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BIFURCATION FROM INTERVALS FOR STURM-LIOUVILLE PROBLEMS AND ITS APPLICATIONS

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Abstract. We study the unilateral global bifurcation for the nonlinear Sturm-Liouville problem

$$\begin{gathered}-(pu')'+qu=\lambda au+af(x,u,u',\lambda)+g(x,u,u',\lambda)\quad x\in(0,1),\\ b_0u(0)+c_0u'(0)=0,\quad b_1u(1)+c_1u'(1)=0,\end{gathered}$$

where $a\in C([0,1],[0,+\infty))$ and $a(x)\not\equiv 0$ on any subinterval of $[0,1]$, $f,g\in C([0,1]\times\mathbb{R}^3,\mathbb{R})$ and f is not necessarily differentiable at the origin or infinity with respect to u. Some applications are given to nonlinear second-order two-point boundary-value problems. This article is a continuation of [8].

1. Introduction

Berestycki [1] considered a class of problems involving a non-differentiable nonlinearity. More precisely, he considered the nonlinear Sturm-Liouville problem

$$\begin{gathered}-(pu')'+qu=\lambda au+\widetilde{F}(x,u,u',\lambda)\quad x\in(0,1),\\ b_0u(0)+c_0u'(0)=0,\quad b_1u(1)+c_1u'(1)=0,\end{gathered}\tag{1.1}$$

where p is a positive, continuously differentiable function on $[0,1]$, q is a continuous function on $[0,1]$, a is a positive continuous function on $[0,1]$ and b_i, c_i are real numbers such that $|b_i|+|c_i|\neq 0$, $i=0,1$. Moreover, the nonlinear term has the form $\widetilde{F}=\widetilde{f}+\widetilde{g}$, where $\widetilde{f}$ and $\widetilde{g}$ are continuous functions on $[0,1]\times\mathbb{R}^3$, satisfying the following conditions:

- (C1) $|\frac{\widetilde{f}(x,u,s,\lambda)}{u}|\leq M$, for all $x\in[0,1]$, $0<|u|\leq 1$, $|s|\leq 1$ and all $\lambda\in\mathbb{R}$, where M is a positive constant;
- (C2) $\widetilde{g}(x,u,s,\lambda)=o(|u|+|s|)$, near $(u,s)=(0,0)$, uniformly in $x\in[0,1]$ and λ on bounded sets.

He obtained a global bifurcation result for (1.1). His result has been extended by Rynne [13] under the assumption that

$$|\widetilde{F}(x,\xi,\eta,\lambda)|\leq M_0|\xi|+M_1|\eta|,\quad (x,\xi,\eta,\lambda)\in[0,1]\times\mathbb{R}^3,$$

as either $|(\xi,\eta)|\to 0$ or $|(\xi,\eta)|\to+\infty$, for some constants M_0 and M_1. Recently, Ma and Dai [8]] improved Berestycki's result to show a unilateral global bifurcation

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result for (1.1). We refer the reader to [3, 4, 5, 7, 11, 14] and their references for information on unilateral global bifurcation.

Of course, the natural question is that what would happen if $a(x)$ is not strictly positive on $[0,1]$? The aim of this article is to consider this case. For this aim, we study the nonlinear Sturm-Liouville problem

$$\begin{gathered} -(pu')' + qu = \lambda au + af(x,u,u',\lambda) + g(x,u,u',\lambda) \quad x \in (0,1), \\ b_0u(0) + c_0u'(0) = 0, \quad b_1u(1) + c_1u'(1) = 0, \end{gathered} \tag{1.2}$$

where p, q, b_i and c_i, $i = 0,1$, are defined as above, $g \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$ satisfies (C2), a and f satisfy the following assumptions:

(C3) $a \in C([0,1],[0,+\infty))$ and $a(x) \not\equiv 0$ on any subinterval of $[0,1]$;

(C4) $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$ is continuous and satisfies $\underline{f}_0, \overline{f}_0 \in \mathbb{R}$, where

$$\underline{f}_0 = \liminf_{|s| \to 0} \frac{f(x,s,y,\lambda)}{s}, \quad \overline{f}_0 = \limsup_{|s| \to 0} \frac{f(x,s,y,\lambda)}{s}$$

uniformly in $x \in [0,1]$, $|y| \le 1$ and for all $\lambda \in \mathbb{R}$.

Under the above assumptions, we shall establish a result involving unilateral global bifurcation of (1.2). Moreover, in line with the global bifurcation from infinity of Rabinowitz [12]], we shall also establish two results involving unilateral global bifurcation of (1.2) from infinity.

Let $Lu := -(pu')' + qu$. It is well known (see [2, 6, 15]) that the linear Sturm-Liouville problem

$$\begin{gathered} Lu = \lambda au, \quad x \in (0,1), \\ b_0u(0) + c_0u'(0) = 0, \quad b_1u(1) + c_1u'(1) = 0 \end{gathered}$$

possesses infinitely many eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_k \to +\infty$, all of which are simple. The eigenfunction φ_k corresponding to λ_k has exactly $k-1$ simple zeros in $(0,1)$. In particular, if b_0, c_0, b_1 and c_1 satisfy

(C5) b_0, $-c_0$, b_1, $c_1 \in [0,+\infty)$ and $b_0c_1 - b_1c_0 + b_0b_1 > 0$,

then $\lambda_1 > 0$ (see [10, 15]).

On the basis of the unilateral global bifurcation results (Theorem 2.1–2.7), we investigate the existence of nodal solutions for the nonlinear second-order two-point boundary-value problem

$$\begin{gathered} Lu = ra(x)F(u), \quad x \in (0,1), \\ b_0u(0) + c_0u'(0) = 0, \quad b_1u(1) + c_1u'(1) = 0, \end{gathered} \tag{1.3}$$

where a satisfies (C3), $r \in (0,+\infty)$, $F \in C(\mathbb{R},\mathbb{R})$, b_i and c_i, $i = 0,1$, satisfy (C5).

In this article, we assume that the nonlinear term has the form $F = f + g$, where f and g are continuous functions on $\mathbb{R}$, satisfying the following conditions:

(C6) $\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty \in \mathbb{R}$ with $\underline{f}_0 \ne \overline{f}_0$ and $\underline{f}_\infty \ne \overline{f}_\infty$, where

$$\underline{f}_0 = \liminf_{|s| \to 0} \frac{f(s)}{s}, \quad \overline{f}_0 = \limsup_{|s| \to 0} \frac{f(s)}{s},$$
$$\underline{f}_\infty = \liminf_{|s| \to +\infty} \frac{f(s)}{s}, \quad \overline{f}_\infty = \limsup_{|s| \to +\infty} \frac{f(s)}{s}.$$

(C7) g satisfies $g(s)s > 0$ for any $s \neq 0$ and there exist $g_0, g_\infty \in (0, +\infty)$ such that
$$g_0 = \lim_{|s|\to 0} \frac{g(s)}{s}, \quad g_\infty = \lim_{|s|\to +\infty} \frac{g(s)}{s}.$$

In particular, we consider the special case of $g \equiv 0$ in (1.3); i.e., consider the problem
$$\begin{gathered} Lu = ra(x)f(u), \quad x \in (0,1), \\ b_0u(0) + c_0u'(0) = 0, \quad b_1u(1) + c_1u'(1) = 0. \end{gathered} \tag{1.4}$$
We shall establish the same results as those in [10]] and some new results for (1.4). Note that the assumption (C6) is weaker than the condition (A2) in [10]] because we do not require $\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty \in [0, +\infty)$ and $f(s)s > 0$ for $s \neq 0$ which are essential in [10]].

The rest of this article is arranged as follows. In Section 2, we establish the unilateral global bifurcation which bifurcates from the trivial solutions axis or from infinity of (1.2), respectively. In Section 3, we determine the interval of r, in which there exist nodal solutions for (1.3) or (1.4).

2. Global bifurcation from an interval

Set
$$E := \big\{u \in C^1[0,1] \big| b_0u(0) + c_0u'(0) = 0,\ b_1u(1) + c_1u'(1) = 0\big\}$$
with the norm $\|u\| = \max_{x\in[0,1]} |u(x)| + \max_{x\in[0,1]} |u'(x)|$. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e., non-degenerate) zeros in $(0,1)$ and are positive near $x = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E. We also let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$ under the product topology. Finally, we use $\mathscr{S}$ to denote the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1.2), and $\mathscr{S}_k^\pm$ to denote the subset of $\mathscr{S}$ with $u \in S_k^\pm$ and $\mathscr{S}_k = \mathscr{S}_k^+ \cup \mathscr{S}_k^-$.

Theorem 2.1. *Let $I_k = [\lambda_k - \overline{f}_0, \lambda_k - \underline{f}_0]$ for every $k \in \mathbb{N}$. The component $\mathscr{D}_k^+$ of $\mathscr{S}_k^+ \cup (I_k \times \{0\})$, containing $I_k \times \{0\}$ is unbounded and lies in $\Phi_k^+ \cup (I_k \times \{0\})$ and the component $\mathscr{D}_k^-$ of $\mathscr{S}_k^- \cup (I_k \times \{0\})$, containing $I_k \times \{0\}$ is unbounded and lies in $\Phi_k^- \cup (I_k \times \{0\})$.*

Remark 2.2. It is easy to verify that [8, Lemma 2.2] is also valid for (1.2). So if (λ, u) is a nontrivial solution of (1.2) under the assumptions of (C2)–(C4), then $u \in \cup_{k=1}^\infty S_k$.

Remark 2.3. It is not difficult to see that condition (C4) is equivalent to (C1) with $M \geq \max\{|\underline{f}_0|, |\overline{f}_0|\}$. If $a(x) > 0$ on $[0,1]$, applying [1, Theorem 1]] to problem (1.2) with $\widetilde{f} = af$, we can obtain that $\widetilde{I}_k = [\lambda_k - \widetilde{M}/a_0, \lambda_k + \widetilde{M}/a_0]$, where $a_0 = \min_{x\in[0,1]} a(x)$ and $\widetilde{M} = a^0M$ with $a^0 = \max_{x\in[0,1]} a(x)$. It is easy to verify that $I_k \subseteq \widetilde{I}_k$. So even in the case of $a(x) > 0$ on $[0,1]$, the conclusion of Theorem 2.1 is better than the corresponding ones in [1, Theorem 1], and [8, Theorem 2.1].

Consider the approximate problem
$$\begin{gathered} -(pu')' + qu = \lambda au + af(x, u|u|^\epsilon, u', \lambda) + g(x, u, u', \lambda) \quad x \in (0,1), \\ b_0u(0) + c_0u'(0) = 0, \quad b_1u(1) + c_1u'(1) = 0. \end{gathered} \tag{2.1}$$

To prove Theorem 2.1, we need the following lemma.

Lemma 2.4. *Let ϵ_n, $0 \le \epsilon_n \le 1$, be a sequence converging to 0. If there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times S_k^\nu$ such that (λ_n, u_n) is a solution of (2.1) corresponding to $\epsilon = \epsilon_n$, and (λ_n, u_n) converges to $(\lambda, 0)$ in $\mathbb{R} \times E$, then $\lambda \in I_k$.*

Proof. By an argument similar to that of [1, Lemma 1], we can show that there are two intervals (ζ_1, η_1) and (ζ_2, η_2) in $(0,1)$ such that

$$\int_{\zeta_2}^{\eta_2} (\lambda - \lambda_k) a w \varphi_k^\nu \, dx + \liminf_{n \to +\infty} \int_{\zeta_2}^{\eta_2} a(x) f_n(x) \varphi_k^\nu \, dx \le 0, \tag{2.2}$$

$$\int_{\zeta_1}^{\eta_1} (\lambda - \lambda_k) a w \varphi_k^\nu \, dx + \limsup_{n \to +\infty} \int_{\zeta_1}^{\eta_1} a(x) f_n(x) \varphi_k^\nu \, dx \ge 0, \tag{2.3}$$

where

$$f_n(x) = \frac{f(x, u_n(x)|u_n(x)|^\epsilon, u_n'(x), \lambda)}{\|u_n\|}.$$

Similar to that of [1, Lemma 1], if w and φ_k^ν have the same sign in (ζ, η), we can easily show that

$$\underline{f}_0 \int_\zeta^\eta a w \varphi_k^\nu \, dx \le \int_\zeta^\eta a f_n(x) \varphi_k^\nu \, dx \le \overline{f}_0 \int_\zeta^\eta a w \varphi_k^\nu \, dx \tag{2.4}$$

for n large enough.

It follows from (2.2) and (2.4) that

$$\int_{\zeta_2}^{\eta_2} (\lambda - \lambda_k + \underline{f}_0) a w \varphi_k^\nu \, dx \le 0,$$

hence $\lambda \le \lambda_k - \underline{f}_0$. Similarly, we from (2.3) and (2.4) we obtain $\lambda \ge \lambda_k - \overline{f}_0$. $\square$

Remark 2.5. Note that we do not need $a(x)$ is strictly positive on $[0,1]$ in the proof of Lemma 2.4 because our nonlinearity is different from that in [1]]. We put the same weight function $a(x)$ before f while this weight function is 1 in [1, 8].

Proof of Theorem 2.1. In view of Remark 2.2 and Lemma 2.4, by an argument similar to that in [8, Theorem 2.1], we can obtain easily the desired conclusions. $\square$

Instead of (C2) and (C4), we assume that f and g satisfy the following conditions:

(C8) $g(x, u, s, \lambda) = o(|u| + |s|)$, near $(u, s) = (\infty, \infty)$, uniformly in $x \in [0,1]$ and on bounded λ intervals;

(C9) $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$ is continuous and satisfies $\underline{f}_\infty, \overline{f}_\infty \in \mathbb{R}$, where

$$\underline{f}_\infty = \liminf_{|s| \to +\infty} \frac{f(x, s, y, \lambda)}{s}, \quad \overline{f}_\infty = \limsup_{|s| \to +\infty} \frac{f(x, s, y, \lambda)}{s}$$

uniformly in $x \in [0,1]$, $|y| \ge C$ for some positive constant C large enough and $\forall \lambda \in \mathbb{R}$.

We use $\mathscr{T}$ to denote the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1.2) under conditions (C3), (C8) and (C9). Applying similar methods to those in [8, Theorem 2.2 and 2.3], with obvious modifications, we obtain the following two results:

Theorem 2.6. *Let $I_k = [\lambda_k - \overline{f}_\infty, \lambda_k - \underline{f}_\infty]$ for every $k \in \mathbb{N}$. There exists a component $\mathscr{D}_k$ of $\mathscr{T} \cup (I_k \times \{\infty\})$, containing $I_k \times \{\infty\}$. Moreover if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap (\cup_{k=1}^{\infty} I_k) = I_k$ and $\mathscr{M}$ is a neighborhood of $I_k \times \{\infty\}$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

1. *$\mathscr{D}_k - \mathscr{M}$ is bounded in $\mathbb{R} \times E$ in which case $\mathscr{D}_k - \mathscr{M}$ meets $\mathscr{R} = \{(\lambda, 0) \big| \lambda \in \mathbb{R}\}$, or*
2. *$\mathscr{D}_k - \mathscr{M}$ is unbounded.*

If (2) occurs and $\mathscr{D}_k - \mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D}_k - \mathscr{M}$ meets $I_j \times \{\infty\}$ for some $j \neq k$.

Theorem 2.7. *There are two subcontinua $\mathscr{D}_k^+$ and $\mathscr{D}_k^-$, consisting of the bifurcation branch $\mathscr{D}_k$, which satisfy the alternatives of Theorem 2.6. Moreover, there exists a neighborhood $\mathscr{N} \subset \mathscr{M}$ of $I_k \times \{\infty\}$ such that $(\mathscr{D}_k^\nu \cap \mathscr{N}) \subset (\Phi_k^\nu \cup (I_k \times \{\infty\}))$ for $\nu = +$ and $\nu = -$.*

3. Applications

In this section, we shall use Theorems 2.1–2.7 to prove the existence of nodal solutions for problem (1.3) under the assumptions of (C3), (C6) and (C7).

The main results of this section are the following theorems.

Theorem 3.1. *For some $k \in \mathbb{N}$, if $g_0 > -\underline{f}_0$ and $g_\infty > -\underline{f}_\infty$, either*

$$\frac{\lambda_k}{g_0 + \underline{f}_0} < r < \frac{\lambda_k}{g_\infty + \overline{f}_\infty} \tag{3.1}$$

or

$$\frac{\lambda_k}{g_\infty + \underline{f}_\infty} < r < \frac{\lambda_k}{g_0 + \overline{f}_0}, \tag{3.2}$$

then problem (1.3) possesses two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in (0,1) and is positive near 0, and u_k^- has exactly $k-1$ zeros in (0,1) and is negative near 0.

Theorem 3.2. *For some $k \in \mathbb{N}$, if $g_0 > -\underline{f}_0$ and $-\overline{f}_\infty < g_\infty \leq -\underline{f}_\infty$, for*

$$\frac{\lambda_k}{g_0 + \underline{f}_0} < r < \frac{\lambda_k}{g_\infty + \overline{f}_\infty},$$

then the conclusion of Theorem 3.1 is valid.

Theorem 3.3. *For some $k \in \mathbb{N}$, if $g_0 > -\underline{f}_0$ and $g_\infty \leq -\overline{f}_\infty$, for*

$$r > \frac{\lambda_k}{g_0 + \underline{f}_0},$$

then the conclusion of Theorem 3.1 is valid.

Theorem 3.4. *For some $k \in \mathbb{N}$, if $-\overline{f}_0 < g_0 \leq -\underline{f}_0$ and $g_\infty > -\underline{f}_\infty$, for*

$$\frac{\lambda_k}{g_\infty + \underline{f}_\infty} < r < \frac{\lambda_k}{g_0 + \overline{f}_0},$$

then the conclusion of Theorem 3.1 is valid.

Theorem 3.5. *For some $k \in \mathbb{N}$, if $g_0 \le -\overline{f}_0$ and $g_\infty > -\underline{f}_\infty$, for*

$$r > \frac{\lambda_k}{g_\infty + \underline{f}_\infty},$$

then the conclusion of Theorem 3.1 is valid.

Proof of Theorem 3.1. Firstly, we study the bifurcation phenomena for the following eigenvalue problem

$$\begin{gathered} Lu = \lambda r a(x) g(u) + r a(x) f(u) \quad x \in (0,1), \\ b_0 u(0) + c_0 u'(0) = 0, \quad b_1 u(1) + c_1 u'(1) = 0, \end{gathered} \tag{3.3}$$

where $\lambda \in \mathbb{R}$ is a parameter. Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that

$$g(s) = g_0 s + \zeta(s) \tag{3.4}$$

with $\lim_{|s|\to 0} \zeta(s)/s = 0$. Let $\widetilde{\zeta}(u) = \max_{0\le |s|\le u} |\zeta(s)|$, then $\widetilde{\zeta}$ is nondecreasing and

$$\lim_{u\to 0^+} \frac{\widetilde{\zeta}(u)}{u} = 0. \tag{3.5}$$

Further it follows from (3.5) that

$$\frac{\zeta(u)}{\|u\|} \le \frac{\widetilde{\zeta}(|u|)}{\|u\|} \le \frac{\widetilde{\zeta}(\|u\|_\infty)}{\|u\|} \le \frac{\widetilde{\zeta}(\|u\|)}{\|u\|} \to 0 \quad \text{as } \|u\| \to 0. \tag{3.6}$$

Hence, (3.3), (3.4) and (3.6) imply that conditions (C2) and (C4) hold. Moreover, we have that

$$I_k = [\frac{\lambda_k}{r g_0} - \frac{\overline{f}_0}{g_0}, \frac{\lambda_k}{r g_0} - \frac{\underline{f}_0}{g_0}] := I_k^0.$$

Using Theorem 2.1, there exist two distinct unbounded components $\mathscr{D}_{k,0}^+$ of $\mathscr{S}_k^+ \cup (I_k^0 \times \{0\})$, containing $I_k^0 \times \{0\}$ and lying in $\Phi_k^+ \cup (I_k^0 \times \{0\})$, and $\mathscr{D}_{k,0}^-$ of $\mathscr{S}_k^- \cup (I_k^0 \times \{0\})$, containing $I_k^0 \times \{0\}$ and lying in $\Phi_k^- \cup (I_k^0 \times \{0\})$.

Next we study the unilateral global bifurcation of (3.3) which bifurcates from infinity. Let $\xi \in C(\mathbb{R}, \mathbb{R})$ be such that

$$g(s) = g_\infty s + \xi(s) \tag{3.7}$$

with $\lim_{|s|\to+\infty} \xi(s)/s = 0$. Let $\widetilde{\xi}(u) = \max_{0\le|s|\le u} |\xi(s)|$, then $\widetilde{\xi}$ is nondecreasing. Define

$$\overline{\xi}(u) = \max_{u/2 \le |s| \le u} |\xi(s)|.$$

Then we can see that

$$\lim_{u\to+\infty} \frac{\overline{\xi}(u)}{u} = 0 \quad \text{and} \quad \widetilde{\xi}(u) \le \widetilde{\xi}(\frac{u}{2}) + \overline{\xi}(u). \tag{3.8}$$

It is not difficult to verify that $\widetilde{\xi}(s)/s$ is bounded in $\mathbb{R}^+$. This fact and (3.8) follows that

$$\limsup_{u\to+\infty} \frac{\widetilde{\xi}(u)}{u} \le \limsup_{u\to+\infty} \frac{\widetilde{\xi}(u/2)}{u} = \limsup_{t\to+\infty} \frac{\widetilde{\xi}(t)}{2t},$$

where $t = u/2$. So we have

$$\lim_{u\to+\infty} \frac{\widetilde{\xi}(u)}{u} = 0. \tag{3.9}$$

Further from (3.9) it follows that

$$\frac{\xi(u)}{\|u\|} \le \frac{\widetilde{\xi}(|u|)}{\|u\|} \le \frac{\widetilde{\xi}(\|u\|_\infty)}{\|u\|} \le \frac{\widetilde{\xi}(\|u\|)}{\|u\|} \to 0 \quad \text{as } \|u\| \to +\infty. \tag{3.10}$$

Hence, (3.3), (3.7) and (3.10) imply that conditions (C8) and (C9) hold. Moreover, we have that

$$I_k = [\frac{\lambda_k}{rg_\infty} - \frac{\overline{f}_\infty}{g_\infty}, \frac{\lambda_k}{rg_\infty} - \frac{\underline{f}_\infty}{g_\infty}] := I_k^\infty.$$

Using Theorem 2.7, we have that there are two components $\mathscr{D}_{k,\infty}^+$ and $\mathscr{D}_{k,\infty}^-$ of $\mathscr{S} \cup (I_k^\infty \times \{\infty\})$, containing $I_k^\infty \times \{\infty\}$, which satisfy the alternates of Theorem 2.6. Moreover, there exists a neighborhood $\mathscr{N} \subset \mathscr{M}$ of $I_k^\infty \times \{\infty\}$ such that $(\mathscr{D}_{k,\infty}^\nu \cap \mathscr{N}) \subset (\Phi_k^\nu \cup (I_k^\infty \times \{\infty\}))$ for $\nu = +$ and $\nu = -$.

We claim that $\mathscr{D}_{k,0}^+ = \mathscr{D}_{k,\infty}^+$ and $\mathscr{D}_{k,0}^- = \mathscr{D}_{k,\infty}^-$. We only prove $\mathscr{D}_{k,0}^+ = \mathscr{D}_{k,\infty}^+$ since the proof of $\mathscr{D}_{k,0}^- = \mathscr{D}_{k,\infty}^-$ is similar. As in [8]], it suffices to show that $\mathscr{D}_{k,\infty}^+$ meets some point $(\lambda_*, 0)$ of $\mathscr{R}$; i.e., (1) of Theorem 2.6 occurs.

Suppose on the contrary that (2) of Theorem 2.6 occurs. Firstly, we shall show that $\mathscr{D}_{k,\infty}^+ - \mathscr{M}$ has a bounded projection on $\mathbb{R}$. By the same method as that of [8]], we can show that $\mathscr{D}_{k,\infty}^+ \subset \Phi_k^+$. On the contrary, we suppose that $(\mu_n, y_n) \in \mathscr{D}_{k,\infty}^+ - \mathscr{M}$ such that

$$\lim_{n\to+\infty} \mu_n = \pm\infty.$$

It follows that

$$Ly_n = \mu_n rag(y_n) + raf(y_n).$$

Let

$$0 < \tau(1, n) < \tau(2, n) < \dots < \tau(k-1, n) < 1$$

denote the zeros of y_n in $(0, 1)$. Let $\tau(0, n = 0$ and $\tau(k, n) = 1$. Then, after taking a subsequence if necessary,

$$\lim_{n\to+\infty} \tau(l, n) = \tau(l, \infty), \quad l \in \{0, 1, \dots, k\}.$$

We claim that there exists $l_0 \in \{0, 1, \dots, k\}$ such that

$$\tau(l_0, \infty) < \tau(l_0 + 1, \infty).$$

Otherwise, we have

$$1 = \Sigma_{l=0}^{k-1}(\tau(l+1, n) - \tau(l, n)) \to \Sigma_{l=0}^{k-1}(\tau(l+1, \infty) - \tau(l, \infty)) = 0.$$

This is a contradiction. Let $(\alpha, \beta) \subset (\tau(l_0, \infty), \tau(l_0 + 1, \infty))$ with $\alpha < \beta$. For all n sufficiently large, we have $(\alpha, \beta) \subset (\tau(l_0, n), \tau(l_0 + 1, n))$. So y_n does not change its sign in (α, β). In view of (C6) and (C7), we have that $\lim_{n\to+\infty} r(\mu_n \frac{g(y_n(x))}{y_n(x)} + \frac{f(y_n(x))}{y_n(x)}) = \pm\infty$ for any $x \in (\alpha, \beta)$. If $\lim_{n\to+\infty} r(\mu_n \frac{g(y_n(x))}{y_n(x)} + \frac{f(y_n(x))}{y_n(x)}) = -\infty$ for any $x \in (\alpha, \beta)$, applying Sturm Comparison Theorem [6, 15] to y_n and φ_1 on $[\alpha, \beta]$, we can get that φ_1 must change its sign in (α, β) for n large enough. While, this is impossible. So we have that $\lim_{n\to+\infty}(\mu_n r \frac{g(y_n(x))}{y_n(x)} + r\frac{f(y_n(x))}{y_n(x)}) = +\infty$ for any $x \in (\alpha, \beta)$. Applying Sturm Comparison Theorem [6, 15]] to φ_1 and y_n, we get that y_n has at least one zero in (α, β) for n large enough, and this contradicts the fact that y_n does not change its sign in (α, β). By an argument similar to that of [8, Theorem 3.1], we can show that the case of $\mathscr{D}_{k,\infty}^+ - \mathscr{M}$ meeting $I_j^\infty \times \{\infty\}$ for some $j \neq k$ is impossible. This is a contradiction.

For simplicity, we write $\mathscr{D}_k^+ := \mathscr{D}_{k,0}^+ = \mathscr{D}_{k,\infty}^+$ and $\mathscr{D}_k^- := \mathscr{D}_{k,0}^- = \mathscr{D}_{k,\infty}^-$. It is clear that any solution of (3.3) of the form $(1, u)$ yields a solution u of (1.3). While, by some simple computations, we can show that assumption (3.1) or (3.2) implies that $\mathscr{D}_k^+$ and $\mathscr{D}_k^-$ cross the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. □

Geometric meaning. The meaning of $\frac{\lambda_k}{rg_0} - \frac{\underline{f}_0}{g_0} < 1 < \frac{\lambda_k}{rg_\infty} - \frac{\overline{f}_\infty}{g_\infty}$ is that subsets $I_k^0 \times E$ and $I_k^\infty \times E$ of $\mathbb{R} \times E$ can be separated by the hyperplane $\{1\} \times E$, and $I_k^0 \times E$ on the left of $\{1\} \times E$ while $I_k^\infty \times E$ on the right of $\{1\} \times E$. Similarly, the meaning of $\frac{\lambda_k}{rg_\infty} - \frac{\underline{f}_\infty}{g_\infty} < 1 < \frac{\lambda_k}{rg_0} - \frac{\overline{f}_0}{g_0}$ is that $I_k^0 \times E$ on the right of $\{1\} \times E$ and $I_k^\infty \times E$ on the left of $\{1\} \times E$.

Proof of Theorems 3.2 and 3.3. The proof is similar to that of Theorem 3.1, we note only that the assumptions of theorems imply $\frac{\lambda_k}{rg_0} - \frac{\underline{f}_0}{g_0} < 1 < \frac{\lambda_k}{rg_\infty} - \frac{\overline{f}_\infty}{g_\infty}$. □

Proof of Theorem 3.4 and 3.5. We note that the assumptions of these theorems imply $\frac{\lambda_k}{rg_\infty} - \frac{\underline{f}_\infty}{g_\infty} < 1 < \frac{\lambda_k}{rg_0} - \frac{\overline{f}_0}{g_0}$. □

Remark 3.6. By some simple computations, we can show that if g_0 and g_∞ satisfy one of the following two cases

- $-\overline{f}_0 < g_0 \le -\underline{f}_0$ and $g_\infty \le -\underline{f}_\infty$, or
- $g_0 \le -\overline{f}_0$ and $g_\infty \le -\underline{f}_\infty$,

then subsets $I_k^0 \times E$ and $I_k^\infty \times E$ of $\mathbb{R} \times E$ cannot be separated by the hyperplane $\{1\} \times E$. So we cannot give a suitable interval of r in which there exist nodal solutions for (1.3) in the above two cases. It would be interesting to have more information about these two cases.

By arguments similar to those of Theorem 3.1–3.5, we can obtain the following more general results.

Theorem 3.7. *For some $k, n \in \mathbb{N}$ with $k \le n$, if $g_0 > -\underline{f}_0$ and $g_\infty > -\underline{f}_\infty$, either*

$$\frac{\lambda_n}{g_0 + \underline{f}_0} < r < \frac{\lambda_k}{g_\infty + \overline{f}_\infty}$$

or

$$\frac{\lambda_n}{g_\infty + \underline{f}_\infty} < r < \frac{\lambda_k}{g_0 + \overline{f}_0},$$

then problem (1.3) *possesses two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in (0,1) and is positive near 0, and u_k^- has exactly $k-1$ zeros in (0,1) and is negative near 0*

Theorem 3.8. *For some $k, n \in \mathbb{N}$ with $k \le n$, if $g_0 > -\underline{f}_0$ and $-\overline{f}_\infty < g_\infty \le -\underline{f}_\infty$, for*

$$\frac{\lambda_n}{g_0 + \underline{f}_0} < r < \frac{\lambda_k}{g_\infty + \overline{f}_\infty},$$

then the conclusion of Theorem 3.4 is valid.

Theorem 3.9. *For some $k \in \mathbb{N}$, if $-\overline{f}_0 < g_0 \le -\underline{f}_0$ and $g_\infty > -\underline{f}_\infty$, for*

$$\frac{\lambda_n}{g_\infty + \underline{f}_\infty} < r < \frac{\lambda_k}{g_0 + \overline{f}_0},$$

then the conclusion of Theorem 3.1 is valid.

Remark 3.10. In view of Remark 2.3, the intervals obtained in Theorem 3.1, 3.2, 3.4, 3.7, 3.8 and 3.9 contain the corresponding intervals in [8, Theorem 3.1–3.6]] in the case of $p \equiv 1$, $q \equiv 0$, $b_0 = b_1 = 1$ and $c_0 = c_1 = 0$. So the results of Theorems 3.1–3.9 are more general than the corresponding ones of [8].

Next, we study problem (1.4). For any function $g \in C(\mathbb{R}, \mathbb{R})$ such that it satisfies (C7), we construct the new problem

$$\begin{gathered} Lu = ra(x)(\widehat{f}(u) + g(u)), \quad x \in (0,1), \\ b_0u(0) + c_0u'(0) = 0, \quad b_1u(1) + c_1u'(1) = 0, \end{gathered} \tag{3.11}$$

where $\widehat{f} = f - g$. Clearly, problem (1.4) is equivalent to problem (3.11). In addition, it is easy to see that $\underline{\widehat{f}}_0 = \underline{f}_0 - g_0$, $\overline{\widehat{f}}_0 = \overline{f}_0 - g_0$, $\underline{\widehat{f}}_\infty = \underline{f}_\infty - g_\infty$ and $\overline{\widehat{f}}_\infty = \overline{f}_\infty - g_\infty$. Applying Theorems 3.7–3.9 to problem (3.11), we obtain the following corollaries.

Corollary 3.11. *For some $k, n \in \mathbb{N}$ with $k \le n$, if $\underline{f}_0 > 0$ and $\underline{f}_\infty > 0$, either*

$$\frac{\lambda_n}{\underline{f}_0} < r < \frac{\lambda_k}{\overline{f}_\infty}$$

or

$$\frac{\lambda_n}{\underline{f}_\infty} < r < \frac{\lambda_k}{\overline{f}_0},$$

then problem (1.4) possesses two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in (0,1) and is positive near 0, and u_k^- has exactly $k-1$ zeros in (0,1) and is negative near 0.

Corollary 3.12. *For some $k, n \in \mathbb{N}$ with $k \le n$, if $\underline{f}_0 > 0$ and $\overline{f}_\infty > 0 \ge \underline{f}_\infty$, for*

$$\frac{\lambda_n}{\underline{f}_0} < r < \frac{\lambda_k}{\overline{f}_\infty},$$

then the conclusion of Corollary 3.11 is valid.

Corollary 3.13. *For some $k, n \in \mathbb{N}$ with $k \le n$, if $\underline{f}_0 > 0$ and $\overline{f}_\infty \le 0$, for*

$$r > \frac{\lambda_n}{\underline{f}_0},$$

then the conclusion of Corollary 3.11 is valid.

Corollary 3.14. *For some $k, n \in \mathbb{N}$ with $k \le n$, if $\overline{f}_0 > 0 \ge \underline{f}_0$ and $\underline{f}_\infty > 0$, for*

$$\frac{\lambda_n}{\underline{f}_\infty} < r < \frac{\lambda_k}{\overline{f}_0},$$

the conclusion of Corollary 3.11 is valid.

Corollary 3.15. *For some $k, n \in \mathbb{N}$ with $k \le n$, if $\overline{f}_0 \le 0$ and $\underline{f}_\infty > 0$, for*

$$r > \frac{\lambda_n}{\underline{f}_\infty},$$

then the conclusion of Corollary 3.11 is valid.

Remark 3.16. If $r = 1$ and $n = k + j$ for any $j \in \{0\} \cup \mathbb{N}$, then Corollary 3.11 reduces to [10, Theorem 2]. If $n = k$, then Corollary 3.11 becomes [10, Corollary 1]. If $n = k$, $\underline{f}_0 = \overline{f}_0$, $\underline{f}_\infty = \overline{f}_\infty$, $p \equiv 1$, $q \equiv 0$, $b_0 = b_1 = 1$ and $c_0 = c_1 = 0$, then Corollary 3.11 reduces to [9, Theorem 1.1]. Note that signum condition is removed in this paper while it is essential in [9, 10]].

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