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ESTIMATES ON POTENTIAL FUNCTIONS AND BOUNDARY BEHAVIOR OF POSITIVE SOLUTIONS FOR SUBLINEAR DIRICHLET PROBLEMS

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Abstract. We give global estimates on some potential of functions in a bounded domain of the Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. These functions may be singular near the boundary and are globally comparable to a product of a power of the distance to the boundary by some particularly well behaved slowly varying function near zero. Next, we prove the existence and uniqueness of a positive solution for the integral equation $u = V(au^\sigma)$ with $0 \leq \sigma < 1$, where V belongs to a class of kernels that contains in particular the potential kernel of the classical Laplacian $V = (-\Delta)^{-1}$ or the fractional laplacian $V = (-\Delta)^{\alpha/2}$, $0 < \alpha < 2$.

1. Introduction

Let D be a $C^{1,1}$-bounded domain in $\mathbb{R}^n$, $n \geq 2$. It is well known that [5, 13, 17] the Green function G_D of the Dirichlet Laplacian $(-\Delta)$ in D satisfies

$$G_D(x,y) \approx H(x,y) = \begin{cases} \frac{1}{|x-y|^{n-2}} \min\big(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\big), & \text{if } n \geq 3, \\ \log\big(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\big), & \text{if } n = 2, \end{cases} \tag{1.1}$$

where $\delta(x)$ denotes the Euclidean distance from x to the boundary of D. Here and throughout the paper, for two nonnegative function f and g defined on a set S, we denote $f(t) \approx g(t)$ and we say that f and g are comparable, if there exists a constant $C > 1$ such that $\frac{1}{C} f(t) \leq g(t) \leq C f(t)$ for all $t \in S$.
On the other hand, if $0 < \alpha < 2$ and $n \geq 2$, then the Green function G_D^α of the operator $(-\Delta)^{\alpha/2}$ in D with Dirichlet conditions, see [4], satisfies

$$G_D^\alpha(x,y) \approx H_\alpha(x,y) = \frac{1}{|x-y|^{n-\alpha}} \min\Big(1, \Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\alpha/2}\Big). \tag{1.2}$$

So, we remark that, if $0 < \alpha \leq 2$, then the Green function of the operator $(-\Delta)^{\alpha/2}$ in D with Dirichlet conditions is comparable to the function

$$\frac{1}{|x-y|^{n-\alpha}} h\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\alpha/2}\Big),$$

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where $h(t)$ is either $\min(1,t)$ or $Log(1+t)$. These global estimates on G_D^α have been exploited by many authors, see [3, 9, 16], to derive estimates on the solutions of the Dirichlet problem

$$\begin{gathered} (-\Delta)^{\alpha/2} u = a(x) u^\sigma, \quad \text{in } D, \\ \lim_{x\to\partial D} (\delta(x))^{1-\frac{\alpha}{2}} u(x) = 0\,, \end{gathered} \tag{1.3}$$

where $\sigma < 1$ and a is a nonnegative measurable function that may be singular at the boundary of D. For instance for $\alpha = 2$, Mâagli in [9] considered the case where $a \in C_{loc}^\gamma(D)$, $0 < \gamma < 1$ such that

$$a(x) \approx (\delta(x))^{-\lambda} L(\delta(x))\,, \tag{1.4}$$

where $\lambda \le 2$ and L belongs to the class $\mathcal{K}$ of Karamata functions defined by

$$L(t) = c\exp\Big(\int_t^\eta \frac{z(s)}{s}\,ds\Big),$$

where $\eta > 0$, $c > 0$ and $z \in C([0,\eta])$ with $z(0) = 0$. Then, he showed in particular the following.

Proposition 1.1. *Let $\lambda \le 2$, $L \in \mathcal{K}$ such that $\int_0^\eta t^{1-\lambda} L(t)\,dt < \infty$ and assume that a satisfies* (1.4)*. Then the Green potential*

$$G_D a(x) := \int_D G_D(x,y) a(y)\,dy$$

is comparable to the function $\psi(\delta(x))$, where

$$\psi(t) = \begin{cases} \int_0^t \frac{L(s)}{s}\,ds & \text{if } \lambda = 2\,, \\ t^{2-\lambda} L(t) & \text{if } 1 < \lambda < 2\,, \\ t\int_t^\eta \frac{L(s)}{s}\,ds & \text{if } \lambda = 1\,, \\ t & \text{if } \lambda < 1. \end{cases}$$

Our aim in this article is two fold, as we explain in what follows. First, we give a unified proof and extend the above estimates for more general potential functions. More precisely, we consider a nonnegative nondecreasing measurable function φ on $[0,\infty)$ satisfying the assumption

(H0) $\varphi(t) \approx t$ for $0 \le t \le 1$ and $\int_1^\infty \frac{\varphi(t)}{t^2}\,dt < \infty$,

Let Γ_D be a measurable function defined in $D \times D$ with values in $[0,\infty]$ such that

$$\Gamma_D(x,y) \approx \frac{1}{|x-y|^{n-\beta}} \varphi\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\beta/2}\Big), \quad \text{with } \beta > 0 \text{ and } n \ge 2,$$

and let q be a nonnegative measurable function satisfying

(H1) $q(x) \approx (\delta(x))^{-\mu} L(\delta(x))$, where $\mu \le \frac{\beta}{2}+1$, $L \in \mathcal{K}$ with $\int_0^\eta t^{-\mu+\frac{\beta}{2}} L(t)\,dt < \infty$ and $\eta > \operatorname{diam}(D)$.

Put $Vq(x) = \int_D \Gamma_D(x,y) q(y) dy$. So, we have the following estimates.

Theorem 1.2. *Assume* (H0), (H1). *Then we have*

$$Vq(x) \approx \begin{cases} (\delta(x))^{\frac{\beta}{2}-1}\Big(\int_0^{\delta(x)} \frac{L(s)}{s}\,ds\Big) & \text{if } \mu = \frac{\beta}{2}+1\,, \\ (\delta(x))^{\beta-\mu} L(\delta(x)) & \text{if } \frac{\beta}{2} < \mu < \frac{\beta}{2}+1\,, \\ (\delta(x))^{\beta/2}\Big(\int_{\delta(x)}^\eta \frac{L(s)}{s}\,ds\Big) & \text{if } \mu = \frac{\beta}{2}\,, \\ (\delta(x))^{\beta/2} & \text{if } \mu < \frac{\beta}{2}\,. \end{cases}$$

Secondly, we fix $\sigma \in [0,1)$ and a nonnegative measurable function a in D satisfying

(H2) $a(x) \approx (\delta(x))^{-\lambda} L(\delta(x))$, where $\lambda \le \frac{\beta}{2}(1+\sigma) + (1-\sigma)$ and $L \in \mathcal{K}$ such that $\int_0^\eta t^{\frac{\beta}{2}(1+\sigma)-\sigma-\lambda} L(t)\, dt < \infty$.

Then we prove the following result.

Theorem 1.3. *Assume that a satisfies* (H2). *Then, the integral equation*

$$u = V(au^\sigma)$$

has a unique solution u satisfying $u(x) \approx \theta_\lambda(x)$, where

$$\theta_\lambda(x) = \begin{cases} (\delta(x))^{\frac{\beta}{2}-1} \Big(\int_0^{\delta(x)} \frac{L(s)}{s}\, ds \Big)^{1/(1-\sigma)} & \text{if } \lambda = \frac{\beta}{2}(1+\sigma) + (1-\sigma)\,, \\ (\delta(x))^{\frac{\beta-\lambda}{1-\sigma}} (L(\delta(x)))^{1/(1-\sigma)} & \text{if } \frac{\beta}{2}(1+\sigma) < \lambda < \frac{\beta}{2}(1+\sigma) + (1-\sigma)\,, \\ (\delta(x))^{\beta/2} \Big(\int_{\delta(x)}^{\eta} \frac{L(s)}{s}\, ds \Big)^{1/(1-\sigma)} & \text{if } \lambda = \frac{\beta}{2}(1+\sigma)\,, \\ (\delta(x))^{\beta/2} & \text{if } \lambda < \frac{\beta}{2}(1+\sigma). \end{cases} \tag{1.5}$$

This paper is organized as follows. Some preliminary lemmas are stated and proved in the next Section, involving some already known results on Karamata functions. In Section 3, we give the proof of Theorems 1.2 and 1.3. The last section is devoted to the study of some examples.

2. The Karamata class

To let the paper be self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory. First, we mention that a function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$ such that

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0. \tag{2.1}$$

Lemma 2.1 ([12, 14])**.** *The following hold*

(i) *Let $L \in \mathcal{K}$ and $\varepsilon > 0$, then $\lim_{t\to 0^+} t^\varepsilon L(t) = 0$.*
(ii) *Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$ and $L_1^p \in \mathcal{K}$.*

Example 2.2. Let $m \in \mathbb{N}^*$. Let $c > 0$, $(\mu_1, \mu_2, \dots, \mu_m) \in \mathbb{R}^m$ and d be a sufficiently large positive real number such that the function

$$L(t) = c \prod_{k=1}^{m} \Big(\log_k \big(\frac{d}{t}\big) \Big)^{-\mu_k}$$

is defined and positive on $(0,\eta]$, for some $\eta > 1$, where $\log_k x = \log \circ \log \circ \dots \circ \log x$ (k times). Then $L \in \mathcal{K}$.

Applying Karamata's theorem (see [12, 14]), we get the following.

Lemma 2.3. *Let $\mu \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. We have*

(i) *If $\mu < -1$, then $\int_0^\eta s^\mu L(s)ds$ diverges and $\int_t^\eta s^\mu L(s)ds \sim_{t\to 0^+} -\frac{t^{1+\mu}L(t)}{\mu+1}$.*
(ii) *If $\mu > -1$, then $\int_0^\eta s^\mu L(s)ds$ converges and $\int_0^t s^\mu L(s)ds \sim_{t\to 0^+} \frac{t^{1+\mu}L(t)}{\mu+1}$.*

Lemma 2.4 ([3])**.** *Let $L \in \mathcal{K}$ be defined on $(0,\eta]$. Then*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s}ds} = 0. \tag{2.2}$$

If further $\int_0^\eta \frac{L(s)}{s}ds$ converges, then

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s}ds} = 0. \tag{2.3}$$

Remark 2.5. Let $L \in \mathcal{K}$ defined on $(0,\eta]$, then using (2.1) and (2.2), we deduce that

$$t \to \int_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s}ds$ converges, by (2.3), we have

$$t \to \int_0^t \frac{L(s)}{s}ds \in \mathcal{K}.$$

3. Proof of main results

We need the following lemmas.

Lemma 3.1. *Let $x \in D$ and let $D_x = \{y \in D : |x-y|^2 \le \delta(x)\delta(y)\}$. Then*

(i) *If $y \in D_x$, then*

$$\frac{3-\sqrt{5}}{2}\delta(x) \le \delta(y) \le \frac{3+\sqrt{5}}{2}\delta(x) \tag{3.1}$$

and

$$|x-y| \le \frac{1+\sqrt{5}}{2}\min(\delta(x),\delta(y)).$$

(ii) *If $y \in D_x^c$, then*

$$\max(\delta(x),\delta(y)) \le \frac{1+\sqrt{5}}{2}|x-y|.$$

In particular,

$$B(x, \frac{\sqrt{5}-1}{2}\delta(x)) \subset D_x \subset B(x, \frac{\sqrt{5}+1}{2}\delta(x)). \tag{3.2}$$

(iii) *If $L \in \mathcal{K}$, then there exists $m \ge 0$ such that for each $y \in D_x$, we have*

$$\Big(\frac{3-\sqrt{5}}{2}\Big)^m L(\delta(x)) \le L(\delta(y)) \le \Big(\frac{3+\sqrt{5}}{2}\Big)^m L(\delta(x)). \tag{3.3}$$

Proof. The proof of (i) and (ii) can be found in [10].

(iii) Let $x \in D$, $y \in D_x$ and $L \in \mathcal{K}$. There exist $c > 0$, $z \in C([0,1])$ such that $z(0) = 0$ and satisfying for each $t \in (0,\eta]$

$$L(t) = c\exp\Big(\int_t^\eta \frac{z(s)}{s}ds\Big).$$

Let $m = \sup_{s\in[0,\eta]} |z(s)|$, then for each $s \in [0,\eta]$, we have $-m \le z(s) \le m$. This together with (3.1) implies

$$m\log\Big(\frac{3-\sqrt{5}}{2}\Big) \le \Big|\int_{\delta(y)}^{\delta(x)} \frac{z(s)}{s}ds\Big| \le m\log\Big(\frac{3+\sqrt{5}}{2}\Big).$$

It follows that

$$\Big(\frac{3-\sqrt{5}}{2}\Big)^m L(\delta(x)) \le L(\delta(y)) \le \Big(\frac{3+\sqrt{5}}{2}\Big)^m L(\delta(x)).$$

□

Lemma 3.2. *Let q be a nonnegative measurable function in D satisfying* (H1) *and assume that φ satisfies* (H0). *Then (1)*

$$\begin{aligned}
&\int_{D_x^c} \frac{1}{|x-y|^{n-\beta}}\varphi\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\beta/2}\Big)q(y)dy\\
&\approx \int_{D_x^c} \frac{(\delta(x))^{\beta/2}(\delta(y))^{\frac{\beta}{2}-\mu}}{|x-y|^n}L(\delta(y))dy\\
&\approx (\delta(x))^{\frac{\beta}{2}-1}\int_{D_x^c} G_D(x,y)(\delta(y))^{\frac{\beta}{2}-\mu-1}L(\delta(y))dy
\end{aligned}$$

(2)

$$\begin{aligned}
&\int_{D_x} \frac{1}{|x-y|^{n-\beta}}\varphi\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\beta/2}\Big)q(y)dy\\
&\approx (\delta(x))^{\beta-\mu}L(\delta(x))\\
&\approx (\delta(x))^{\frac{\beta}{2}-1}\int_{D_x} G_D(x,y)(\delta(y))^{\frac{\beta}{2}-\mu-1}L(\delta(y))dy
\end{aligned}$$

Proof. (1) If $y \in D_x^c$, then $0 < \frac{\delta(x)\delta(y)}{|x-y|^2} \le 1$, so since $\beta > 0$,

$$\varphi\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\beta/2}\Big) \approx \frac{(\delta(x)\delta(y))^{\beta/2}}{|x-y|^\beta}.$$

By (1.1), it follows that

$$\begin{aligned}
&\int_{D_x^c} \frac{1}{|x-y|^{n-\beta}}\varphi\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\beta/2}\Big)q(y)dy\\
&\approx \int_{D_x^c} \frac{(\delta(x))^{\beta/2}(\delta(y))^{\frac{\beta}{2}-\mu}}{|x-y|^n}L(\delta(y))dy\\
&\approx (\delta(x))^{\frac{\beta}{2}-1}\int_{D_x^c} G_D(x,y)(\delta(y))^{\frac{\beta}{2}-\mu-1}L(\delta(y))dy.
\end{aligned}$$

(2) If $y \in D_x$, then $\frac{3-\sqrt{5}}{2}\delta(x) \le \delta(y) \le \frac{3+\sqrt{5}}{2}\delta(x)$. So

$$\varphi\Big(\Big(\frac{(\sqrt{5}-1)\delta(x)}{2|x-y|}\Big)^\beta\Big) \le \varphi\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\beta/2}\Big) \le \varphi\Big(\Big(\frac{(\sqrt{5}+1)\delta(x)}{2|x-y|}\Big)^\beta\Big). \tag{3.4}$$

On the one hand, using (3.1) and (3.3), for $c > 0$ we have

$$\int_{B(x,c\delta(x))} \frac{1}{|x-y|^{n-\beta}}\varphi\Big(\Big(\frac{c\delta(x)}{|x-y|}\Big)^\beta\Big)(\delta(y))^{-\mu}L(\delta(y))\,dy$$

$$
\begin{aligned}
&\approx (\delta(x))^{-\mu} L(\delta(x)) \int_0^{c\delta(x)} r^{\beta-1} \varphi\Big(\Big(\frac{c\delta(x)}{r}\Big)^{\beta}\Big) dr \\
&\approx (\delta(x))^{\beta-\mu} L(\delta(x)) \Big(\int_1^{\infty} \frac{\varphi(t)}{t^2}\, dt \Big) \\
&\approx (\delta(x))^{\beta-\mu} L(\delta(x)).
\end{aligned}
$$

By using (3.2), we have also that

$$
\int_{D_x} G(x,y)\, dy \approx (\delta(x))^2.
$$

It follows by (3.1) and (3.2) that

$$
\begin{aligned}
&\int_{D_x} \frac{1}{|x-y|^{n-\beta}} \varphi\Big(\Big(\frac{\delta(x)\delta(y)}{|x-y|^2}\Big)^{\beta/2}\Big) q(y) dy \\
&\approx (\delta(x))^{\beta-\mu} L(\delta(x)) \\
&\approx (\delta(x))^{\beta-\mu-2} L(\delta(x)) \int_{D_x} G(x,y)\, dy \\
&\approx (\delta(x))^{\frac{\beta}{2}-1} \int_{D_x} G_D(x,y) (\delta(y))^{-(\mu-\frac{\beta}{2}+1)} L(\delta(y)) dy.
\end{aligned}
$$

□

Proof of Theorem 1.2. By using Lemma 3.2 and Proposition 1.1 with $\lambda = \mu - \frac{\beta}{2} + 1$, we obtain

$$
\begin{aligned}
Vq(x) &= \int_D \Gamma_D(x,y) q(y) dy \\
&\approx (\delta(x))^{\frac{\beta}{2}-1} \int_D G_D(x,y)(\delta(y))^{-(\mu-\frac{\beta}{2}+1)} L(\delta(y)) dy \\
&\approx \begin{cases}
(\delta(x))^{\frac{\beta}{2}-1} \Big(\int_0^{\delta(x)} \frac{L(s)}{s}\, ds \Big) & \text{if } \mu = \frac{\beta}{2}+1\,, \\
(\delta(x))^{\beta-\mu} L(\delta(x)) & \text{if } \frac{\beta}{2} < \mu < \frac{\beta}{2}+1\,, \\
(\delta(x))^{\beta/2} \Big(\int_{\delta(x)}^{\eta} \frac{L(s)}{s}\, ds \Big) & \text{if } \mu = \frac{\beta}{2} \\
(\delta(x))^{\beta/2} & \text{if } \mu < \frac{\beta}{2}.
\end{cases}
\end{aligned}
$$

□

Corollary 3.3. *Let $\sigma \in [0,1)$ and assume that a satisfies* (H2). *Let θ_λ be the function defined by* (1.5). *Then $V(a\theta_\lambda^\sigma)(x) \approx \theta_\lambda(x)$.*

Proof. We have

$$
a(x)\theta_\lambda^\sigma(x) = \begin{cases}
(\delta(x))^{(\frac{\beta}{2}-1)\sigma-\lambda} L(\delta(x)) \Big(\int_0^{\delta(x)} \frac{L(s)}{s}\, ds \Big)^{\frac{\sigma}{1-\sigma}} \\
\quad \text{if } \lambda = \frac{\beta}{2}(1+\sigma) + (1-\sigma)\,, \\
(\delta(x))^{\frac{(\beta-\lambda)\sigma}{1-\sigma}-\lambda} (L(\delta(x)))^{1/(1-\sigma)} \\
\quad \text{if } \frac{\beta}{2}(1+\sigma) < \lambda < \frac{\beta}{2}(1+\sigma) + (1-\sigma)\,, \\
(\delta(x))^{\frac{\beta}{2}\sigma-\lambda} L(\delta(x)) \Big(\int_{\delta(x)}^{\eta} \frac{L(s)}{s}\, ds \Big)^{\frac{\sigma}{1-\sigma}} & \text{if } \lambda = \frac{\beta}{2}(1+\sigma)\,, \\
(\delta(x))^{\frac{\beta}{2}\sigma-\lambda} L(\delta(x)) & \text{if } \lambda < \frac{\beta}{2}(1+\sigma).
\end{cases}
$$

So, we see that
$$a(x)\theta_\lambda^\sigma(x) = (\delta(x))^{-\mu}\,\widetilde{L}(\delta(x))\,,$$
where $\mu \le \frac{\beta}{2}+1$ and according to Lemma 2.1 and Lemma 2.3, we have $\widetilde{L} \in \mathcal{K}$ with $\int_0^\eta t^{\frac{\beta}{2}-\mu}\,\widetilde{L}(t)\,dt < \infty$. Hence the result follows from Theorem 1.2. $\square$

Proof of Theorem 1.3. Let $\sigma \in [0,1)$ and assume that a satisfies (H_2). Then by Corollary 3.3, there exists a positive constant M such that
$$\frac{1}{M}\theta_\lambda \le V(a\theta_\lambda^\sigma) \le M\theta_\lambda \tag{3.5}$$
Put $c_0 = M^{1/(1-\sigma)}$ and consider the nonempty closed convex set
$$\Lambda = \left\{u \in B^+(D):\ \frac{1}{c_0}\theta_\lambda \le u \le c_0\theta_\lambda\right\},$$
where $B^+(D)$ denotes the set of nonnegative Borel measurable functions in D. Let T be the operator defined on Λ by $Tu = V(au^\sigma)$. Since $\sigma \in [0,1)$, then T is nondecreasing on Λ. Now, using (3.5) we deduce that $T\Lambda \subset \Lambda$. Consider the sequence (u_k) defined by $u_0 = \frac{1}{c_0}\theta_\lambda$ and $u_{k+1} = Tu_k$ for $k \ge 0$. Then, using again (3.5) and the monotonicity of T, we obtain
$$\frac{1}{c_0}\theta_\lambda = u_0 \le u_1 \le \dots \le u_k \le c_0\theta_\lambda.$$
Hence, it follows from the monotone convergence theorem that the sequence $(u_k)_k$ converges to a function $u \in \Lambda$ satisfying the integral equation
$$u = V(au^\sigma). \tag{3.6}$$
Finally, we aim at proving that the integral equation (3.6) has a unique solution comparable to θ_λ. Let $u, v \in B^+(D)$ such that $u = V(au^\sigma)$, $v = V(av^\sigma)$ and $u \approx \theta_\lambda \approx v$. Then there exists $k > 0$ such that
$$\frac{1}{k} \le \frac{u}{v} \le k.$$
So the set $J = \{t \in (0,1] : tu \le v\}$ is nonempty. Let $c = \sup(J)$ and assume that $c < 1$. Then, we have $cu \le v$ and $v - c^\sigma u = V(a(v^\sigma - c^\sigma u^\sigma)) \ge 0$. Which implies that
$$cu \le c^\sigma u \le v \quad \text{in } D.$$
This contradicts the fact that $c = \sup(J)$. So $c = 1$ and $u \le v$. By symmetry, we deduce that $u = v$. $\square$

4. Examples

Example 4.1. Let $0 < \sigma < 1$ and let a be a nonnegative measurable function such that $a(x) \approx (\delta(x))^{-\lambda}L(\delta(x))$, with $\lambda \le 2$ and $L \in \mathcal{K}$ such that $\int_0^\eta t^{1-\lambda}L(t)\,dt < \infty$. Then the Dirichlet problem
$$\begin{gathered} -\Delta u = a(x)u^\sigma \quad \text{in } D, \\ u = 0 \quad \text{on } \partial D \end{gathered} \tag{4.1}$$

has a unique positive continuous solution u satisfying $u(x) \approx \theta_\lambda(x)$, where

$$\theta_\lambda(x) = \begin{cases} \Big(\int_0^{\delta(x)} \frac{L(s)}{s} ds\Big)^{1/(1-\sigma)} & \text{if } \lambda = 2\,, \\ (\delta(x))^{\frac{2-\lambda}{1-\sigma}} \big(L(\delta(x))\big)^{1/(1-\sigma)} & \text{if } 1+\sigma < \lambda < 2, \\ \delta(x)\Big(\int_{\delta(x)}^{\eta} \frac{L(s)}{s} ds\Big)^{1/(1-\sigma)} & \text{if } \lambda = 1+\sigma\,, \\ \delta(x) & \text{if } \lambda < 1+\sigma\,. \end{cases}$$

Indeed we deduce by [1] that if a satisfies (H_2), then $V(a)$ is continuous in $\overline{D}$ with boundary value zero. This together with the boundedness of u and the fact that $0 < \sigma < 1$ give that u is a solution of (4.1) if and only if u satisfies (3.6). So the result follows from Theorem 1.3.

Example 4.2. Let $0 < \sigma < 1$, $0 < \alpha < 2$ and a be a nonnegative measurable function such that $a(x) \approx (\delta(x))^{-\lambda} L(\delta(x))$, with $\lambda \leq \frac{\alpha}{2}(1+\sigma) + (1-\sigma)$ and $L \in \mathcal{K}$ such that $\int_0^\eta t^{\frac{\alpha}{2}(1+\sigma)-\sigma-\lambda} L(t)\,dt < \infty$. Then the Dirichlet problem

$$\begin{gathered} (-\Delta)^{\alpha/2} u = a(x) u^\sigma \quad \text{in } D, \\ \lim_{x\to\partial D} (\delta(x))^{1-\frac{\alpha}{2}} u(x) = 0 \quad \text{on } \partial D \end{gathered} \tag{4.2}$$

has a unique positive continuous solution u satisfying $u(x) \approx \theta_\lambda(x)$, where

$$\theta_\lambda(x) = \begin{cases} \big(\delta(x)\big)^{\frac{\alpha}{2}-1}\Big(\int_0^{\delta(x)} \frac{L(s)}{s} ds\Big)^{1/(1-\sigma)} & \text{if } \lambda = \frac{\alpha}{2}(1+\sigma) + (1-\sigma)\,, \\ \big(\delta(x)\big)^{\frac{\alpha-\lambda}{1-\sigma}} \big(L(\delta(x))\big)^{1/(1-\sigma)} & \text{if } \frac{\alpha}{2}(1+\sigma) < \lambda < \frac{\alpha}{2}(1+\sigma) + (1-\sigma)\,, \\ \big(\delta(x)\big)^{\alpha/2}\Big(\int_{\delta(x)}^{\eta} \frac{L(s)}{s} ds\Big)^{1/(1-\sigma)} & \text{if } \lambda = \frac{\alpha}{2}(1+\sigma)\,, \\ \big(\delta(x)\big)^{\alpha/2} & \text{if } \lambda < \frac{\alpha}{2}(1+\sigma)\,, \end{cases}$$

Indeed we deduce by [3] that u is a solution of (4.2) if and only if u satisfies (3.6). So the result follows from Theorem 1.3.

Example 4.3. Let $0 < \sigma < 1$, $0 < \alpha < 2$ and a be a nonnegative measurable function such that $a(x) \approx (\delta(x))^{-\lambda} L(\delta(x))$, where $\lambda < \alpha + (2-\alpha)(1-\sigma)$ and L is defined on $(0, \eta]$ belongs to $\mathcal{K}$. Consider the Dirichlet problem

$$\begin{gathered} (-\Delta_{/D})^{\alpha/2} u = a(x) u^\sigma \quad \text{in } D, \\ \lim_{x\to\partial D} \big(\delta(x)\big)^{2-\alpha} u(x) = 0. \end{gathered} \tag{4.3}$$

Since by [15] the Green function G_α^D of $(-\Delta_{/D})^{\alpha/2}$ satisfies

$$G_\alpha^D(x,y) \approx \frac{1}{|x-y|^{n-\alpha}} \min\Big(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\Big)\,.$$

Then we deduce that (4.3) has a unique positive continuous solution u in D satisfying $u(x) \approx \theta_\lambda(x)$, where

$$\theta_\lambda(x) = \begin{cases} \big(\delta(x)\big)^{\frac{\alpha-\lambda}{1-\sigma}} \big(L(\delta(x))\big)^{1/(1-\sigma)} & \text{if } \alpha-(1-\sigma) < \lambda < \alpha + (2-\alpha)(1-\sigma)\,, \\ \big(\delta(x)\big)\Big(\int_{\delta(x)}^{\eta} \frac{L(s)}{s} ds\Big)^{1/(1-\sigma)} & \text{if } \lambda = \alpha-(1-\sigma)\,, \\ \big(\delta(x)\big) & \text{if } \lambda < \alpha-(1-\sigma)\,. \end{cases}$$

Indeed we deduce by [11] that u is a solution of (4.3) if and only if u satisfies (3.6). So the result follows from Theorem 1.3.

Example 4.4. Let $0 < \sigma < 1$, m a positive integer and let a be a nonnegative measurable function in $B(0,1)$ such that $a(x) \approx (\delta(x))^{-\lambda} L(\delta(x))$, where $\lambda \leq m(1+\sigma) + (1-\sigma)$, $L \in \mathcal{K}$ with $\int_0^\eta t^{m(1+\sigma)-\sigma-\lambda} L(t)\, dt < \infty$. Consider the following Dirichlet problem

$$\begin{gathered}
(-\Delta)^m u = a(x) u^\sigma \quad \text{in } B(0,1), \\
\lim_{|x|\to 1} \frac{u(x)}{(1-|x|)^{m-1}} = 0 \quad \text{on } \partial B(0,1).
\end{gathered} \tag{4.4}$$

Let $G_{m,n}$ be the Green function of the polyharmonic operator $(-\Delta)^m$ on $B(0,1)$ with Dirichlet boundary conditions $u = \frac{\partial}{\partial\nu} u = \dots = \frac{\partial^{m-1}}{\partial\nu^{m-1}} u = 0$. Then by [1],

$$G_{m,n}(x,y) \approx \begin{cases} |x-y|^{2m-n} \min\big(1, \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\big) & \text{if } n > 2m\,, \\ \log\big(1 + \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\big) & \text{if } n = 2m, \\ (\delta(x)\delta(y))^{m-\frac{n}{2}} \min\big(1, \frac{(\delta(x)\delta(y))^{\frac{n}{2}}}{|x-y|^n}\big) & \text{if } n < 2m. \end{cases}$$

So we can apply our results to deduce that (4.4) has a positive continuous solution u satisfying $u(x) \approx \theta_\lambda(x)$, where

$$\theta_\lambda(x) = \begin{cases} \big(\delta(x)\big)^{m-1} \Big(\int_0^{\delta(x)} \frac{L(s)}{s} ds \Big)^{1/(1-\sigma)} & \text{if } \lambda = m(1+\sigma) + (1-\sigma)\,, \\ \big(\delta(x)\big)^{\frac{2m-\lambda}{1-\sigma}} \big(L(\delta(x))\big)^{1/(1-\sigma)} & \text{if } m(1+\sigma) < \lambda < m(1+\sigma) + (1-\sigma)\,, \\ \big(\delta(x)\big)^m \Big(\int_{\delta(x)}^\eta \frac{L(s)}{s} ds \Big)^{1/(1-\sigma)} & \text{if } \lambda = m(1+\sigma)\,, \\ (\delta(x))^m & \text{if } \lambda < m(1+\sigma)\,. \end{cases}$$

In this case, we refer to [6] to deduce that if a satisfies (H2) and $0 < \sigma < 1$, then the m-potential $V(a\theta_\lambda)$ is continuous in $\overline{D}$ with boundary value zero. Hence if u satisfies (3.6), then u is a solution of (4.4). So the result follows from theorem 1.3.

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