f_1(s) ds \|, & \mu \notin \mathcal{A} \\ \frac{1}{|H_2(\mu)|} \| \int_0^{nq} e^{i\mu s} U(nq, s) f_2(s) ds \|, & \mu \in \mathcal{A}. \end{cases}
$$

On the other hand

$$
\begin{aligned}
\| \int_0^{nq} e^{i\mu s} U(nq, s) f_j(s) ds \| &= \| \frac{1}{q} \sum_{k\in\mathbb{Z}} \int_0^{nq} e^{i(\mu + 2\pi k/q)s} U(nq, s) c_k(f_j) \| \\
&\le \frac{K}{q} \sum_{k\in\mathbb{Z}} \|c_k(f_j)\| \\
&\le \frac{K}{q} \Big(L_1(x) + \sum_{k\in\{-1,0,1\}} \|c_k(f_j)\| \Big).
\end{aligned}
$$

□

Proof of Theorem 2.2. Let $x \in D(A^2)$ be fixed. Obviously, the Fourier coefficients $c_k(f_j), k \in \mathbb{Z}, j \in \{1, 2\}$, belong to $D(A^2)$. On the other hand

$$
\sup_{\mu\in\mathbb{R}} \sup_{t\ge 0} \| \int_0^t e^{i\mu s} T(t-s) Ay \, ds \| \le R_1 \|Ay\|_{D(A)} < \infty, \quad \forall y \in D(A^2),
$$

and then

$$
\sup_{\mu\in\mathbb{R}} \sup_{t\ge 0} \| \int_0^t e^{i\mu s} T(t-s) y \, ds \|_{D(A)} \le R_1 \|y\|_{D(A)} < \infty, \quad \forall y \in D(A^2).
$$

Now, we can apply Theorem 2.1, for $H = (D(A), \|\cdot\|_{D(A)}), D = D(A^2)$ and $U(t,s) = T(t-s)$ for $t \ge s$. □

4. Examples

Example 4.1. Let $H := L^2([0,\pi], \mathbb{C})$ be endowed with the usual norm and let $\{T(t)\}_{t\ge 0}$ be the semigroup defined on H, by

$$
\big(T(t)x\big)(\xi) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-tn^2} \sin(n\xi) \Big(\int_0^{\pi} x(s) \sin(ns) ds \Big), \quad \xi \in [0,\pi], t \ge 0,
$$

having A as infinitesimal generator. Further details about this semigroup and its infinitesimal generator may be find, for example, in [23, pp. 179, 199]. We recall that the domain of A consists by all absolutely continuous functions $x(\cdot)$ such that $x'(\cdot)$ is absolutely continuous, $x''(\cdot) \in H$ and $x(0) = x(\pi) = 0$.

Also, consider the map $a : \mathbb{R}_+ \to (0, \infty)$ verifying the following conditions:

(i) $a(\cdot)$ is a π-periodic map,
(ii) $a(t) \ge 1$ for every $t \ge 0$,
(iii) there exist $\alpha \in (0, 1]$ and $c > 0$ such that $|a(t) - a(s)| \le c|t-s|^{\alpha}$, for all $t, s \ge 0$.

Set $A(t) = a(t)A$, $t \geq 0$. The family $\{A(t)\}_{t\geq 0}$ is well-posed (i.e. there exists a strongly continuous and periodic evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ such that any solution $u(\cdot)$ of the system $\dot{x}(t) = A(t)x(t)$, verifies $u(t) = U(t,s)u(s)$ for all $t \geq s \geq 0$). See, [19, 17, 14], for further details concerning well-posedness. In this case, the evolution family is given by, [9, Example 2.9b],

$$U(t,s)x = T\Big(\int_s^t a(r)dr\Big)x, \quad \forall x \in H,\ t \geq s \geq 0.$$

Obviously, every continuous function $f \in H$ verifies the inequality $\|f\|_{L^2([0,\pi],\mathbb{C})} \leq \sqrt{\pi}\|f\|_\infty$. Set $x(\cdot) \in H$. Taking into account that $\hat{x}_n := \int_0^\pi x(s)\sin nsds$ satisfies the estimation $\|\hat{x}_n\| \leq \sqrt{\pi}\|x\|_2$, and denoting by $F(\cdot)$ a primitive function of $a(\cdot)$, we obtain

$$\begin{aligned}
\|\int_0^t e^{i\mu\tau}T\Big(\int_\tau^t a(r)dr\Big)x(\xi)d\tau\| &= \frac{2}{\pi}\|\int_0^t e^{i\mu\tau}\Big(\sum_{n=1}^\infty e^{-n^2\int_\tau^t a(r)dr}\sin(n\xi)\Big)\hat{x}_n d\tau\| \\
&\leq \frac{2}{\sqrt{\pi}}\|x\|_2\sum_{n=1}^\infty\int_0^t e^{-n^2\int_\tau^t a(r)dr}d\tau \\
&\leq 2\|x\|_\infty\sum_{n=1}^\infty e^{-n^2F(t)}\int_0^t a(\tau)e^{n^2F(\tau)}d\tau \\
&\leq \frac{2}{\sqrt{\pi}}\|x\|_2\sum_{n=1}^\infty\frac{1}{n^2}\big(1 - e^{n^2(F(0)-F(t))}\big) \\
&\leq \frac{2}{\sqrt{\pi}}\|x\|_2\sum_{n=1}^\infty\frac{1}{n^2} < \infty.
\end{aligned}$$

Thus, for each $x = x(\cdot) \in H$, have

$$\|\int_0^t e^{i\mu\tau}T\Big(\int_\tau^t a(r)dr\Big)x\,d\tau\|_{L^2([0,\pi],\mathbb{C})} \leq 2\|x\|_2\sum_{n=1}^\infty\frac{1}{n^2} < \infty.$$

On the other hand, the map

$$t \mapsto U(t,0)x = T\Big(\int_0^t a(s)ds\Big)x$$

is differentiable for all $x \in D(A)$, and its derivative is bounded by $|a(\cdot)|_\infty \times \|Ax\|$. Then, it satisfies a Lipschitz condition on the interval $(0,q)$, and the Corollaries 2.4 and 2.5 above can be applied in this particular case.

Mention that Corollary 2.7 above cannot be applied to the evolution family in this example, because, in the special case when $a(t) = 1$ for every $t \geq 0$, there exists at least one $x(\cdot)$ such that the trajectory $U(\cdot,0)x$ does not satisfy any Lipschitz condition on the interval $(0,q)$. Thus, our theoretical results allow us to establish the strong stability of the periodic evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ rather than its uniform exponential stability. As is well-known, the semigroup $\{T(t)\}$ is uniformly exponentially stable. Combining this with the inequality $\int_s^t a(\tau)d\tau \geq (t-s)$ for $t \geq s$, is easily to see that the evolution family $\{T(\int_s^t a(\tau)d\tau)\}_{t\geq s\geq 0}$ is uniformly exponentially stable as well.

The next example shows that the boundedness integral conditions (1.1) and (2.6) are not equivalent. More exactly, there exist semigroups which verify (2.6) and does not verify (1.1).

Example 4.2. Let $\mathbb{Z}_+$ the set of all nonnegative integers and let $H := l^2(\mathbb{Z}_+, \mathbb{C})$ endowed with the usual norm denoted by $\|\cdot\|_2$. Let $\alpha_n := -\frac{1}{n} + in$, $n \in \mathbb{Z}_+$, $x = (x_n)_{n\in\mathbb{Z}_+}$ be a sequence in H and let $T(t)x := (e^{\alpha_n t}x_n)$. Obviously, the one parameter family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous and uniformly bounded semigroup having as infinitesimal generator the "diagonal" operator defined by $((Ax)_n) := (\alpha_n x_n)$. The maximal domain of A consists by all sequences $(x_n) \in H$ which verify the condition $\sum_{n=0}^{\infty} |\alpha_n x_n|^2 < \infty$. Obviously, the semigroup $\mathbf{T}$ is not uniformly exponentially stable. Indeed, supposing the contrary, there are two positive constants K and ν such that

$$e^{\frac{-2}{N}t}|x_N|^2 \leq K^2 e^{-2\nu t}\|x\|_2^2$$

holds for every $t \geq 0$, $N \in \mathbb{Z}_+$ and $x \in H$. This provides a contradiction when N is large enough and $x_N \neq 0$. Then (1.1) is not fulfilled (cf. [15]).

In the following we prove that (2.6) is fulfilled. Let $x = (x_n)$ in $D(A)$ and μ be any real number. Then

$$\Big(\int_0^t e^{i\mu s}e^{(t-s)A}xds\Big)(n) = \frac{e^{i\mu t} - e^{-\frac{1}{n}+int}}{\frac{1}{n} + i(\mu - n)}x_n.$$

Therefore,

$$\|\int_0^t e^{i\mu s}e^{(t-s)A}x\|_2^2 \leq 2\sum_{n=1}^{\infty} n^2|x_n|^2 < 2\|Ax\|^2 \leq 2\|x\|_{D(A)}^2 < \infty,$$

i.e. (2.6) holds.

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