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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A QUASILINEAR SCHRÖDINGER EQUATIONS WITH SIGN-CHANGING POTENTIAL

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Abstract. In this article we consider the quasilinear Schrödinger equation where the potential is sign-changing. We employ a mountain pass argument without compactness conditions to obtain the existence of a nontrivial solution.

1. Introduction

In this paper we are concerned with the existence of a nontrivial solution for the quasilinear Schrödinger equation

$$-\Delta u+V(x)u-\Delta(u^2)u=f(x,u),\quad x\in\mathbb{R}^N \tag{1.1}$$

These type of equations come from the study of the standing wave solutions of quasilinear Schrödinger equations derived as models for several physical phenomena; see [12]. The case $\inf_{\mathbb{R}^N}V(x)>0$ has been extensively studied in recent years. However, to our best knowledge, there is no result for the other important case $\inf_{\mathbb{R}^N}V(x)<0$. The various methods developed for the quasilinear Schrödinger equations do not seem to apply directly in this case.

In this article, we assume that the potential is sign-changing and the nonlinearity is more general than in other articles. Some authors recover the compactness by assuming that the potential $V(x)$ is either coercive or has radial symmetry, see [4, 9, 11, 12]. Here we do not need the compactness, but we assume the potential bounded from above, but may be unbounded from below. We consider the case $\mathbb{N}\geq 3$. This work is motivated by the ideas in [3, 14, 15, 18].

First we consider the problem

$$-\Delta u+V(x)u-\Delta(u^2)u=g(x,u)+h(x),\quad u\in H^1(\mathbb{R}^N). \tag{1.2}$$

We suppose that V and g satisfy the following assumptions:

- (G1) g is continuous, and $|g(x,u)|\leq a(1+|u|^{p-1})$ for some $a>0$ and $4<p<2\cdot 2^*$, where $2^*:=2N/(N-2)$.
- (G2) $g(x,u)=o(u)$ uniformly in x as $u\to 0$.

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- (G3) There exists $\theta > 4$ such that $0 < \theta G(x,u) \le g(x,u)u$, for $x \in \mathbb{R}^N$, $u \in \mathbb{R}\backslash\{0\}$, where $G(x,u) := \int_0^u g(x,s)ds$.
- (V1) $V(x)$ is sign-changing, $V^+(x) \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to\infty} V^+(x) = a_0 > 0$ and $|V^-|_{L^{N/2}(\mathbb{R}^N)} < \frac{\theta-4}{S(\theta-2)}$, where $V^\pm(x) := \max\{\pm V(x), 0\}$, and S denotes the Sobolev optimal constant.
- (V2) $\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 > 0$ for every $u \in E \setminus \{0\}$.
- (H1) $h \neq 0$ and $|h|_{L^{2N/(N+2)}} < S^{-1/2}k\rho$, where k and ρ are given in Lemma 3.2.

We remark that (V2) means that either V^- is small or V^- has a small support. It is also easy to give a concrete condition on V^- satisfying (V2). Note that (H1) is similar to condition (H8) in [18]. Let

$$E := \big\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+(x)u^2dx < \infty\big\},$$

with norm $\|u\| := \big(\int_{\mathbb{R}^N} |\nabla u|^2 + V^+(x)u^2\big)^{1/2}$. It is obvious that $\|\cdot\|$ is an equivalent norm with the standard one by (V1). In Section 3 we prove one of the main result of this article:

Theorem 1.1. *Suppose that* (V1), (V2), (G1)–(G3), (H1) *are satisfied. Then* (1.1) *has a nontrivial solution.*

Remark 1.2. Under stronger conditions on V, we can derive the existence of a nontrivial solution without the small perturbation term h. For example, if V is continuous, 1-periodic in x_i, $1 \le i \le N$, and there exists a constant $a_0 > 0$ such that $V(x) \ge a_0$ for all $x \in \mathbb{R}^N$. It is obvious that the condition in Theorem 1.1 is satisfied, so we have a bounded $(C)_c$ sequence by the proof of Theorem 1.1. Similarly as in [14, Lemma 1.2], under a translation if necessary, we get a nontrivial solution.

Also, we consider the problem

$$-\Delta u + V(x)u - \Delta(u^2)u = g(u), \quad u \in H^1(\mathbb{R}^N), \tag{1.3}$$

where the nonlinearity g satisfies (G1)–(G3). We assume that

- (V1') $V(x)$ is sign-changing, $\lim_{|x|\to\infty} V^+(x) = V^+(\infty) > 0$, $V^+(x) \le V^+(\infty)$ on $\mathbb{R}^N$ and $|V^-|_{L^{N/2}(\mathbb{R}^N)} < (\theta-4)/(S(\theta-2))$.

Note that (V1') implies (V1). The second main result of this paper, which we prove in Section 4, is the following.

Theorem 1.3. *Suppose that* (V1'), (V2) *are satisfied. Then* (1.3) *admits a nontrivial solution.*

Remark 1.4. We would like to point out that (V1') is weaker than the assumptions (V0) and (V1) in [3]. But they obtain the existence of a positive solution, while we do not.

Positive constants will be denoted by $C, C_1, C_2, \dots$, while $|A|$ will denote the Lebesgue measure of a set $A \subset \mathbb{R}^N$.

2. Preliminary results

We observe that (1.1) is the Euler-Lagrange equation associated with the energy functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} (1+2u^2)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} (G(x,u) + h(x)u). \tag{2.1}$$

To use the usual argument, we make a change of variables $v := f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{(1+2f^2(t))^{1/2}} \text{on } [0,+\infty) \quad \text{and} \quad f(t) = -f(-t) \text{on } (-\infty, 0].$$

Below we summarize the properties of f, whose can be found in [3, 6, 7].

Lemma 2.1. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*
(2) *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*
(3) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
(4) *$f(t)/t \to 1$ as $t \to 0$;*
(5) *$f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$;*
(6) *$f(t)/2 \le tf'(t) \le f(t)$ for all $t \in \mathbb{R}$;*
(7) *$|f(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*
(8) *$f^2(t) - f(t)f'(t)t \ge 0$ for all $t \in \mathbb{R}$;*
(9) *there exists a positive constant C such that $|f(t)| \ge C|t|$ for $|t| \le 1$ and $|f(t)| \ge C|t|^{1/2}$ for $|t| \ge 1$;*
(10) *$|f(t)f'(t)| < 1/\sqrt{2}$ for all $t \in \mathbb{R}$.*

Consider the functional

$$I(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \int_{\mathbb{R}^N} (G(x, f(v)) + h(x)f(v)).$$

Then I is well-defined on E and $I \in C^1(E,\mathbb{R})$ under the hypotheses (V1), (G1) and (G2). It is easy to see that

$$\langle I'(v), w\rangle = \int_{\mathbb{R}^N} \nabla v \nabla w + \int_{\mathbb{R}^N} V(x)f(v)f'(v)w - \int_{\mathbb{R}^N} (g(x, f(v)) + h(x))f'(v)w$$

for all $v, w \in E$ and the critical points of I are weak solutions of the problem

$$-\Delta v + V(x)f(v)f'(v) = (g(x, f(v)) + h(x))f'(v), \quad v \in E.$$

If $v \in E$ is a critical point of the functional I, then $u = f(v) \in E$ and u is a solution of (1.1) (cf: [3]).

3. Proof of Theorem 1.1

In the following we assume that (V1), (V2), (G1)–(G3) and (H1) are satisfied. First, (G1) and (G2) imply that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|g(x, u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \quad \text{for all } u \in \mathbb{R}. \tag{3.1}$$

Lemma 3.1. *There exist $\xi, \alpha > 0$ such that $\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(x)f^2(u) \ge \alpha\|u\|^2$, if $\|u\| = \xi$.*

Proof. Arguing by contradiction, there exist $u_n \to 0$ in E, such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + V(x)\frac{f^2(u_n)}{u_n^2}v_n^2 \to 0.$$

where $v_n := \frac{u_n}{\|u_n\|}$. We have that $u_n \to 0$ in $L^2(\mathbb{R}^N)$, $u_n \to 0$ a.e., $v_n \rightharpoonup v$ in E, $v_n \to v$ in L^2_{loc}, $v_n \to v$ a.e. up to a subsequence.

If $v \neq 0$, then we claim that
$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 + V(x)\frac{f^2(u_n)}{u_n^2}v_n^2 \geq \int_{\mathbb{R}^N} |\nabla v|^2 + V(x)v^2.$$
Indeed, we have
$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} V^+(x)\frac{f^2(u_n)}{u_n^2}v_n^2 \geq \int_{\mathbb{R}^N} V^+(x)v^2$$
due to Fatou's lemma and Lemma 2.1-(4). Since $v_n^2 \rightharpoonup v^2$ in $L^{N/(N-2)}$ and $V^-(x) \in L^{N/2}$, we obtain
$$\int_{\mathbb{R}^N} V^-(x)\frac{f^2(u_n)}{u_n^2}v_n^2 \leq \int_{\mathbb{R}^N} V^-(x)v_n^2 \to \int_{\mathbb{R}^N} V^-(x)v^2$$
by Lemma 2.1(3) and the definition of the weak convergence. We have a contradiction to (V2).

The other case is $v = 0$. Note that $\lim_{n\to\infty} \int_{\mathbb{R}^N} V^-(x)\frac{f^2(u_n)}{u_n^2}v_n^2 = 0$, then
$$\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V^+(x)v_n^2) + \int_{\mathbb{R}^N} V^+(x)\Big(\frac{f^2(u_n)}{u_n^2} - 1\Big)v_n^2 \to 0.$$
We use a similar argument as in [8, Lemma 3.3]. Since $u_n \to 0$ in $L^2(\mathbb{R}^N)$, for every $\varepsilon > 0$, $|\{x \in \mathbb{R}^N : |u_n(x)| > \varepsilon\}| \to 0$ as $n \to \infty$. We have by (V1), Lemma 2.1(3) and the Hölder inequality,
$$\begin{aligned} \Big|\int_{|u_n|>\varepsilon} V^+(x)\Big(\frac{f^2(u_n)}{u_n^2} - 1\Big)v_n^2\Big| &\leq C\int_{|u_n|>\varepsilon} v_n^2 \\ &\leq |\{x \in \mathbb{R}^N : |u_n(x)| > \varepsilon\}|^{2/N}|v_n|_{2^*}^2 \to 0. \end{aligned}$$
Now it follows from Lemma 2.1(4) and $\int_{\mathbb{R}^N} V^+(x)v_n^2 \leq C_1$ that
$$\int_{|u_n|<\varepsilon} V^+(x)\Big(\frac{f^2(u_n)}{u_n^2} - 1\Big)v_n^2$$
is small as ε is small. So $v_n \to 0$ in E which contradicts to $\|v_n\| = 1$. We finish the proof. □

Lemma 3.2. *There exist $k, \rho > 0$(small) such that $\inf_{\|u\|=\rho} I_1(u) \geq k\rho^2$, where $I_1(u) := I(u) + \int_{\mathbb{R}^N} h(x)f(u)$.*

Proof. Due to (G1) and (G2), we have for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that $|g(x,u)| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1}$. So it follows from a standard argument by Lemma (2.1)(3),(7) and Lemma 3.1 that $I_1(u) \geq k\|u\|^2 = k\rho^2$. □

Lemma 3.3. *For the above ρ, $\inf_{\|u\|=\rho} I(u) > 0$.*

Proof. By Lemma 3.2 and Lemma 2.1-(3), we derive
$$\begin{aligned} I(u) &\geq k\|u\|^2 - \int_{\mathbb{R}^N} h(x)f(u) \\ &\geq k\|u\|^2 - |h|_{L^{2N/(N+2)}}S^{1/2}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^{1/2} \\ &\geq k\|u\|^2 - |h|_{L^{2N/(N+2)}}S^{1/2}\|u\| \\ &= \|u\|(k\|u\| - |h|_{L^{2N/(N+2)}}S^{1/2}) > 0 \end{aligned}$$

□

Lemma 3.4. *There exists $u_0 \neq 0$, such that $I(u_0) \leq 0$.*

Proof. We have by condition (G3) and Lemma 2.1(3),

$$\int_{u\neq 0} \frac{G(x, f(tu))}{t^4} = \int_{u\neq 0} \frac{G(x, f(tu))}{f^4(tu)} \frac{f^4(tu)}{t^4u^4} u^4 \to \infty.$$

Hence $\lim_{t\to\infty} \frac{I(tu)}{t^4} = -\infty$. □

Since the functional I satisfies the mountain pass geometry, the $(C)_c$ sequence exists, where $c := \inf_{r\in\Gamma} \max_{t\in[0,1]} I(r(t))$ and $\Gamma := \{r \in C([0, 1], E) : r(0) = 0, r(1) = u_0\}$.

Lemma 3.5. *The $(C)_c$ sequence (u_n) is bounded.*

Proof. We employ a similar argument as in [14, Lemma 3.3]. First we claim

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V^+(x)f^2(u_n) \leq C_1.$$

Indeed, we have

$$I(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)f^2(u_n) - \int_{\mathbb{R}^N} (G(x, f(u_n)) + h(x)f(u_n)) \to c\,,$$

$$\begin{aligned} I'(u_n)u_n &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)f(u_n)f'(u_n)u_n \\ &\quad - \int_{\mathbb{R}^N} (g(x, f(u_n)) + h(x))f'(u_n)u_n \to 0\,. \end{aligned}$$

Hence

$$I(u_n) - \frac{2}{\theta} I'(u_n)u_n = c + o(1).$$

By Lemma 2.1(6),(3) and (G3) we obtain

$$\begin{aligned} &C_2 + C_3\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^{1/2} \\ &\geq C_2 + (1 - \frac{1}{\theta})\int_{\mathbb{R}^N} h(x)f(u_n) \\ &\geq (\frac{1}{2} - \frac{2}{\theta})\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + V^+(x)f^2(u_n)\Big) - (\frac{1}{2} - \frac{1}{\theta})\int_{\mathbb{R}^N} V^-(x)f^2(u_n) \\ &\geq (\frac{1}{2} - \frac{2}{\theta})\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + V^+(x)f^2(u_n)\Big) - (\frac{1}{2} - \frac{1}{\theta})\int_{\mathbb{R}^N} V^-(x)u_n^2 \\ &\geq (\frac{1}{2} - \frac{2}{\theta})\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + V^+(x)f^2(u_n)\Big) - (\frac{1}{2} - \frac{1}{\theta})|V^-|_{L^{N/2}} S\int_{\mathbb{R}^N} |\nabla u_n|^2. \end{aligned}$$

It follows from (V1) that $(\frac{1}{2} - \frac{2}{\theta}) - (\frac{1}{2} - \frac{1}{\theta})|V^-|_{L^{N/2}} S > 0$. The claim is proved.

To prove that (u_n) is bounded in E, we only need to show that $\int_{\mathbb{R}^N} V^+(x)u_n^2$ is bounded. Due to Lemma 2.1(9), (V1) and the Sobolev embedding theorem, there exists $C > 0$ such that

$$\int_{|u_n|\leq 1} V^+(x)u_n^2 \leq \frac{1}{C^2}\int_{|u_n|\leq 1} V^+(x)f^2(u_n) \leq C_3$$

and

$$\int_{|u_n|\geq 1} V^+(x)u_n^2 \leq C_4 \int_{|u_n|\geq 1} u_n^{2^*} \leq C_4\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^{2^*/2} \leq C_5.$$

□

Proof of Theorem 1.1. Assume that (u_n) is a $(C)_c$ sequence. Then (u_n) is bounded by Lemma 3.5. Going if necessary to a subsequence, $u_n \rightharpoonup u$ in E. It is obvious that $I'(u) = 0$, and $u \neq 0$. The proof is complete. □

4. Proof of Theorem 1.3

In this section we look for nontrivial critical points of the functional $I_1 : E \to R$ given by

$$I_1(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(u) - \int_{\mathbb{R}^N} G(f(u)),$$

where $G(u) := \int_0^u g(s)ds$. And we also denote the corresponding limiting functional

$$\tilde{I}_1(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V^+(\infty)f^2(u) - \int_{\mathbb{R}^N} G(f(u)).$$

Lemma 4.1. *If $\{v_n\} \subset E$ is a bounded Palais-Smale sequence for I_1 at level $c > 0$, then, up to a subsequence, $v_n \rightharpoonup v \neq 0$ with $I_1'(v) = 0$.*

Proof. Since $\{v_n\}$ is bounded, going if necessary to a subsequence, $v_n \rightharpoonup v$ in E. It is obvious that $I_1'(v) = 0$. If $v \neq 0$, then the proof is complete.

If $v = 0$, we claim that $\{v_n\}$ is also a Palais-Smale sequence for $\tilde{I}_1$. Indeed,

$$\tilde{I}_1(v_n) - I_1(v_n) = \int_{\mathbb{R}^N} (V^+(\infty) - V^+(x))f^2(v_n) + \int_{\mathbb{R}^N} V^-(x)f^2(v_n) \to 0,$$

by (V1'), Lemma 2.1(3) and $v_n^2 \rightharpoonup 0$ in $L^{N/(N-2)}$. Similarly we derive

$$\begin{aligned}\sup_{\|u\|\leq 1} |\langle \tilde{I}_1'(v_n) - I_1'(v_n), u\rangle| &= \sup_{\|u\|\leq 1} \Big|\int_{\mathbb{R}^N} (V^+(\infty) - V^+(x))f(v_n)f'(v_n)u\Big| \\ &\quad + \sup_{\|u\|\leq 1} \Big|\int_{\mathbb{R}^N} V^-(x)f(v_n)f'(v_n)u\Big| \to 0.\end{aligned}$$

In the following we use a similar argument as in [3, lemma 4.3]. If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} v_n^2 dx = 0$$

for all $R > 0$, then we obtain a contradiction with the fact that $I_1(v_n) \to c > 0$. So there exist $\alpha > 0$, $R < \infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_R(y^n)} v_n^2 dx \geq \alpha > 0.$$

Denote $\tilde{v}_n(x) = v_n(x + y_n)$, then $\{\tilde{v}_n(x)\}$ is also a Palais-Smale sequence for $\tilde{I}_1$. We have that $\tilde{v}_n \rightharpoonup \tilde{v}$ and $\tilde{I}_1(\tilde{v}) = 0$ with $\tilde{v} \neq 0$. We obtain

$$c = \limsup_{n\to\infty} [\tilde{I}(\tilde{v}_n) - \frac{1}{2}\tilde{I}'(\tilde{v}_n)\tilde{v}_n] \geq \tilde{I}(\tilde{v}) - \frac{1}{2}\tilde{I}'(\tilde{v})\tilde{v} = \tilde{I}(\tilde{v}),$$

by Fatou's lemma. We could find a path $r(t) \in \Gamma$ such that $r(t)(x) > 0$ for all $x \in \mathbb{R}^N$, and all $t \in (0,1]$, $\tilde{\omega} \in r([0,1])$ and $\max_{t\in[0,1]} \tilde{I}_1(r(t)) = \tilde{I}_1(\tilde{\omega}) \le c$. Thus $I_1(r(t)) < \tilde{I}_1(r(t))$ for all $t \in (0,1]$, and then

$$c \le \max_{t\in[0,1]} I_1(r(t)) < \max_{t\in[0,1]} \tilde{I}_1(r(t)) \le c,$$

a contradiction. □

Proof of Theorem 1.3. The argument is the same as in [3]. By Lemmas 3.2 and 3.4, the functional I_1 has a mountain pass geometry. So the $(C)_c$-sequence $\{u_n\}$ exists, where $c := \inf_{r\in\Gamma} \max_{t\in[0,1]} I_1(r(t))$ and $\Gamma := \{r \in C([0,1], E) : r(0) = 0, I_1(r(1)) < 0\}$. It follows from Lemma 3.5 that $\{u_n\}$ is bounded. Hence $\{u_n\}$ is a bounded Palais-Smale sequence for I_1 at level $c > 0$. Due to Lemma 4.1, we have $I_1'(v) = 0$ and $v \ne 0$. □

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