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# ASYNCHRONOUS EXPONENTIAL GROWTH OF A BACTERIAL POPULATION

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*Dedicated to Samir Noui Mehidi who died prematurely*

ABSTRACT. In this work, we complete a study started earlier in [1, 2] wherein a model of growing bacterial population has been the matter of a mathematical analysis. We show that the full model is governed by a strongly continuous semigroup. Beside the positivity and the irreducibility of the generated semigroup, we describe its asymptotic behavior in the uniform topology which leads to the asynchronous exponential growth of the bacterial population.

## 1. INTRODUCTION

In this work, we continue a study started earlier in [1, 2] wherein a model of growing bacterial population, originally proposed in [6], has been the matter of a mathematical analysis. We have then considered a bacterial population in which, each bacteria is distinguished by its degree of maturity $0 \leq \mu \leq 1$ and its maturation velocity $0 < v < \infty$. The degree of maturity of a daughter bacteria is $\mu = 0$ while the degree of maturity of a mother bacteria is $\mu = 1$. If $f = f(t, \mu, v)$ denotes the bacterial density with respect to the degree of maturity $\mu$ and the maturation velocity $v$ at time $t$, then

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \mu} - \sigma(\mu, v)f + \int_0^\infty r(\mu, v, v')f(t, \mu, v')dv' \tag{1.1}$$

where, $r(\mu, v, v')$ stands for the *transition rate* at which bacteria change their velocity from $v'$ to $v$, while $\sigma(\mu, v)$ denotes the bacterial *mortality rate* or bacteria loss due to causes other than division.

In most observed bacterial populations, there is often a correlation between the maturation velocity of a bacteria mother $v'$ and that of its bacteria daughter $v$. So, let us consider a correlation whose kernel is $\alpha(v)\beta(v')$. The bacterial mitotic obeys then to the biological *transition law* mathematically described by

$$vf(t, 0, v) = p\alpha(v) \int_0^\infty \beta(v')f(t, 1, v')v'dv' \tag{1.2}$$

where, $p \geq 0$ denotes the average number of bacteria daughter viable per mitotic.

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In this article, we focus on the full model (1.1),(1.2). We consider then the suitable framework $L_1 := L_1(\Omega)$ $(\Omega := (0,1) \times (0,\infty))$ whose norm

$$\|f(t,\cdot,\cdot)\|_1 = \int_0^1 \int_0^\infty |f(t,\mu,v)|\, d\mu\, dv$$

denotes the bacteria number at time $t$. We firstly show that the full model (1.1),(1.2) is governed by a strongly continuous semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ on $L_1$. Beside the positivity and the irreducibility of the generated semigroup, we also prove that $\omega_{\rm ess}(\mathbb{U}_K) < \omega_0(\mathbb{U}_K)$. Consequently, a lot of interesting properties of the generated semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ can be listed. The most interesting one is the asymptotic behavior described in the uniform topology of $L_1$ and therefore for any initial data into $L_1$ as follows.

**Lemma 1.1** ([4, Theorems 9.10 and 9.11])**.** *Let $\mathbb{U} = (\mathbb{U}(t))_{t\geq 0}$ be a positive and irreducible strongly continuous semigroup, on the Banach lattice space $X$, satisfying the inequality $\omega_{\rm ess}(\mathbb{U}) < \omega_0(\mathbb{U})$. Then, there exist a rank one projector $\mathbb{P}$ into $X$ and an $\varepsilon > 0$ such that : for any $\eta \in (0,\varepsilon)$, there exists $M_\eta \geq 1$ satisfying*

$$\|e^{-\omega_0(\mathbb{U})t}\mathbb{U}(t) - \mathbb{P}\|_{\mathcal{L}(X)} \leq M_\eta e^{-\eta t} \quad t \geq 0.$$

Thanks to [4, Thereom 8.7], the projector $\mathbb{P}$ can be written as $\mathbb{P}\varphi = \langle \varphi, \varphi_0^* \rangle \varphi_0$, where, $\varphi_0 \in X_+$ is a quasi-interior vector and $\varphi_0^* \in (X^*)_+$ is a strictly positive functional such that $\langle \varphi_0, \varphi_0^* \rangle = 1$. A strongly continuous semigroup $\mathbb{U} = (\mathbb{U}(t))_{t\geq 0}$ fulfilling Lemma 1.1 possesses the *asynchronous exponential growth property* with the *intrinsic growth density* $\varphi_0^*$. Moreover, Lemma 1.1 describes the bacterial profile whose privileged direction is mathematically interpreted by the quasi-interior vector $\varphi_0$. This is what the biologist observes in his laboratory.

## 2. Preliminaries

Let $Y_1$, $Z_1$ and $Z_\infty^\omega$ $(\omega \geq 0)$ be the following Banach spaces

$$Y_1 := L^1(\mathbb{R}_+,\ vdv) \quad \text{whose norm is } \|\psi\|_{Y_1} = \int_0^\infty |\psi(v)| v\, dv$$

$$Z_1 := L^1(\mathbb{R}_+,\ dv) \quad \text{whose norm is } \|\psi\|_{Z_1} = \int_0^\infty |\psi(v)|\, dv$$

$$Z_\infty^\omega := L^\infty(\omega,\infty) \quad \text{whose norm is } \|\psi\|_{Z_\infty^\omega} = \operatorname{ess\,sup}_{v\geq\omega} |\psi(v)|$$

and let $K$ be the following linear operator

$$K\psi(v) := p\frac{\alpha(v)}{v}\int_0^\infty \beta(v')\psi(v')v'dv'$$

where, $p \geq 0$ and $\alpha$ and $\beta$ are subject to the following assumptions

$(H^1_{\alpha,\beta})$ $\|\alpha\|_{Z_1} < \infty$ and $\|\beta\|_{Z^0_\infty} < \infty$

$(H'^1_{\alpha,\beta})$ There exists $\omega_0 \geq 0$ such that $p\|\alpha\|_{Z_1}\|\beta\|_{Z^{\omega_0}_\infty} < 1$

$(H^2_{\alpha,\beta})$ $\alpha \geq 0$ and $\beta \geq 0$

$(H'^2_{\alpha,\beta})$ $\alpha(v) > 0$ and $\beta(v) > 0$ for almost all $v \geq 0$.

The most interesting properties of the operator $K$ are listed as follows.

**Lemma 2.1.** *Suppose that $p \geq 0$. If $(H^1_{\alpha,\beta})$ holds, then $K$ is a compact linear operator from $Y_1$ into itself, whose norm is*

$$\|K\|_{\mathcal{L}(Y_1)} = p\|\alpha\|_{Z_1}\|\beta\|_{Z^0_\infty}. \tag{2.1}$$

*Furthermore,*

(1) *If* $(H'^1_{\alpha,\beta})$ *holds, then* $\|K\mathbb{I}_\omega\psi\|_{Y_1} < 1$ *for all* $\omega > \overline{\omega}_{\alpha,\beta}$*, where,* $\mathbb{I}_\omega$ *denotes the characteristic function of the set* $(\omega,\infty)$ *and*
$$\overline{\omega}_{\alpha,\beta} := \inf\big\{\omega \ge 0 : p\|\alpha\|_{Z_1}\|\beta\|_{Z^\omega_\infty} < 1\big\}. \tag{2.2}$$
(2) *If* $(H^2_{\alpha,\beta})$ *holds, then* $K$ *is a positive operator on* $Y_1$*.*
(3) *If* $(H'^2_{\alpha,\beta})$ *holds and* $p > 0$*, then* $K$ *is a strongly positive operator; i.e., for all* $\psi \in (Y_1)_+$ *and* $\psi \ne 0$*, we have* $K\psi(v) > 0$ *for almost all* $v > 0$*.*

*Proof.* Let $\psi \in Y_1$. Writhing,
$$K\psi(v) = \underbrace{p\frac{\alpha(v)}{v}}_{f}\ \underbrace{\int_0^\infty \beta(v')\psi(v')v'dv'}_{C_\psi}$$
Note that $f \in Y_1$ because
$$\|f\|_{Y_1} = \int_0^\infty |p\frac{\alpha(v)}{v}|v\,dv = p\|\alpha\|_{Z_1} < \infty$$
and $C_\psi$ is a finite constant because
$$|C_\psi| \le \int_0^\infty |\beta(v')||\psi(v')|v'\,dv' = \big[\operatorname{ess\,sup}_{v'\ge 0}|\beta(v')|\big]\|\psi\|_{Y_1} = \|\beta\|_{Z^0_\infty}\|\psi\|_{Y_1} < \infty.$$
then $K$ is obviously a rank one operator into $Y_1$ and therefore compact. Furthermore,
$$\begin{aligned}\|K\psi\|_{Y_1} &= \int_0^\infty \Big|p\frac{\alpha(v)}{v}\int_0^\infty \beta(v')\psi(v')v'dv'\Big|v\,dv\\ &= p\Big[\int_0^\infty |\alpha(v)|dv\Big]\Big|\int_0^\infty \beta(v')\psi(v')v'\,dv'\Big|\end{aligned}$$
and therefore,
$$\|K\|_{\mathcal{L}(Y_1)} = p\Big[\int_0^\infty |\alpha(v)|dv\Big]\big[\operatorname{ess\,sup}_{v'\ge 0}|\beta(v')|\big] = p\|\alpha\|_{Z_1}\|\beta\|_{Z^0_\infty}.$$
(1) Firstly, (2.2) is well defined because of $(H'^1_{\alpha,\beta})$. Next, for all $\omega > \overline{\omega}_{\alpha,\beta}$,
$$\begin{aligned}\|K\mathbb{I}_\omega\psi\|_{Y_1} &= \int_0^\infty \Big|p\frac{\alpha(v)}{v}\int_0^\infty \beta(v')\mathbb{I}_\omega(v')\psi(v')v'dv'\Big|vdv\\ &\le p\Big[\int_0^\infty |\alpha(v)|dv\Big]\int_\omega^\infty |\beta(v')||\psi(v')|v'dv'\\ &\le p\Big[\int_0^\infty |\alpha(v)|dv\Big]\big[\operatorname{ess\,sup}_{v'\ge\omega}|\beta(v')|\big]\int_\omega^\infty |\psi(v')|v'dv'\\ &= p\|\alpha\|_{Z_1}\|\beta\|_{Z^\omega_\infty}\|\psi\|_{Y_1}\end{aligned}$$
and therefore $\|K\mathbb{I}_\omega\|_{\mathcal{L}(Y_1)} \le p\|\alpha\|_{Z_1}\|\beta\|_{Z^\omega_\infty} < 1$.

Items (2) and (3) are obvious; we omit their proofs. $\square$

Let $K_\lambda$ be the linear operator
$$K_\lambda\psi(v) := p\frac{\alpha(v)}{v}\int_0^\infty e^{-\lambda/v'}\beta(v')\psi(v')v'\,dv'. \tag{2.3}$$
Some of its properties are as follows.

**Lemma 2.2.** *Suppose that $p \geq 0$. If $(H^1_{\alpha,\beta})$ holds, then $K_\lambda$ ($\lambda \geq 0$) is a compact linear operator from $Y_1$ into itself. Furthermore,*

1. *If $(H^2_{\alpha,\beta})$ holds, then $K_\lambda$ is a positive operator on $Y_1$*
2. *If $(H'^2_{\alpha,\beta})$ holds and $p > 0$, then $K_\lambda$ is a strongly positive operator.*

*Proof.* Let $\lambda \geq 0$. As $K_\lambda = K \circ E_\lambda$ where, $E_\lambda \psi = e^{-\frac{\lambda}{\bullet}} \psi$ with $\|E_\lambda\|_{\mathcal{L}(Y_1)} \leq 1$, then the compactness of $K_\lambda$ follows from that of the operator $K$ (Lemma 2.1). Then items (1) and (2) are obvious. $\square$

## 3. Full model (1.1), (1.2)

The goal of this section is to prove that the full model (1.1), (1.2) is well-posed. So, let $T_K$ be the unbounded linear operator

$$T_K\varphi = -v\frac{\partial \varphi}{\partial \mu} \quad \text{on} \quad D(T_K) = \{\varphi \in W_1 : \varphi(0,\cdot) = K\varphi(1,\cdot)\}, \tag{3.1}$$

where,

$$W_1 = \{\varphi \in L_1 : v\frac{\partial \varphi}{\partial \mu} \in L_1 \text{ and } v\varphi \in L_1\}$$

whose norm is $\|\varphi\|_{W_1} = \|v\varphi\|_1 + \|v\frac{\partial \varphi}{\partial \mu}\|_1$. The generation property of $T_K$ is given by the lemma.

**Lemma 3.1.** *Suppose that $(H^1_{\alpha,\beta})$ holds and let $p \geq 0$. If $(H'^1_{\alpha,\beta})$ holds, then $T_K$ generates, on $L_1$, a strongly continuous semigroup $\mathbb{T}_K = (\mathbb{T}_K(t))_{t\geq 0}$ defined by*

$$\mathbb{T}_K(t) = \mathbb{T}_0(t) + \overline{\mathbb{T}}_K(t) \quad t \geq 0, \tag{3.2}$$

*where*

$$\mathbb{T}_0(t)\varphi(\mu, v) := \begin{cases} \varphi(\mu - tv, v) & \text{if } \mu \geq tv \\ 0 & \text{if } \mu < tv \end{cases} \tag{3.3}$$

*and*

$$\overline{\mathbb{T}}_K(t)\varphi(\mu, v) := \begin{cases} 0 & \text{if } \mu \geq tv \\ p\frac{\alpha(v)}{v}\int_0^\infty \beta(v')\mathbb{T}_K\big(t - \frac{\mu}{v}\big)\varphi(1, v')v'dv' & \text{if } \mu < tv. \end{cases} \tag{3.4}$$

*Furthermore,*

1. *If $p\|\alpha\|_{Z_1}\|\beta\|_{Z^0_\infty} < 1$, then $\mathbb{T}_K = (\mathbb{T}_K(t))_{t\geq 0}$ is contractive.*
2. *If $(H^2_{\alpha,\beta})$ holds, then $\mathbb{T}_K = (\mathbb{T}_K(t))_{t\geq 0}$ is positive.*
3. *If $(H'^2_{\alpha,\beta})$ holds and $p > 0$, then $\mathbb{T}_K = (\mathbb{T}_K(t))_{t\geq 0}$ is irreducible.*

*Proof.* Firstly, according to [3, Theorem 2.2] we have that $\varphi \to \varphi(0,\cdot)$ and $\varphi \to \varphi(1,\cdot)$ are bounded linear mappings from $W_1$ into $Y_1$ and therefore (3.1) is well defined.

Next, by [1, Remark 3.1] and by Lemma 2.1(1), we have that $K$ is an admissible operator whose abscissa is (2.2). Hence we infer, by [1, Theorem 3.2], that $T_K$ generates, on $L_1$, a strongly continuous semigroup satisfying

$$\|\mathbb{T}_K(t)\|_{\mathcal{L}(L_1)} \leq \delta_{\alpha,\beta}e^{(\overline{\omega}_{\alpha,\beta} \ln \delta_{\alpha,\beta})t} \tag{3.5}$$

where, $\overline{\omega}_{\alpha,\beta}$ is given by (2.2) and $\delta_{\alpha,\beta} = \max\{p\|\alpha\|_{Z_1}\|\beta\|_{Z^0_\infty}, 1\}$.

(1) If $p\|\alpha\|_{Z_1}\|\beta\|_{Z^0_\infty} < 1$, then $\delta_{\alpha,\beta} = 1$ and

$$p\|\alpha\|_{Z_1}\|\beta\|_{Z^\omega_\infty} \leq p\|\alpha\|_{Z_1}\|\beta\|_{Z^0_\infty} < 1 \quad \text{for all } \omega \geq 0$$

and therefore $\overline{\omega}_{\alpha,\beta} = 0$ because of (2.2). Now, it suffices to put $\delta_{\alpha,\beta} = 1$ and $\overline{\omega}_{\alpha,\beta} = 0$ into (3.5).

(2) This follows from Lemma 2.1(2) together with [1, Proposition 6.1(1)].

(3) This follows from Lemma 2.1(3) together with [1, Proposition 6.1(2)]. $\square$

Now, let us define two perturbation operators:

$$S\varphi(\mu, v) = -\sigma(\mu, v)\varphi(\mu, v)$$
$$R\varphi(\mu, v) = \int_0^\infty r(\mu, v, v')\varphi(\mu, v')dv'$$

where, $\sigma$ and $r$ are subject to the following assumptions:

$(H_\sigma)$ $\sigma \in \big(L^\infty(\Omega)\big)_+$,

$(H_r^1)$ $\overline{r} := \operatorname{ess\,sup}_{(\mu,v')\in\Omega} \int_0^\infty |r(\mu, v, v')|dv < \infty$,

$(H_r^2)$ $r$ is positive,

$(H_{\sigma-r})$ $\int_0^\infty |r(\mu, v', v)|dv' \le \sigma(\mu, v)$ for almost all $(\mu, v) \in \Omega$

**Lemma 3.2.** (1) *If $(H_\sigma)$ holds, then $S$ is a bounded linear operator from $L_1$ into itself. Furthermore, $-S$ is a positive operator.*

(2) *If $(H_r^1)$ holds, then $R$ is a bounded linear operator from $L_1$ into itself. Furthermore, if $(H_r^2)$, then $R$ is a positive operator.*

(3) *Suppose that $(H_\sigma)$ and $(H_r^1)$ hold. If $(H_{\sigma-r})$ holds, then $S + R$ is a dissipative operator on $L_1$.*

*Proof.* Firstly, $-S$ and $R$ are obviously positive operators. Let $\varphi \in L_1$. On the one hand,

$$\begin{aligned}
\|S\varphi\|_1 &= \int_\Omega |\sigma(\mu, v)\varphi(\mu, v)|\, d\mu\, dv \\
&\le \big[\operatorname{ess\,sup}_{(\mu,v)\in\Omega} |\sigma(\mu, v)|\big] \int_\Omega |\varphi(\mu, v)|\, d\mu\, dv \\
&= \|\sigma\|_\infty \|\varphi\|_1
\end{aligned}$$

which proves (1). On the other hand,

$$\begin{aligned}
\|R\varphi\|_1 &= \int_\Omega \Big| \int_0^\infty r(\mu, v, v')\varphi(\mu, v')dv' \Big|\, d\mu\, dv \\
&\le \int_0^1 \int_0^\infty \Big[ \int_0^\infty |r(\mu, v, v')|dv \Big] |\varphi(\mu, v')|\, d\mu\, dv' \\
&\le \Big[ \operatorname{ess\,sup}_{(\mu,v')\in\Omega} \int_0^\infty |r(\mu, v, v')|dv \Big] \int_\Omega |\varphi(\mu, v')|\, d\mu\, dv' \\
&= \overline{r}\|\varphi\|_1
\end{aligned}$$

which proves (2). Finally,

$$\begin{aligned}
&\langle \operatorname{sgn}\varphi, (S + R)\varphi \rangle \\
&= \int_\Omega \operatorname{sgn}\varphi(\mu, v) \left(S\varphi(\mu, v) + R\varphi(\mu, v)\right)\, d\mu\, dv \\
&= -\int_\Omega \sigma(\mu, v)|\varphi(\mu, v)|\, d\mu\, dv + \int_\Omega \operatorname{sgn}\varphi(\mu, v) \Big[ \int_0^\infty r(\mu, v, v')\varphi(\mu, v')dv' \Big]\, d\mu\, dv \\
&\le -\int_\Omega \sigma(\mu, v)|\varphi(\mu, v)|\, d\mu\, dv + \int_\Omega \Big[ \int_0^\infty |r(\mu, v, v')|dv \Big] |\varphi(\mu, v')|\, d\mu\, dv'
\end{aligned}$$

$$\leq -\int_\Omega \sigma(\mu,v)|\varphi(\mu,v)|\,d\mu\,dv + \int_\Omega \sigma(\mu,v')|\varphi(\mu,v')|\,d\mu\,dv' = 0$$

which proves (3). □

Now, let $V_K$ be the first linear perturbation of (3.1); i.e.,

$$V_K := T_K + S \quad \text{on } D(V_K) = D(T_K). \tag{3.6}$$

The generation property of $V_K$ is as follows.

**Lemma 3.3.** *Suppose that $(H^1_{\alpha,\beta})$ holds and let $p \geq 0$ be such that $(H'^1_{\alpha,\beta})$ holds. If $(H_\sigma)$ holds, then $V_K$ generates, on $L_1$, a strongly continuous semigroup $\mathbb{V}_K = (\mathbb{V}_K(t))_{t\geq 0}$ whose essential type,*

$$\omega_{\text{ess}}(\mathbb{V}_K) \leq -\underline{\sigma}, \tag{3.7}$$

*where*

$$\underline{\sigma} := \operatorname{ess\,inf}_{(\mu,v)\in\Omega} |\sigma(\mu,v)|. \tag{3.8}$$

*Furthermore,*

(1) *If $(H^2_{\alpha,\beta})$ holds, then $\mathbb{V}_K = (\mathbb{V}_K(t))_{t\geq 0}$ is positive,*

$$0 \leq \mathbb{V}_K(t) \leq \mathbb{T}_K(t) \quad t \geq 0 \tag{3.9}$$

$$-\overline{\sigma} + \omega_0(\mathbb{T}_K) \leq \omega_0(\mathbb{V}_K) \tag{3.10}$$

*where*

$$\overline{\sigma} := \operatorname{ess\,sup}_{(\mu,v)\in\Omega} |\sigma(\mu,v)|. \tag{3.11}$$

(2) *If $(H'^2_{\alpha,\beta})$ holds and $p > 0$, then $\mathbb{V}_K = (\mathbb{V}_K(t))_{t\geq 0}$ is irreducible.*

*Proof.* Lemma 2.1(1) implies that $K$ is an admissible operator in the sense of [1, Remark 3.1]. Therefore, the generation property of $V_K$ follows from [1, Theorem 5.1] while (3.7) follows from [2, (5.18)]

(1) The positivity of $\mathbb{V}_K = (\mathbb{V}_K(t))_{t\geq 0}$ follows from Lemma 2.1(2) together with [1, Theorem 6.1(1)]. Furthermore, Trotter Formula leads to

$$\mathbb{V}_K(t)\varphi = \lim_{n\to\infty} \Big[e^{\left(\frac{t}{n}S\right)}\mathbb{T}_K(\tfrac{t}{n})\Big]^n \varphi \quad t \geq 0 \tag{3.12}$$

for all $\varphi \in L_1$. However, if $\varphi \in (L_1)_+$ then for all integers $n$,

$$0 \leq \Big[e^{\frac{t}{n}S}\mathbb{T}_K(\tfrac{t}{n})\Big]^n \varphi \leq [\mathbb{T}_K(\tfrac{t}{n})]^n\varphi = \mathbb{T}_K(t)\,\varphi.$$

Passing to the limit $n \to \infty$ and using (3.12), we infer (3.9).

Rewriting [1, (6.9)]; i.e.,

$$e^{-\overline{\sigma}t}\mathbb{T}_K(t) \leq \mathbb{V}_K(t) \quad t \geq 0$$

we obtain

$$-\overline{\sigma} + \frac{\ln \|\mathbb{T}_K(t)\|_{\mathcal{L}(L_1)}}{t} \leq \frac{\ln \|\mathbb{V}_K(t)\|_{\mathcal{L}(L_1)}}{t} \quad t > 0.$$

Passing to the limit $t \to \infty$, the desired (3.10) follows.

(3) This follows from Lemma 2.1(3) together with [1, Theorem 6.1(2)]. □

To study the well posedness of the full model (1.1), (1.2), we consider the unbounded linear operator

$$U_K := V_K + S = T_K + S + R \quad \text{on } D(U_K) = D(T_K). \tag{3.13}$$

**Theorem 3.4.** *Suppose that $(H^1_{\alpha,\beta})$ and $(H_\sigma)$ hold and let $p \geq 0$ be such that $(H'^1_{\alpha,\beta})$ holds. If $(H^1_r)$ holds, then $U_K$ generates, on $L_1$, a strongly continuous semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$. Furthermore,*

1. *If $(H_{\sigma-r})$ holds and $p\|\alpha\|_{Z_1}\|\beta\|_{Z^0_\infty} < 1$, then*
$$\|\mathbb{U}_K(t)\|_{\mathcal{L}(L_1)} \leq 1 \quad t \geq 0. \tag{3.14}$$
2. *If $(H^2_{\alpha,\beta})$ and $(H^2_r)$ hold, then $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ is positive and*
$$0 \leq \mathbb{V}_K(t) \leq \mathbb{U}_K(t) \quad t \geq 0. \tag{3.15}$$
3. *If $(H'^2_{\alpha,\beta})$ and $(H^2_r)$ hold and $p > 0$, then $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ is irreducible.*

*Proof.* Due to the boundedness of the linear operator $R$ (Lemma 3.2(2)), we infer that $U_K = V_K + R$ is a bounded linear perturbation of the generator $V_K$ (Lemma 3.3). Hence, $U_K$ is itself a generator, on $L_1$ of a strongly continuous semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$.

(1) Lemma 3.1 (1) implies that $T_K$ is a dissipative generator. Thanks to Lemma 3.2 (3), the generator $U_K = T_K + (S + R)$ appears then as a 0-bounded linear dissipative perturbation of the dissipative generator $T_K$ and therefore $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ is contractive.

(2) Let $\varphi \in L_1$. According to Trotter Formula, we have
$$\mathbb{U}_K(t)\varphi = \lim_{n\to\infty} \Big[e^{\left(\frac{t}{n}R\right)}\mathbb{V}_K(\tfrac{t}{n})\Big]^n \varphi \quad t \geq 0$$
which leads, by virtue of the positivity of $\mathbb{V}_K = (\mathbb{V}_K(t))_{t\geq 0}$ (Lemma 3.2(1)) and that of the operator $R$ (Lemma 3.3(2)), to the desired positivity of $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$. Now, Duhamel Formula implies, for all $t \geq 0$, that
$$\mathbb{U}_K(t) = \mathbb{V}_K(t) + \int_0^t \mathbb{U}_K(s) R \mathbb{V}_K(t-s)ds \geq \mathbb{V}_K(t) \geq 0$$
which proves (3.15).

(3) This follows from Lemma 3.3(2) together with (3.15). □

## 4. Asynchronous Growth Property

The aim of this section is to describe the asymptotic behavior of the generated semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ like in Lemma 1.1, which leads to the asynchronous exponential growth of the studied bacterial population. According to the positivity and to the irreducibility of the generated semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ (see Theorem 3.4(2) and Theorem 3.4(3)), it remains to prove that $\omega_{\rm ess}(\mathbb{U}_K) < \omega_0(\mathbb{U}_K)$. So, let us consider

$(H^3_r)$ $\widetilde{r} = \operatorname{ess\,sup}_{(\mu,v,v')\in(0,1)\times\mathbb{R}_+\times\mathbb{R}_+}(\frac{1}{\mu}|r(\mu,v,v')|) < \infty$.

Let us recall the following useful results.

**Lemma 4.1** ([5]). *Let $A$ and $B$ be linear and bounded operators on $L^1(\Omega)$.*

1. *If $A$ is weakly compact and $0 \leq A \leq B$, then $B$ is weakly compact.*
2. *If $A$ and $B$ are weakly compact, then $AB$ is compact.*
3. *The set of all weakly compact operators is norm-closed subset in $L^1(\Omega)$.*

**Lemma 4.2** ([7])**.** *Let $R \in \mathcal{L}(L^1(\Omega))$ and let $\mathbb{V} = (\mathbb{V}(t))_{t\geq 0}$ and $\mathbb{W} = (\mathbb{W}(t))_{t\geq 0}$ be two strongly continuous semigroups on $L^1(\Omega)$ whose generators are $V$ and $V + R$. Suppose that $R\mathbb{V}(t_1)R \cdots R\mathbb{V}(t_n)R$ is compact for all $t_1, \cdots, t_n > 0$ (for some integer $n$). Then, $\omega_{\rm ess}(\mathbb{W}) = \omega_{\rm ess}(\mathbb{V})$.*

**Proposition 4.3.** *Suppose that $(H^1_{\alpha,\beta})$, $(H^2_{\alpha,\beta})$, $(H_\sigma)$, $(H^1_r)$ and $(H^2_r)$ hold and let $p \geq 0$ be such that $(H'^1_{\alpha,\beta})$ holds. If $(H^3_r)$ holds, then*

$$\omega_{\rm ess}(\mathbb{U}_K) \leq -\underline{\sigma} \tag{4.1}$$

*where, $\underline{\sigma}$ is defined by* (3.8)*.*

*Proof.* Let us divide the proof in several steps.

**Step I.** Let $t > 0$ and let $\varphi \in (L_1)_+$. Using (3.3), easy computations show, for all $(\mu, v) \in \Omega$, that

$$R\mathbb{T}_0(t)R\varphi(\mu, v) = \int_0^{\mu/t}\int_0^\infty r(\mu, v, v')r(\mu - tv', v', v'')\varphi(\mu - tv', v'')dv''\, dv'.$$

Due to $(H^3_r)$, it follows that $r \in L^\infty((0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+)$ and therefore

$$R\mathbb{T}_0(t)R\varphi(\mu, v) \leq \|r\|^2_\infty \int_0^{\mu/t}\int_0^\infty \varphi(\mu - tv', v'')dv''\, dv'.$$

The change of variables $(\mu' = \mu - tv')$ yields

$$R\mathbb{T}_0(t)R\varphi(\mu, v) \leq \frac{\|r\|^2_\infty}{t}\int_\Omega \varphi(\mu', v'')dv''\, d\mu'$$

which can be written as

$$R\mathbb{T}_0(t)R \leq \frac{\|r\|^2_\infty}{t}\mathbf{1} \otimes \mathbf{1} \tag{4.2}$$

where, $\mathbf{1} \otimes \mathbf{1}\varphi = \big[\int_\Omega \varphi(\mu, v)\, d\mu\, dv\big]\mathbf{1}$, with, $\mathbf{1}(\mu, v) = 1$ for all $(\mu, v) \in \Omega$.

**Step II.** Suppose that $\alpha \in C_c(\mathbb{R}_+) \subset Z_1$. There exists $0 < b < \infty$ such that

$$\mathrm{supp}(\alpha) \subset (0, b). \tag{4.3}$$

Let $t > 0$ and let $\varphi \in (L_1)_+$. Using (3.4), easy computations show, for almost all $(\mu, v) \in \Omega$, that

$$\begin{aligned} &R\overline{\mathbb{T}}_K(t)R\varphi(\mu, v)\\ &= p\int_{\frac{\mu}{t}}^\infty r(\mu, v, v')\frac{\alpha(v')}{v'}\int_0^\infty \beta(v'')\mathbb{T}_K\left(t - \frac{\mu}{v'}\right)R\varphi(1, v'')\mu v''dv''dv' \end{aligned}$$

which by (4.3) and $(H^1_{\alpha,\beta})$ and $(H^3_r)$, lead to

$$\begin{aligned} R\overline{\mathbb{T}}_K(t)R\varphi(\mu, v) &\leq pb\widetilde{r}\|\alpha\|_{Z^0_\infty}\|\beta\|_{Z^0_\infty}\int_{\frac{\mu}{t}}^b\int_0^\infty |\mathbb{T}_K\left(t - \frac{\mu}{v'}\right)R\varphi(1, v'')|\frac{\mu}{v'^2}v''dv''\, dv'\\ &:= C\int_{\frac{\mu}{t}}^b\int_0^\infty |\mathbb{T}_K\left(t - \frac{\mu}{v'}\right)R\varphi(1, v'')|\frac{\mu}{v'^2}v''dv''\, dv' \end{aligned}$$

where, $C := pb\widetilde{r}\|\alpha\|_{Z^0_\infty}\|\beta\|_{Z^0_\infty}$ is obviously a finite constant. The following change of variables $(s = t - \frac{\mu}{v'})$ yields that

$$R\overline{\mathbb{T}}_K(t)R\varphi(\mu, v) \leq C\int_0^t\int_0^\infty \mathbb{T}_K(s)R\varphi(1, v'')v''dv''ds$$

$$= C\int_0^t \|\mathbb{T}_K(s)R\varphi(1,\cdot)\|_{Y_1}ds$$

and therefore, for all $\omega \geq 0$,

$$\begin{aligned} R\overline{\mathbb{T}}_K(t)R\varphi(\mu,v) &\leq C\int_0^t e^{\omega s}e^{-\omega s}\|\mathbb{T}_K(s)R\varphi(1,\cdot)\|_{Y_1}ds \\ &\leq Ce^{\omega t}\int_0^\infty e^{-\omega s}\|\mathbb{T}_K(s)R\varphi(1,\cdot)\|_{Y_1}ds. \end{aligned} \tag{4.4}$$

However, if $\omega > 0$ is large, then, by [2, (3.11) and (4.1)], there exists a constant $c_\omega$ ($0 < c_\omega < \infty$) such that

$$\int_0^\infty e^{-\omega t}\|\mathbb{T}_K(t)\varphi(1,\cdot)\|_{Y_1}dt \leq c_\omega\|\varphi\|_1.$$

Accordingly, (4.4) becomes

$$R\overline{\mathbb{T}}_K(t)R\varphi(\mu,v) \leq Ce^{\omega t}c_\omega\|R\varphi\|_1 \leq Ce^{\omega t}c_\omega\|R\|_{\mathcal{L}(L_1)}\int_\Omega \varphi(\mu',v')dv'd\mu'$$

which can be written as

$$R\overline{\mathbb{T}}_K(t)R \leq Ce^{\omega t}c_\omega\|R\|\mathbf{1}\otimes\mathbf{1} \tag{4.5}$$

where, $\mathbf{1}\otimes\mathbf{1}$ is already defined in Step I.

Finally, according to (3.2) we have

$$0 \leq R\mathbb{T}_K(t)R = R\mathbb{T}_0(t)R + R\overline{\mathbb{T}}_K(t)R$$

in which we put (4.2) together with (4.5) to infer that

$$0 \leq R\mathbb{T}_K(t)R \leq \underbrace{\Big(\frac{\|r\|_\infty^2}{t} + Ce^{\omega t}c_\omega\|R\|\Big)}_{C'_t}\mathbf{1}\otimes\mathbf{1} \tag{4.6}$$

where, $C'_t$ is obviously a finite constant. As $\mathbf{1}\otimes\mathbf{1}$ is a rank one operator into $L_1$, then Lemma 4.1(1) leads us to say that: *if $\alpha \in C_c(\mathbb{R}_+)$, then $R\mathbb{T}_K(t)R$ is weakly compact, into $L_1$, for all $t > 0$.*

**Step III.** Due to $\alpha \in Z_1$ (because of $(H^1_{\alpha,\beta})$), there exists $(\alpha_n)_n \subset C_c(\mathbb{R}_+)$ such that

$$\lim_{n\to\infty}\|\alpha_n - \alpha\|_{Z_1} = 0. \tag{4.7}$$

Let then $K_n$ be defined as operator

$$K_n\psi(v) := p\frac{\alpha_n(v)}{v}\int_0^\infty \beta(v')\psi(v')v'\,dv'.$$

Accordingly,

$$\|K_n - K\|_{\mathcal{L}(Y_1)} \leq p\|\alpha_n - \alpha\|_{Z_1}\|\beta\|_{Z^0_\infty}$$

which leads, by (4.7), to

$$\lim_{n\to\infty}\|K_n - K\|_{\mathcal{L}(Y_1)} = 0. \tag{4.8}$$

Now, let $\varepsilon$ be such that

$$0 < \varepsilon < \frac{1 - p\|\alpha\|_{Z_1}\|\beta\|_{Z^{\omega_0}_\infty}}{p\|\beta\|_{Z^{\omega_0}_\infty}}$$

where, $\omega_0$ is given in $(H'^1_{\alpha,\beta})$. There exists then an integer $N$ such that

$$\|\alpha_n\|_{Z_1} \leq \|\alpha\|_{Z_1} + \varepsilon \quad \text{for all} \quad n > N$$

and therefore $(H^1_{\alpha_n,\beta})$ holds for all $n > N$. On the other hand,

$$\begin{aligned}
p\|\alpha_n\|_{Z_1}\|\beta\|_{Z_\infty^{\omega_0}} &\leq p\left(\|\alpha\|_{Z_1} + \varepsilon\right)\|\beta\|_{Z_\infty^{\omega_0}} \\
&< p\|\alpha\|_{Z_1}\|\beta\|_{Z_\infty^{\omega_0}} + p\frac{1 - p\|\alpha\|_{Z_1}\|\beta\|_{Z_\infty^{\omega_0}}}{p\|\beta\|_{Z_\infty^{\omega_0}}}\|\beta\|_{Z_\infty^{\omega_0}} \\
&= p\|\alpha\|_{Z_1}\|\beta\|_{Z_\infty^{\omega_0}} + 1 - p\|\alpha\|_{Z_1}\|\beta\|_{Z_\infty^{\omega_0}} = 1
\end{aligned}$$

and therefore $(H'^1_{\alpha_n,\beta})$ holds true for all $n > N$. Thanks to Lemma 3.1, we infer that $T_{K_n}$ $(n > N)$ generates, on $L_1$, a strongly continuous semigroup $\mathbb{T}_{K_n} = (\mathbb{T}_{K_n}(t))_{t\geq 0}$. Furthermore, Step II implies that $R\mathbb{T}_{K_n}(t)R$ is a weakly compact, into $L_1$, for all $t > 0$ because of $\alpha_n \in C_c(\mathbb{R}_+)$. On the other hand, [2, Theorem 4.1] together with (4.8) imply that

$$\lim_{n\to\infty} \|\mathbb{T}_{K_n}(t) - \mathbb{T}_K(t)\|_{\mathcal{L}(L_1)} = 0$$

which leads to

$$\|R\mathbb{T}_K(t)R - R\mathbb{T}_{K_n}(t)R\|_{\mathcal{L}(L_1)} \leq \|R\|^2_{\mathcal{L}(L_1)}\|\mathbb{T}_K(t) - \mathbb{T}_{K_n}(t)\|_{\mathcal{L}(L_1)} \to \quad \text{as } n \to \infty 0.$$

Finally, thanks to Lemma 4.1(3) we can say that: *$R\mathbb{T}_K(t)R$ is a weakly compact operator, into $L_1$, for all $t > 0$.*

**Step IV.** According to the positivity of the operator $R$ (Lemma 3.2(2)) together with (3.9), we obtain

$$0 \leq R\mathbb{V}_K(t)R \leq R\mathbb{T}_K(t)R \quad t \geq 0. \tag{4.9}$$

However, as $R\mathbb{T}_K(t)R$ is weakly compact for all $t > 0$ (Step III), then Lemma 4.1(2) together with (4.9) imply that $R\mathbb{V}_K(t)R$ is also weakly compact for all $t > 0$. Accordingly,

$$R\mathbb{V}_K(t_1)R\mathbb{V}_K(t_2)R\mathbb{V}_K(t_3)R = \big(R\mathbb{V}_K(t_1)R\big)\mathbb{V}_K(t_2)\big(R\mathbb{V}_K(t_3)R\big)$$

is a compact operator, into $L_1$, for all $t_1,\ t_2,\ t_3 > 0$, because of Lemma 4.1(3). Finally, Lemma 4.2 and (3.7) lead to

$$\omega_{\rm ess}(\mathbb{W}_K) = \omega_{\rm ess}(\mathbb{V}_K) \leq -\underline{\sigma}$$

which proves (4.1) and completes the proof. □

**Proposition 4.4.** *Suppose that $(H^1_{\alpha,\beta})$, $(H'^2_{\alpha,\beta})$, $(H_\sigma)$, $(H^1_r)$ and $(H^2_r)$ hold and let $p \geq 0$ be such that $(H'^1_{\alpha,\beta})$ holds. If the spectral radius*

$$\rho(K_{\overline{\sigma}-\underline{\sigma}}) > 1$$

*then*

$$-\,\underline{\sigma} < \omega_0\left(\mathbb{U}_K\right) \tag{4.10}$$

*where, the operator $K_{\overline{\sigma}-\underline{\sigma}}$ is defined by* (2.3) *and $\underline{\sigma}$ and $\overline{\sigma}$ are given by* (3.8) *and* (3.11).

*Proof.* Firstly, due to Lemma 2.2 we infer that $K_{\overline{\sigma}-\underline{\sigma}}$ is a positive, irreducible and compact operator and therefore its spectral radius $\rho(K_{\overline{\sigma}-\underline{\sigma}}) \neq 0$. It suffices then to find suitable assumptions on $\alpha$ and $\beta$ such that $\rho(K_{\overline{\sigma}-\underline{\sigma}}) > 1$.

Next. On one hand, (3.15) leads to

$$\lim_{t\to\infty} \frac{\ln \|\mathbb{V}_K(t)\|_{\mathcal{L}(L_1)}}{t} \leq \lim_{t\to\infty} \frac{\ln \|\mathbb{U}_K(t)\|_{\mathcal{L}(L_1)}}{t}$$

which is nothing else but $\omega_0(\mathbb{V}_K) \leq \omega_0(\mathbb{U}_K)$ and by (3.10) we infer that

$$-\overline{\sigma} + \omega_0(\mathbb{T}_K) \leq \omega_0(\mathbb{U}_K). \tag{4.11}$$

On the other hand, Lemma 2.1 means that $K$ is a positive, irreducible and compact operator. Hence, all the required conditions of [1, Theorem 7.1] are fulfilled and therefore [1, (7.18)] holds; i.e.,

$$-\underline{\sigma} < -\overline{\sigma} + \omega_0(\mathbb{T}_K). \tag{4.12}$$

Finally, the desired (4.10) obviously follows from (4.11) and (4.12). □

Now, we are ready to describe the asymptotic behaviour of the generated semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ like in Lemma 1.1 and to get, by the way, the asynchronous exponential growth property of the full model (1.1), (1.2).

Before we start, we notice that the full model (1.1),(1.2) is well posed because of Theorem 3.4 and therefore the unique solution $f$ is given by

$$f(t,\cdot,\cdot) = \mathbb{U}_K(t) f(0,\cdot,\cdot) \quad t \geq 0 \tag{4.13}$$

for all initial data $f(0,\cdot,\cdot) \in L_1$.

Hence, if $p$ (the average number of bacteria daughter viable per mitotic) is small enough (i.e., $p < \|\alpha\|_{Z_1}^{-1} \|\beta\|_{Z_\infty^0}^{-1}$), then the bacterial density (4.13) is decreasing and therefore the full model (1.1),(1.2) becomes biologically uninteresting. Indeed, for all $t$ and all $s$ with $t > s$, (4.13) and (3.14) lead to

$$\begin{aligned} \|f(t,\cdot,\cdot)\|_1 &= \|\mathbb{U}_K(t) f(0,\cdot,\cdot)\|_1 \\ &= \|\mathbb{U}_K(t-s)\mathbb{U}_K(s) f(0,\cdot,\cdot)\|_1 \\ &\leq \|\mathbb{U}_K(s) f(0,\cdot,\cdot)\|_1 \\ &\leq \|f(s,\cdot,\cdot)\|_1. \end{aligned}$$

We understand then that $p\|\alpha\|_{Z_1}\|\beta\|_{Z_\infty^0} > 1$ is closely related to an increasing number of bacteria during each mitotic. This situation is the most biologically observed for which the asynchronous exponential growth property is given by the following theorem.

**Theorem 4.5.** *Suppose that $(H^1_{\alpha,\beta})$, $(H'^2_{\alpha,\beta})$, $(H_\sigma)$, $(H^1_r)$, $(H^2_r)$ and $(H^3_r)$ hold and let $p \geq 0$ be such that $(H'^1_{\alpha,\beta})$ holds. Suppose also that $\rho\left(K_{\overline{\sigma}-\underline{\sigma}}\right) > 1$ where, $\underline{\sigma}$ and $\overline{\sigma}$ are given by (3.8) and (3.11). There exist then a rank one projector $\mathbb{P}$ into $L_1$ and an $\varepsilon > 0$ such that: for every $\eta \in (0,\varepsilon)$ there exist $M_\eta \geq 1$ satisfying*

$$\|e^{-\omega_0(\mathbb{U}_K)t}\mathbb{U}_K(t) - \mathbb{P}\|_{\mathcal{L}(L_1)} \leq M_\eta e^{-\eta t} \quad t \geq 0. \tag{4.14}$$

*Proof.* The generated semigroup $\mathbb{U}_K = (\mathbb{U}_K(t))_{t\geq 0}$ exists because of Theorem 3.4. Moreover, it is positive (Theorem 3.4(2)) and irreducible (Theorem 3.4(3)). Finally, (4.1) and (4.10) lead to $\omega_{\text{ess}}(\mathbb{U}_K) < \omega_0(\mathbb{U}_K)$. Now, all the required conditions of Lemma 1.1 are fulfilled. □

**Remark 4.6.** Thanks to [4, Thereom 8.7], there exist a quasi-interior vector $\varphi_0 \in X_+$ and a strictly positive functional $\varphi_0^* \in (X^*)_+$ with $\langle \varphi_0, \varphi_0^* \rangle = 1$ such that $\mathbb{P}\varphi = \langle \varphi, \varphi_0^* \rangle \varphi_0$. Hence, (4.14) together with (4.13) lead to

$$\left\| e^{-\omega_0(\mathbb{U}_K)t} f(t,\cdot,\cdot) - \langle f(0,\cdot,\cdot), \varphi_0^* \rangle \varphi_0 \right\|_{L_1} \leq M_\eta e^{-\eta t} \| f(0,\cdot,\cdot) \|_{L_1} \quad t \geq 0.$$

for all initial data $f(0,\cdot,\cdot) \in L_1$.

We noticed that there are some mistakes in [1, 2], but they do not have any consequence for the main results. An addendum are already published on June 24, 2013.

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## Addendum posted on October 29, 2014

The author would like to make the following changes in the Proof of Proposition 4.3:

**1.** Assumption $(H_r^3)$ must be replaced by

$$\widetilde{r}(\mu, v) := \frac{1}{\mu} \operatorname*{ess\,sup}_{v' \in (0,\infty)} |r(\mu, v, v')| \in L^1(\Omega) \cap L^\infty(\Omega).$$

**2.** Operator $\mathbf{1} \otimes \mathbf{1}$ must be replaced by

$$\mathbf{1} \otimes \mathbf{1}\varphi := \Big[ \int_\Omega \varphi(\mu, v)\, d\mu\, dv \Big] \widetilde{r},$$

which is rank one into $L^1(\Omega)$.

**3.** Step I must be updated. Formula (4.2) becomes

$$RT_0(t)R \leq \frac{\|r\|_\infty}{t} \mathbf{1} \otimes \mathbf{1}.$$

**4.** Step II: take $C := pb\|\alpha\|_{Z_\infty^0} \|\beta\|_{Z_\infty^0}$ and update (4.6) because of (4.2).

End of addendum.


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